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Cyclic sieving for reduced reflection factorizations of the Coxeter element

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Abstract. In a seminal work, Bessis gave a geometric interpretation of the noncrossing lattice $NC(W)$, associated to a well-generated complex reflection group W. We use this framework to prove, in a unified way, various instances of the cyclic sieving phenomenon on the set of reduced reflection factorizations of the Coxeter element. These include in particular the conjectures of N. Williams on the actions $\mathfrak{Pro}$ and $\mathfrak{Twist}$.

Résumé. Dans un travail pionnier, Bessis a donné une interprétation géométrique du treillis $NC(W)$ des partitions non-croisées, associé à un groupe de réflexion complexe W irréductible et bien engendré. Nous utilisons ce cadre pour démontrer, de manière unifiée, diverses instances du phénomène du crible cyclique sur l'ensemble des factorisations réduites de l'élément de Coxeter en réflexions. Ces résultats incluent notamment les conjectures de N. Williams sur les actions $\mathfrak{Pro}$ et $\mathfrak{Twist}$.

Keywords: Cyclic sieving phenomenon, Coxeter element, reflection factorizations, Hurwitz action, Lyashko–Looijenga morphism

1 Introduction

Already in 1892, Hurwitz knew that there exist n^{n-2} minimal length factorizations of the long cycle $(12\cdots n) \in S_n$ into transpositions. This same number counts maximal chains of the noncrossing lattice $NC(n)$, labeled trees on n vertices, topologically distinct branched coverings of the sphere by itself, and[1] classes of degree n polynomials with prescribed critical values.

It often happens that the most exciting phenomena first observed in the symmetric group S_n, have suitable analogs for other reflection groups W. Our main object of study in this note is the set $\mathrm{Red}_W(c)$ of minimal length reflection factorizations of a Coxeter element c of W. It is enumerated by the Hurwitz number

$$|\mathrm{Red}_W(c)| = \frac{h^n n!}{|W|},$$

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[1] Although these last two objects are naturally overcounted by a factor of n.

where h is the order of the Coxeter element.

Bessis used the noncrossing lattice $NC(W)$ as a combinatorial recipe to build up the universal covering spaces of reflection arrangement complements. In the most difficult part of his work, he had to refine this construction to work with centralizer subgroups of W. The insight for that came from a topological understanding of his and Reiner's work [2] on the CSP:

The cyclic sieving phenomenon (CSP)

In 2004, the cyclic sieving phenomenon was first observed in Minnesota [5]. It occurs when a polynomial $X(q)$ carries orbital information about the action of a cyclic group C on a space X. More precisely, and if C is generated by an element c of order n, we say that the triple $(X, X(q), C)$ *exhibits the cyclic sieving phenomenon (CSP)* if for all integers d,

$$|X^{c^d}| = X(\zeta^d), \tag{1.1}$$

where $\zeta = e^{2\pi i/n}$ and X^{c^d} denotes the set of elements of X fixed by c^d.

Already in his thesis Drew Armstrong had conjectured cyclic sieving phenomena for the poset $NC^{(k)}(W)$ of generalized noncrossing partitions. Bessis and Reiner however, were the first [2] to prove a CSP for $NC(W)$. Their work was later used by Krattenthaler and Müller to prove Armstrong's conjectures, albeit with an approach relying on case by case calculations.

Below we recall the cyclic actions Φ_{cyc} and Ψ_{cyc} that have been considered originally in the context of $NC^{(k)}(W)$, and the action $\mathfrak{Twist}$ on factorizations of the Coxeter element:

Definition 1.1 ([2, 6])**.** For a well-generated group W (see Section 2), we consider the following cyclic actions on the set $\text{Red}_W(c)$ of reduced reflection factorizations of the Coxeter element c of W:

$$\Phi_{\text{cyc}} : (t_1, \cdots t_n) \to \left({}^{({}^c t_n)}t_1, {}^c t_n, t_2, \cdots, t_{n-1}\right) \qquad \Psi_{\text{cyc}} : (t_1, \cdots, t_n) \to ({}^c t_n, t_1, \cdots, t_{n-1})$$

$$\mathfrak{Twist} : (t_1, \cdots, t_n) \to \Big({}^{(t_1 \cdots t_{n-1})}t_n, {}^{(t_1 \cdots t_{n-2})}t_{n-1}, \cdots, {}^{t_1} t_2, t_1 \Big), \tag{1.2}$$

where ${}^w t := wtw^{-1}$ stands for conjugation.

