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On the cone of f-vectors of cubical polytopes

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Abstract. What is the minimal closed cone containing all f-vectors of cubical d-polytopes? We construct cubical polytopes showing that this cone, expressed in the cubical g-vector coordinates, contains the nonnegative g-orthant, thus verifying one direction of the Cubical Generalized Lower Bound Conjecture of Babson, Billera and Chan. Our polytopes also show that a natural cubical analogue of the simplicial Generalized Lower Bound Theorem does not hold.

Keywords: cubical polytope, cubical g-vector

1 Introduction

Understanding the possible face numbers of polytopes, and of subfamilies of interest, is a fundamental question, dealt with since antiquity. The celebrated g-theorem, conjectured by McMullen [8] and proved by Stanley [13] (necessity) and by Billera and Lee [4] (sufficiency), characterizes the f-vectors of simplicial polytopes. Here we consider f-vectors of cubical polytopes; a d-polytope Q is *cubical* if all its facets are combinatorially isomorphic to the $(d-1)$-cube. Adin [1] proved analogues of the Dehn–Sommerville relations for cubical polytopes, thus encoding the f-vector of Q by its (long) cubical g-vector

$$g^c(Q) = \left(g^c_1(Q), g^c_2(Q), \ldots, g^c_{\lfloor d/2 \rfloor}(Q)\right)$$

(with the constant $g^c_0(Q) = 2^{d-1}$ omitted). The construction of neighborly cubical d-polytopes by Joswig and Ziegler [6], where the number of vertices varies, shows that the linear span of the vectors $g^c(Q)$, over all cubical d-polytopes, is the entire vector space $\mathbb{R}^{\lfloor d/2 \rfloor}$. Adin [1, Question 2] asked whether $g^c(Q)$ is always in the nonnegative orthant, and Babson, Billera and Chan [3, Conjecture 5.2] conjectured further that the minimal closed cone $\mathcal{C}_d$ containing all the vectors $g^c(Q)$ corresponding to cubical d-polytopes is exactly this nonnegative orthant $\mathcal{A}_d$.

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Denote by e_i the i-th unit vector in $\mathbb{R}^{\lfloor d/2 \rfloor}$. Stacked cubical polytopes show that the ray spanned by e_1 is in $\mathcal{C}_d$, and neighborly cubical polytopes show that the ray spanned by $e_{\lfloor d/2 \rfloor}$ is in $\mathcal{C}_d$. Our main result is that all the rays spanned by the vectors e_i are in $\mathcal{C}_d$:

Theorem 1.1. $\mathcal{A}_d \subseteq \mathcal{C}_d$.

The conjecture of Adin and Babson–Billera–Chan is

Conjecture 1.2. $\mathcal{A}_d = \mathcal{C}_d$.

An analogue of Theorem 1.1 was previously known for the much wider class of PL cubical spheres [3, Theorem 5.7]. Also, Conjecture 1.2 holds for $d \leq 5$, by combining the constructions above with Steve Klee's result [7, Prop.3.7] asserting that $g_k^c(Q) \geq 0$ for any cubical polytope Q of dimension $2k+1$.

Sanyal and Ziegler [12] showed how to construct, from any simplicial $(d-2)$-polytope P on $n-1$ vertices and a total order $v_1 < v_2 < \ldots < v_{n-1}$ on its vertices, a cubical d-polytope $Q = Q(P, <)$ on 2^n vertices; it is the projection of a deformed n-cube in $\mathbb{R}^n$ onto the last d coordinates. Further, they showed that if P is k-neighborly then the k-skeleton of Q is isomorphic to the k-skeleton of the n-cube. We apply their construction to the case where P_n is the k-neighborly k-stacked $(d-2)$-polytope on $n-1$ vertices constructed by McMullen and Walkup [9], with $1 \leq k \leq \lfloor \frac{d-2}{2} \rfloor$, and with a suitable total order $<$. Analyzing the cubical g-vectors of the resulting polytopes $Q(k,d,n) = Q(P_n, <)$, as n tends to infinity, gives Theorem 1.1. See Theorem 5.4 and Corollary 5.5 for the exact values and asymptotic behavior of the cubical g-vectors.