In fact, Williams [6] was the first to conjecture CSP's for the set $\text{Red}_W(c)$. He considered the action $\mathfrak{Pro}$, which is the inverse of Ψ_{cyc} above, and the action $\mathfrak{Twist}$. We are now ready to state our main theorem:

Theorem 1.2. *For an irreducible, well-generated, complex reflection group W, with degrees $d_1, \cdots, d_n$ (see Section 2), Coxeter element c and Coxeter number $h = d_n$, the triple*

$$\left(\text{Red}_W(c), \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}, C \right)$$

exhibits the cyclic sieving phenomenon, where C may be generated by any of the actions Φ_{cyc}, Ψ_{cyc}, *or* $\mathfrak{Twist}$ *given in Definition 1.1 and where* $[n]_q := \dfrac{1-q^n}{1-q}$.

We prove this theorem (see Section 6 after Lemma 6.1) by interpreting the three combinatorial cyclic actions on factorizations, as Galois actions in a topological covering induced by the Lyashko–Looijenga morphism LL (see Section 4 and Section 5). These, it turns out, correspond to scalar actions on certain symmetric fibers of the LL map. Finally a geometric Lemma 6.1 provides a CSP on the latter ones with respect to the Hilbert series of the special fiber $LL^{-1}(\mathbf{0})$, which is exactly the polynomial that appears in Theorem 1.2.

2 Background on reflection groups and their complex geometry

Given a complex vector space $V \cong \mathbb{C}^n$, we say that a *finite* subgroup W of $\text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ is a *complex reflection group* if it is generated by pseudo-reflections. Those are $\mathbb{C}$-linear maps t whose fixed spaces $V^t := \ker(t - \text{id})$ are hyperplanes (i.e. $\text{codim}(V^t) = 1$).

Complex reflection groups act naturally on the polynomial algebra $\mathbb{C}[V] := \text{Sym}(V^*)$ via precomposition (that is, $(w * f)(v) := f(w^{-1} \cdot v)$). The Shephard–Todd–Chevalley theorem states that their invariant algebra $\mathbb{C}[V]^W$, is itself a polynomial algebra. It is generated by n algebraically independent polynomials $(f_1, \cdots, f_n)$, which can be chosen to be homogeneous. Even though there is no canonical selection for them, their degrees $d_i := \deg(f_i)$ are uniquely determined.

One of the most important applications of the Shephard–Todd–Chevalley theorem is on the *geometric invariant theory* (GIT) of W. It implies that the orbit space $W \setminus V$ can be identified with an n-dimensional complex space, and that we can think of the *fundamental invariants* f_i as its coordinates (i.e. $\text{Spec}(\mathbb{C}[f_1, \cdots, f_n]) = W \setminus V$). Furthermore it gives an algebraic description on the abstract quotient map $\rho : V \to W \setminus V$:

$$\mathbb{C}^n \cong V \ni \mathbf{x} := (x_1, \cdots, x_n) \xrightarrow{\rho} \mathbf{f}(\mathbf{x}) := \big(f_1(\mathbf{x}), \cdots, f_n(\mathbf{x})\big) \in W \backslash V \cong \mathbb{C}^n \tag{2.1}$$

In this note, we will be interested in the subclass of complex reflection groups that are generated by $n = \dim(V)$-many quasi-reflections. Those are called *well-generated* and they possess good analogs of the Coxeter elements of real reflection groups. For us, and if $h := d_n$ and $\zeta = e^{2\pi i/h}$, a *Coxeter element* will be a Springer ζ-regular element. That is, an element $c \in W$ with a ζ-eigenvector v that is regular (i.e. lies in a free orbit of W).

2.1 Braid groups and discriminant hypersurfaces

Each complex reflection group W comes with an associated arrangement $\mathcal{A}_W := \bigcup_{H\in R} H$ of its reflection hyperplanes $H \subset V$. It is a theorem of Steinberg that the restriction of the quotient map $\rho : V \to W \setminus V$ on the complement $V^{\text{reg}} := V \setminus \mathcal{A}_W$ is a Galois covering. We define the *braid group* $B(W)$ to be the fundamental group of the base of this covering and use the following s.e.s. to produce a surjection $B(W) \overset{\pi}{\twoheadrightarrow} W$.

$$\begin{array}{ccccccc} 1 \hookrightarrow & \pi_1(V^{\text{reg}}) & \xrightarrow{\rho_*} & \pi_1(W\backslash V^{\text{reg}}) & \overset{\pi}{\twoheadrightarrow} & W \to 1 \\ & \shortparallel & & \shortparallel & & \\ & P(W) & & B(W) & & \end{array} \tag{2.2}$$

Given a choice of a basepoint $v \in V^{\text{reg}}$, a loop $b \in B(W)$ lifts to a path that connects v to $b_*(v)$ (we call this the *Galois action* of b). Then, we define $w := \pi(b)$ to be the *unique* element $w \in W$ such that $w \cdot v = b_*(v)$. We make now such a choice of a basepoint v and in what follows we consider the surjection π fixed.