The generalized lower bound theorem for simplicial polytopes (GLBT), conjectured by McMullen and Walkup [9] and proved by Murai and Nevo [10], asserts that for $1 \leq k < \lfloor d/2 \rfloor$, a simplicial d-polytope P is k-stacked if and only if $g_{k+1}(P) = 0$. The polytopes $Q(k,d,n)$ demonstrate that the natural cubical analogue of the GLBT is false:

Theorem 1.3. *For any $k \geq 1$ and $n \geq d \geq 2k+2$, we have $g_{k+2}^c(Q(k,d,n)) = 0$, and $Q(k,d,n)$ is not cubical $(k+1)$-stacked.*

This is an extended abstract. For the complete paper, see [2].

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2 Preliminaries

The purpose of this section is mainly to set the notation that we will use throughout the paper. For undefined terminology we refer the reader to [14]. A **d-dimensional polytope** P is the convex hull of a finite set of points in $\mathbb{R}^d$ which affinely span $\mathbb{R}^d$. A (proper) **face** σ of P is the intersection of P with one of its supporting hyperplanes, the **dimension** $\dim \sigma$ of σ is then the dimension of the affine span of that intersection. The faces of dimensions 0, 1, and $d-1$ are called **vertices**, **edges**, and **facets**, respectively. The empty set $\emptyset$ and the polytope P itself are called **trivial faces** and have dimensions -1 and d, respectively. We will abbreviate and write d-polytope and i-face to denote dimension. We denote by $\mathrm{Vert}(P)$ the set of vertices of P, and for a vertex $v \in \mathrm{Vert}(P)$, we denote by P/v the **vertex figure** of P at v, that is, P/v is a $(d-1)$-polytope obtained as the intersection of P with a hyperplane which strictly separates v from $\mathrm{Vert}(P) \setminus \{v\}$; the face lattice of P/v does not depend on the separating hyperplane chosen.

A **polytopal complex** K is a finite collection of polytopes in $\mathbb{R}^d$ such that

(i) the empty polytope is in K,

(ii) if $P \in K$ then all faces of P are also in K,

(iii) if $P, Q \in K$ then $P \cap Q$ is a face of both P and Q.

The **dimension** $\dim K$ of K is the maximum of $\dim P$ over all $P \in K$; we say that K is a $\dim K$-complex. The elements in K are called **faces**; the faces of dimension $\dim K$ are called **facets**. For $F \in K$ we define the (open) **star** of F and the **antistar** of F, respectively, by

$$\mathrm{st}_F(K) = \{G \in K \mid F \subseteq G\},$$
$$\mathrm{ast}_F(K) = \{G \in K \mid F \nsubseteq G\}.$$

The number of i-faces in K is denoted by $f_i(K)$, and the **f-vector** of K is $f(K) = (f_0(K), f_1(K), \ldots, f_{\dim K}(K))$. The **$f$-polynomial** of K is defined by

$$f(K,t) = \sum_{i=0}^{\dim K+1} f_{i-1}(K) t^i,$$

where $f_{-1}(K) = 1$.

For a polytope P we denote by $\langle P \rangle$ the complex of all faces of P. The **boundary complex** ∂P is the complex formed by all the proper faces of P, that is $\partial P = \langle P \rangle \setminus \{P\}$. We also define the f-vector and f-polynomial of P by $f(P) = f(\partial P)$ and $f(P,t) = f(\partial P, t)$. We use $\mathrm{lk}_v(P)$ to denote the boundary complex $\partial\,(P/v)$ of the vertex figure of P at v.