It is well understood that much of the combinatorics of real reflection groups W is determined by the associated arrangement $\mathcal{A}_W$. As we transition to the complex case, $\mathcal{A}_W$ is replaced by its image $W \setminus \mathcal{A}_W$ in the quotient space. It is (yet another) consequence of the GIT of W that this image is an algebraic variety in $W \setminus V \cong \mathbb{C}^n$ which we call the *discriminant hypersurface* of W and denote by $\mathcal{H}$.

Furthermore, $\mathcal{H}$ is cut out by a single polynomial in the f_i's, which we also call the *discriminant* of W, and which we denote by $\Delta(W, \mathbf{f})$. When W is a well-generated group, we can always choose a system of fundamental invariants f_i, such that the discriminant takes the form

$$\Delta(W, \mathbf{f}) = f_n^n + \alpha_2 f_n^{n-2} + \cdots + \alpha_n, \tag{2.3}$$

where $\alpha_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$ are quasi-homogeneous polynomials of weighted degree hi.

3 Geometric factorizations of a Coxeter element

Recall that we have identified (see (2.1)) the orbit space $W \setminus V$ with an n-dimensional complex vector space whose coordinates are the fundamental invariants f_i. Let us now define the *base space* $Y := \operatorname{Spec} \mathbb{C}[f_1, \cdots, f_{n-1}]$, so that $W \setminus V \cong Y \times \mathbb{C}$ with coordinates written (y, x) or sometimes (y, f_n).

The special slice $L_0 := \mathbf{0} \times \mathbb{C}$, given by $f_1 = \cdots = f_{n-1} = 0$ and f_n arbitrary, is independent of the choice of f_i's (see [3, Remark 25]). Because of the monicity of the discriminant (2.3) in f_n, L_0 intersects the hypersurface $\mathcal{H}$ only at the origin $(\mathbf{0}, 0)$. In particular, the loop inside L_0 given by $f_n(t) = e^{2\pi i t}$, $t \in [0, 1]$ (and the rest of the f_i's equal to 0), is an element of the braid group $B(W) = \pi_1(W \setminus V - \mathcal{H})$. We denote it by δ (see Figure 1).

We can map δ to an element of W via the fixed surjection $\pi : B(W) \twoheadrightarrow W$ from (2.2). Because of our explicit knowledge of the quotient map ρ, it is easy to see that the Galois action of δ on the basepoint $v \in V^{\text{reg}}$ is multiplication by $e^{2\pi i/h}$:

Proposition 3.1 ([1, Lemma 6.13]). *The element $c := \pi(\delta)$ is a Coxeter element for W.*

As a loop around the unique point of contact $\{(\mathbf{0}, 0)\} = \mathcal{H} \bigcap L_0$ gives rise to a *geometrically constructed* Coxeter element, we might consider nearby "vertical" slices $L_y := y \times \mathbb{C} \subset W \setminus V \cong Y \times \mathbb{C}$. In that case, the intersection $L_y \bigcap \mathcal{H}$ consists of n points (counted with multiplicity), which are the solutions (see (2.3)) of the equation

$$\big(\Delta(W, \mathbf{f}); (y, t)\big) := t^n + \alpha_2(y)t^{n-2} + \cdots + \alpha_n(y) = 0. \tag{3.1}$$

Here $y \in Y \cong \mathbb{C}^{n-1} = \operatorname{Spec}(\mathbb{C}[f_1, \cdots, f_{n-1}])$ is fixed and t is the parameter, and we write (y, x_i) for the solutions.

Bessis describes a way of drawing loops around these points (y, x_i) and shows that, via the fixed surjection π, they map to factorizations of the Coxeter element. This process of assigning factorizations to points y of the base space will be called a *labeling map*. The construction depends nontrivially (and this is a subtlety that is important for us) on a choice of ordering of the complex numbers. In what follows, we will consider the following two:

Definition 3.2.

1. *Complex lexicographic order (clo)*: We order the points $x_i \in \mathbb{C}$ by increasing real part first and we cut ties by increasing imaginary part.

2. *Polar lexicographic order (plo)*: We order the points $x_i \in \mathbb{C}$ via their polar coordinates $(r, \theta) = (|x|, \arg(x))$, where $r \geq 0$ and $\theta \in [-3\pi/2, \pi/2]$. We order them by increasing length r first, and by **decreasing** argument θ to cut ties (where θ starts at the 12:00 position).