2.1 Simplicial complexes and polytopes

A **simplicial complex** is a polytopal complex in which all polytopes are simplices. Let K be a simplicial $(D-1)$-complex; the **h-polynomial** of K is defined by

$$h(K,t) = h_0(K) + h_1(K)t + \cdots + h_D(K)t^D$$
$$:= (1-t)^D \cdot f\left(K, \frac{t}{1-t}\right),$$

and the **h-vector** of K is $(h_0(K), \ldots, h_D(K))$. If $K = \partial P$ for a simplicial D-polytope P then the **Dehn–Sommerville relations** assert that $h_i(K) = h_{D-i}(K)$ for any $0 \le i \le D$. The corresponding **g-vector** $\left(g_0(K), \ldots, g_{\lfloor D/2 \rfloor}(K)\right)$ of K is then defined by

$$g_0(K) = h_0(K) = 1,$$
$$g_i(K) = h_i(K) - h_{i-1}(K), \quad \text{for all} \quad 1 \le i \le \lfloor D/2 \rfloor .$$

For two simplicial complexes K and L we define the **join** $K * L$ to be the simplicial complex whose simplices are the disjoint unions of simplices of K and simplices of L.

A polytope is **simplicial** if each of its proper faces is a simplex. For a simplicial polytope P we write $h(P,t)$ to mean $h(\partial P, t)$, and similarly $h_i(P) := h_i(\partial P)$ and $g_i(P) := g_i(\partial P)$.

A simplicial d-polytope P is called **k-stacked** if P has a triangulation in which there are no interior faces of dimension less than $d-k$. A simplicial polytope P is called **k-neighborly** if each subset of at most k vertices forms the vertex set of a face of P. We denote by $C(d,n)$ the **cyclic d-polytope with n vertices**:

$$C(d,n) := \operatorname{conv}\{x(t_1), x(t_2) \ldots, x(t_n)\},$$

where $t_1 < t_2 < \cdots < t_n$ and $x(t) := \left(t, t^2, \ldots, t^d\right)$ is the moment curve in $\mathbb{R}^d$.

2.2 Cubical complexes and polytopes

A **cubical complex** is a polytopal complex in which all polytopes are combinatorially isomorphic to cubes. Let Q be a cubical $(d-1)$-complex, the **short cubical h-polynomial** is defined by

$$h^{sc}(Q,t) = \sum_{i=0}^{d-1} h_i^{sc}(Q)t^i = \sum_{j=0}^{d-1} f_j(Q)(2t)^j(1-t)^{d-1-j}.$$

When Q is clear from the context, we may sometimes drop it from the notation, as we do in the following few definitions. The **(long) cubical h-vector** $(h_0^c, h_1^c, \ldots, h_d^c)$ is defined by

$$h_0^c = 2^{d-1},$$
$$h_i^{sc} = h_i^c + h_{i+1}^c, \quad \text{for} \quad 0 \le i \le d-1,$$

and the corresponding (**short** and **long**) **cubical** g**-vector**s are defined, as in the simplicial case, by

$$g_0^{sc} = h_0^{sc} = f_0, \qquad g_i^{sc} = h_i^{sc} - h_{i-1}^{sc} \quad \text{for } 1 \le i \le \lfloor (d-1)/2 \rfloor\,;$$
$$g_0^{c} = h_0^{c} = 2^{d-1}, \qquad g_i^{c} = h_i^{c} - h_{i-1}^{c} \quad \text{for } 1 \le i \le \lfloor d/2 \rfloor\,.$$

A polytope is **cubical** if each of its proper faces is combinatorially a cube. Adin [1] showed that any cubical d-polytope Q satisfies an analogue of the Dehn–Sommerville relations: $h_i^c(Q) = h_{d-i}^c(Q)$ for all $0 \le i \le d$.

In analogy with the simplicial case, [3] defined cubical neighborliness and cubical stackedness: a cubical d-polytope is k**-neighborly** if it has the k-skeleton of a cube (of some dimension); it is k**-stacked** if it has a cubical subdivision with no interior faces of dimension less than $d-k$.

Each vertex figure in a cubical d-polytope P is a simplicial $(d-1)$-polytope; we finish this section with the relation known as **Hetyei's observation**:

$$h^{sc}(P,t) := h^{sc}(\partial P, t) = \sum_{v \in \mathrm{Vert}(P)} h(\mathrm{lk}_v(P), t). \tag{2.1}$$

It shows that the cubical Dehn–Sommerville relations follow from the simplicial ones.

3 The McMullen–Walkup polytopes

In section 3 of [9], McMullen and Walkup describe the construction of k-neighborly k-stacked simplicial D-polytopes with N vertices, for any set of parameters satisfying $2 \le 2k \le D < N$. Their construction takes a k-neighborly $2k$-polytope C with $N-D+2k$ vertices (e.g. the cyclic $2k$-polytope with $N-D+2k$ vertices), and a $(D-2k)$-simplex T, both lying in $\mathbb{R}^D$ in such a way that the relative interior of T intersects the affine hull $\mathrm{Aff}(C)$ in a vertex x of C. Then the convex hull $\mathrm{conv}(C \cup T)$ is the desired polytope. We define a slightly more general construction.

Definition 3.1. Let $2 \le K \le D < N$. Let $C = C(K, N-D+K)$ be the cyclic K-polytope with $N-D+K$ vertices, and let T be a $(D-K)$-simplex, both lying in $\mathbb{R}^D$ in such a way that the relative interior of T intersects $\mathrm{Aff}(C)$ in a vertex x of C. The polytope $\mathrm{conv}(C \cup T)$ is a D-dimensional simplicial polytope with N vertices, denoted $\mathsf{MW}(K, D, N; x)$.

The boundary complex of $\mathsf{MW}(K, D, N; x)$ is thus described by

$$\partial \mathsf{MW}(K, D, N; x) = \langle T \rangle * \mathrm{lk}_x(C) \bigcup_{\partial T * \mathrm{lk}_x(C)} \partial T * \mathrm{ast}_x(C). \tag{3.1}$$

McMullen and Walkup have shown that $\mathsf{MW}(2k, D, N; x)$ is k-neighborly as well as k-stacked, thus satisfying

$$g_i(\mathsf{MW}(2k, D, N; x)) = \begin{cases} 0 & i > k, \\ \left(\!\binom{N-D-1}{i}\!\right) = \binom{N-D+i-2}{i} & i \leq k. \end{cases} \tag{3.2}$$

In fact, the proof that $\mathsf{MW}(2k, D, N; x)$ is k-neighborly and k-stacked given in [9, p. 269] shows:

Lemma 3.2. *The polytope* $\mathsf{MW}(K, D, N; x)$ *is* $\lfloor K/2 \rfloor$*-neighborly and* $\lfloor K/2 \rfloor$*-stacked. In particular,*

$$g_i(\mathsf{MW}(2k-1, D, N; x)) = \begin{cases} 0 & i > k-1, \\ \left(\!\binom{N-D-1}{i}\!\right) = \binom{N-D+i-2}{i} & i \leq k-1. \end{cases} \tag{3.3}$$

The vertices of C come with a natural order $v_1 < v_2 < \ldots v_{N-D+K}$ (inherited from the order of the parameters $t_1 < t_2 < \cdots < t_{N-D+K}$ in the definition of C). We will take x to be the last vertex in that ordering, denoting the resulting polytope simply by $\mathsf{MW}(K, D, N)$. Removing $x = v_{N-D+K}$, we extend the order $v_1 < \cdots < v_{N-D+K-1}$ of the remaining vertices of C to an order $v_1 < \cdots < v_{N-D+K-1} < v_{N-D+K} < \cdots < v_N$ of the vertices of $\mathsf{MW}(K, D, N)$, where $v_{N-D+K}, \ldots, v_N$ are the vertices of the $(D-K)$-simplex T. We will use the following result:

Lemma 3.3. $\mathsf{MW}(2k, D, N) \big/ v_1$ *is combinatorially isomorphic to* $\mathsf{MW}(2k-1, D-1, N-1)$.

Proof sketch. For $C = C(2k, N-D+2k)$ denote $C' = C \big/ v_1$, and note that $C' \cong C(2k-1, N-D+2k-1)$. Applying the construction in Definition 3.1 with C' produces an $\mathsf{MW}(2k-1, D-1, N-1)$ with boundary complex

$$\langle T \rangle * \mathrm{lk}_x(C') \bigcup_{\partial T * \mathrm{lk}_x(C')} \partial T * \mathrm{ast}_x(C'). \tag{3.4}$$

Now one shows that the complex above is equal to $\mathrm{lk}_{v_1}(\mathsf{MW}(2k, D, N))$. □

4 The Sanyal–Ziegler construction

We give a very brief sketch of the construction, focusing on the combinatorial description of links of vertices. The reader is prompted to confer with the paper [12], or with Sanyal's diploma thesis [11].

Let $(P, <)$ be a simplicial $(d-2)$-polytope with $n-1$ vertices. Label the vertices $v_1, \ldots, v_{n-1} \in \mathbb{R}^{d-2}$ according to the given order $v_1 < v_2 < \cdots < v_{n-1}$, and assume that the vertices are in general position, i.e., no $d-1$ vertices of P lie on a hyperplane. We start by defining the lexicographic diamonds of P.

4.1 Lexicographic diamonds

Let $w_1, \dots, w_{n-1} \in \mathbb{R}$ be a set of **heights**, and denote by $V^w = \{(w_i, v_i) \mid 1 \le i \le n-1\} \subset \mathbb{R}^{d-1}$ the set of **lifted vertices**. Let $p = (w_0, v_0) \in \mathbb{R}^{d-1}$ be an arbitrary point with $w_0 \gg |w_i|$ for every $1 \le i \le n-1$, and consider the $(d-1)$-polytope $D(P, w) = \operatorname{conv}(V^w, p)$. We call $D(P, w)$ the **diamond** over P with subdivision w, noting that, for w_0 big enough, the combinatorial type of $D(P, w)$ is independent of the choice of point p.

Of special interest are the subdivisions of P induced by the heights

$$(w_1, w_2, \dots, w_{n-1}) = (\pm h, 0, \dots, 0) \quad \text{with } h > 0.$$

The subdivision induced by $(-h, 0, \dots, 0)$ is obtained by **pulling** v_1, it is a triangulation of P, and its cells are the pyramids with apex v_1 over facets in $P \cap P_1$ with $P_1 = \operatorname{conv}(v_2, \dots, v_{n-1})$. The subdivision induced by $(h, 0, \dots, 0)$ is obtained by **pushing** v_1, and it consists of the pyramids with apex v_1 over facets in $P_1 \setminus P$, and one (possibly non-simplex) cell P_1. The ***a*-th lexicographic subdivision** $\operatorname{Lex}_a(P)$ of P is obtained by successively pushing the vertices $v_1, \dots, v_{a-1}$, and then pulling v_a. That is, pushing v_1 creates a subdivision with one non-simplex cell P_1, which we replace by its subdivision obtained by pushing v_2, which, in turn, has only one non-simplex cell $P_2 = \operatorname{conv}\{v_3, \dots, v_{n-1}\}$, and so on, until we finally replace $P_{a-1} = \operatorname{conv}\{v_a, \dots, v_{n-1}\}$ by its triangulation obtained by pulling v_a.

The above iterative procedure amounts to choosing a set of heights $w_1, \dots w_{n-1}$ with

$$w_1 > \cdots > w_{a-1} > 0 > w_a, \quad \text{and} \quad w_{a+1} = \cdots = w_{n-1} = 0.$$

The resulting diamond, denoted D_a, is called the ***a*-th lexicographic diamond**. Its vertices are labeled $v_0, v_1, \dots, v_{n-1}$, with v_0 corresponding to the apex p.

Remark 4.1. Note that pushing or pulling a vertex in a simplex has no effect, thus the (possibly) different diamonds are D_a with $1 \le a \le n-d+1$.

4.2 The vertex figures of Q

Take a Gale transform $G \in \mathbb{R}^{(n-1)\times(n-d)}$ of P that has the form $G = \begin{bmatrix} I_{n-d} \\ \overline{G} \end{bmatrix}$, where $\overline{G} \in \mathbb{R}^{(d-1)\times(n-d)}$. Plugging $\overline{G}$ into the **deformed cube template** (see [12, Definition 3.1]) produces a combinatorial n-cube $C = C_n(\overline{G})$. The projection of C onto the last d coordinates $\pi_d(C)$ is the cubical polytope $Q = Q(P, <)$ mentioned in the introduction.

The following key result from [12][1] states that each vertex figure of Q is combinatorially equivalent to some diamond D_a, and moreover, it tells us which diamond corresponds to a given vertex v of Q.

[1]Theorem 3.7 in [12] actually contains a typo, having $n-d-1$ instead of the correct value $n-d+1$. Their proof, however, does give the correct value.

Lemma 4.2 ([12, Theorem 3.7])**.** *Let v be a vertex of C labeled by $\sigma \in \{+,-\}^n$. Then the vertex figure of $\pi_d(v)$ in Q is isomorphic to D_a with*

$$a = \min\left(\{i \in [n] \mid \sigma_i = +\} \bigcup \{n-d+1\}\right).$$

The isomorphism $D_a \cong Q/v$ is given by: v_i (in D_a) corresponds to the neighbor of v obtained by flipping the $(i+1)$-st coordinate of v $(0 \le i \le n-1)$.

5 The cubical polytopes $Q(k,d,n)$

Fix positive integers $k \ge 1$ and $n \ge d \ge 2k+2$. We apply the Sanyal–Ziegler construction to the McMullen–Walkup polytope $P = \mathrm{MW}(2k, d-2, n-1)$, with a total order $<$ on its vertices as described after Lemma 3.2 above. The result is a d-dimensional cubical polytope $Q = Q(k,d,n) = Q(P,<)$ with 2^n vertices. We now compute its cubical g-vector $g^c(Q)$, in stages.

5.1 Computing $g^{sc}(Q(k,d,n))$

By Hetyei's observation (2.1) we have

$$h_i^{sc}(Q) = \sum_{v \in \mathrm{Vert}(Q)} h_i(\mathrm{lk}_v(Q)) \qquad (0 \le i \le d-1).$$

Therefore, for $1 \le i \le \lfloor \frac{d-1}{2} \rfloor$:

$$g_i^{sc}(Q) = \sum_{v \in \mathrm{Vert}(Q)} g_i(\mathrm{lk}_v(Q)) = \sum_{a=1}^{n-d} 2^{n-a} g_i(D_a) + 2^d g_i(D_{n-d+1}). \tag{5.1}$$

We compute the g-vectors of the diamonds D_a at hand, i.e. for our choice of $(P,<)$.

Proposition 5.1. *For each $1 \le a \le n-d+1$ and $0 \le i \le \lfloor \frac{d-1}{2} \rfloor$:*

$$g_i(D_a) = \begin{cases} 0, & \text{if } i > k+1; \\ \left(\!\binom{n-d-a+1}{k}\!\right), & \text{if } i = k+1; \\ \left(\!\binom{n-d}{i}\!\right), & \text{if } i \le k. \end{cases}$$

Proof sketch. For $a = 1$ this follows from the f-polynomials identity

$$f(D_1,t) = f(\mathrm{Lex}_1(P),t) + t \cdot f(P,t) = (1+t)(f(P,t) - t \cdot f(\mathrm{lk}_{v_1}(P),t)) + t \cdot f(P,t)$$

combined with Lemma 3.2, Lemma 3.3 and the transformation to h-polynomials.

For $a > 1$, contract the edge v_0v_1 in P to obtain the $(a-1)$-lexicographic diamond $D_{a-1}(P_1)$ over $P_1 = \mathsf{MW}(2k, d-2, n-2)$. Use the relation

$$h(D_a(P), t) = h(D_{a-1}(P_1), t) + t \cdot h(\mathrm{lk}_{v_0v_1}(D_a(P), t))$$

and iterate by edge contraction in P_1 etc. □

Combining (5.1) with Proposition 5.1, and noting that $\left(\!\binom{0}{k}\!\right) = 0$ for $k \geq 1$, we conclude

Corollary 5.2. *For each* $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$,

$$g_i^{sc}(Q) = \begin{cases} 0, & \text{if } i > k+1; \\ \sum_{a=1}^{n-d} 2^{n-a} \left(\!\binom{n-d-a+1}{k}\!\right), & \text{if } i = k+1; \\ 2^n \left(\!\binom{n-d}{i}\!\right), & \text{if } i \leq k. \end{cases}$$

5.2 Computing $g^c(Q(k, d, n))$

In order to compute the cubical g-vector of Q, and in particular $g^c_{k+2}(Q)$, we need the following binomial identity; its proof is omitted here.

Lemma 5.3. *For any integers $k \geq 1$ and $m \geq 0$,*

$$\sum_{a=1}^{m} 2^{m-a} \left(\!\binom{m-a+1}{k}\!\right) = (-1)^{k+1} + 2^m \sum_{j=0}^{k} (-1)^j \left(\!\binom{m}{k-j}\!\right).$$

Theorem 5.4. *For each* $1 \leq i \leq \lfloor d/2 \rfloor$,

$$g_i^c(Q) = \begin{cases} 0, & \text{if } i > k+1; \\ 2^n \sum_{j=1}^{i} (-1)^{j-1} \left(\!\binom{n-d}{i-j}\!\right) + (-1)^i 2^d, & \text{if } i \leq k+1. \end{cases}$$

Proof sketch. From the definitions of g^c and g^{sc}, we have

$$g_i^c(Q) = \sum_{j=1}^{i} (-1)^{j-1} g_{i-j}^{sc}(Q) + (-1)^i 2^d \qquad (1 \leq i \leq \lfloor d/2 \rfloor). \tag{5.2}$$

The values of $g_i^c(Q)$ for $i \leq k+1$ now follow easily from Corollary 5.2. It also follows that

$$g_i^c(Q) + g_{i+1}^c(Q) = 0 \qquad (i \geq k+2),$$

and all that remains is to show that $g^c_{k+2}(Q) = 0$. Indeed, by (5.2) and Corollary 5.2,

$$g^c_{k+2}(Q) = \sum_{a=1}^{n-d} 2^{n-a} \left(\!\!\binom{n-d-a+1}{k}\!\!\right) - \left[2^n \sum_{j=0}^{k} (-1)^j \left(\!\!\binom{n-d}{k-j}\!\!\right) + (-1)^{k+1} 2^d\right].$$

Using Lemma 5.3 with $m = n - d$ gives, indeed, $g^c_{k+2}(Q) = 0$ as claimed. □

Corollary 5.5. *Fix $k \geq 1$ and $d \geq 2k+2$, and let $\{Q_n\}_{n=d}^{\infty} = \{Q(k,d,n)\}_{n=d}^{\infty}$. Then*

$$\lim_{n\to\infty} \frac{g^c_{k+1}(Q_n)}{2^n \left(\!\binom{n-d}{k}\!\right)} = 1, \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{g^c_i(Q_n)}{2^n \left(\!\binom{n-d}{k}\!\right)} = 0 \quad (\forall i \neq k).$$

Corollary 5.5 shows that the ray spanned by e_k $(2 \leq k \leq \lfloor d/2 \rfloor)$ in $\mathbb{R}^{\lfloor d/2 \rfloor}$ belongs to the closed cone $\mathcal{C}_d$. Note that this was already known for the ray spanned by $e_{\lfloor d/2 \rfloor}$ because of the existence of neighborly cubical d-polytopes, such as $Q(k, 2k+2, n)$). The ray spanned by e_1 belongs to this cone because of the existence of stacked cubical d-polytopes. Thus $\mathcal{C}_d$ contains $\mathcal{A}_d$, and Theorem 1.1 is proved.

6 No obvious cubical GLBT

In [3], after introducing the definitions of cubical stackedness and cubical neighborliness, the authors show that cubical 1-stacked d-polytopes with at least n vertices exist, for any $n \geq 2^d$ (see [3, Corollary 5.6]). It is also shown (see [3, proof of Proposition 5.5]) that if Q is a cubical k-stacked d-polytope, then $g^c_i(Q) = 0$ for $k < i \leq \lfloor d/2 \rfloor$. The converse claim, namely, that $g^c_{k+1}(Q) = 0$ implies that Q is cubical k-stacked, is false, as our analysis of $Q(k,d,n)$ below shows. This is in apparent contrast with the simplicial GLBT.

Theorem 6.1. *The polytope $Q = Q(k,d,n)$ is not cubical $(k+1)$-stacked.*

Proof sketch. Assume by contradiction that Q is cubical $(k+1)$-stacked, so Q has some cubulation Q', namely a subdivision into (combinatorial) cubes, without interior $(d-k-2)$-faces. Let C_n be the deformed n-cube that Q is a projection of.

Lemma 6.2. *All faces of Q' must be faces of C_n.*

To see this, note that any 1-dimensional subcomplex of C_n which is isomorphic to the graph of an m-cube, is the 1-skeleton of an m-face of C_n.

Each $\mathrm{lk}_v(Q')$ (the simplicial complex whose face lattice is the ideal above the vertex v in the face lattice of Q') is a triangulation of Q/v with no interior $(d-k-3)$-faces. Thus the vertex figure of v in Q — isomorphic to some diamond D_a — is $(k+1)$-stacked, and by the GLBT, $\mathrm{lk}_v(Q')$ is the triangulation obtained from D_a by inserting all $(d-1)$-simplices whose $(d-k-3)$-skeleton is contained in the boundary of the diamond D_a. (We abuse notation and identify ∂D_a with $\mathrm{lk}_v(Q)$ by the isomorphism given in Lemma 4.2.) This description allows us to determine, for each vertex v of Q, the set of d-cubes in Q' that contain v. The compatibility condition mentioned above is the requirement that if u is a vertex in a d-cube from the list of v, then the list of u must contain this d-cube too.

Omitting many details, here is a description of the facets in the $(k+1)$-stacked triangulation of D_a, after introducing some terminology: let G be a subset of vertices of the cyclic polytope C. A *block* $B \subset G$ is a maximal subset of G (w.r.t inclusion) of consecutive vertices, and B is even / odd if its size is. Recall that p denotes the apex of D_a.

Proposition 6.3. *The sets of d vertices that form the $(d-1)$-simplices of the $(k+1)$-stacked triangulation of D_a are exactly the ones of one of the following two types:*

(i) $\{p\} \cup \{2k\text{-set in } C \setminus \{x\} \text{ consisting of even blocks}\} \cup T$,

(ii) $\{a\} \cup \{2k\text{-set } F \text{ in } C \setminus \{x\} \text{ consisting of even blocks}, \min F > a\} \cup T$.

Consider a vertex v_σ of Q with $\sigma \in \{+,-\}^n$, and with $a = \min\{i \mid \sigma_i = +\} < n-d+1$; then $Q/v_\sigma \cong D_a$. Now $v_{\sigma'}$, where σ' is obtained by flipping the $(a+1)$-th coordinate of σ, is a neighbor of v_σ with $Q/v_{\sigma'} \cong D_b$, for some $b > a$. Since $\{v_\sigma, v_{\sigma'}\}$ forms an edge of Q, it must be contained in some d-cube of the cubulation of Q, and, by the isomorphism $Q/v_\sigma \cong D_a$ given after Lemma 4.2, the d-set F of D_a corresponding to this d-cube must be of type (ii). But F is not a d-set of either type (i) or (ii) in D_b, and so the triangulations of Q/v_σ and $Q/v_{\sigma'}$ are incompatible.

This contradicts the assumption that Q' exists. □

7 Concluding remarks

The following question, implicit in [3], asks for a sequence of cubical k-stacked d-polytopes with g^c-vector approaching the ray spanned by e_k. It is still unanswered.

Question 7.1. *Let $2 \le k \le \lfloor d/2 \rfloor - 1$. Does there exist a sequence of cubical k-stacked d-polytopes such that the k-th coordinate of g^c dominates the other coordinates?*

Jockusch studied the lower and upper bound problems for cubical polytopes in [5], where he stated a *Cubical Lower Bound Conjecture*:

Conjecture 7.2 (CLBC, [5])**.** *Let Q be a cubical d-polytope with n vertices. Then*

$$f_k(Q) \ge \left(2^{d-k}\binom{d}{k} - 2^{d-k-1}\binom{d-1}{k}\right)\left(\frac{n}{2^{d-1}} - 2\right) + 2^{d-k}\binom{d}{k} \qquad (1 \le k \le d-1).$$

We prove a cubical version of the *MPW-reduction* stating that the case $k = 1$ of Conjecture 7.2 implies that it holds for all k. The case $k = 1$ is equivalent to $g_2^c \ge 0$.

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