After we have decided on such an ordering, we proceed by picking a path θ in Y that connects $\mathbf{0}$ to y. We lift it to a path β_θ in $Y \times \mathbb{C}$, starting at $(\mathbf{0}, 1)$, which always stays "above" (i.e. has bigger imaginary part than) all the points in the intersections $L_{y'} \bigcap \mathcal{H}$ (see Figure 1). We call the endpoint of this path (y, x_∞) to indicate that it lies in the slice L_y and above all points (y, x_i).

From x_∞ we now construct paths β_i in L_y to the (ordered) points x_i such that they never cross each other (or themselves), and their order as they leave x_∞ is given by their indices (i.e. β_1 is the leftmost one).

Given this information, we can now easily construct elements $b_{(y,x_i)}$ of $B(W)$: First, we follow the path β_θ from the basepoint $(\mathbf{0}, 1)$ to (y, x_∞), then we go down β_i but before we reach its end, we trace a small counterclockwise circle around x_i, and finally we return by the same route (see Figure 1).

The product $\delta_y = b_{(y,x_1)} \cdots b_{(y,x_k)}$ (where k is the number of *geometrically distinct* points x_i) surrounds $\mathcal{H}$ and is, of course, homotopic to our loop δ. We have finally made it; from each slice L_y, we have constructed a geometric factorization of δ. The fixed surjection $\pi : B(W) \twoheadrightarrow W$ allows us to turn this into a factorization of the Coxeter element c:

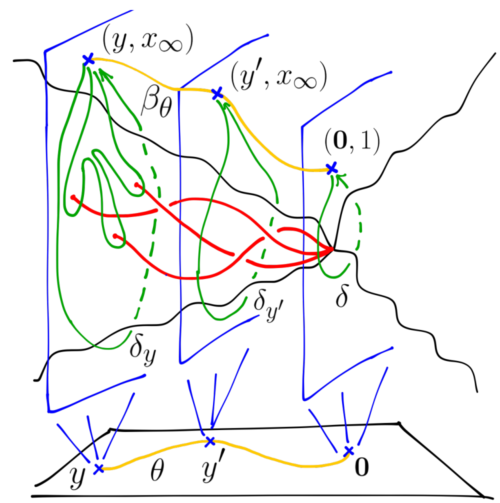


Figure 1: Factorizations via the slice L_y.

Definition 3.3. There is a collection of labeling maps which, to any point $y \in Y$ assign a factorization $c_1 \cdots c_k = c$, where the factors $c_i := \pi\big(b_{(y,x_i)}\big)$ are defined as above. Such labeling maps are differentiated by the choice of ordering of the complex numbers as discussed before Definition 3.2.

We denote the labeling maps corresponding to the orders (plo) and (clo) as clbl and rlbl respectively (referring to "cyclic" labels vs "reduced"[2] labels).

4 The Lyashko–Looijenga morphism (LL) and trivialization theorems

In the previous section we described a way to produce factorizations of the Coxeter element, by intersecting the discriminant hypersurface $\mathcal{H}$ with the slices L_y. The most striking fact of this theory though, is that this geometric construction is sufficient to produce *all* reduced block factorizations of the Coxeter element c. In fact, if we additionally keep track of the intersection $L_y \bigcap \mathcal{H}$, each such factorization is attained exactly once (Theorem 4.3 records this for reflection factorizations, which is sufficient for us).

The geometric object that keeps track of the point configurations $L_y \bigcap \mathcal{H}$ for the various $y \in Y$ is the Lyashko–Looijenga morphism. We use the notation E_n for the space of *centered configurations* of n points (that need not be distinct):

Definition 4.1. For an irreducible well-generated complex reflection group W, we define the *Lyashko–Looijenga* map by:

$$\begin{array}{ccc} Y & \xrightarrow{LL} & E_n \\ y = (f_1, \cdots, f_{n-1}) & \longrightarrow & \text{multiset of roots of } \big(\Delta(W, \mathbf{f}); (y,t)\big) = 0 \end{array}$$

[2] This term is not related to some length function; it is only here to allow the exposition be consistent with [1, Definition 7.14].

and denote it by LL. Notice that there is a simple description of LL as an algebraic morphism. Indeed the (multiset of) roots of a polynomial is completely determined by its coefficients, therefore we can express LL as the map:

$$\begin{array}{ccc} Y \cong \mathbb{C}^{n-1} & \xrightarrow{LL} & E_n \cong \mathbb{C}^{n-1} \\ y = (f_1, \cdots, f_{n-1}) & \longrightarrow & \big(\alpha_2(f_1, \cdots, f_{n-1}), \cdots, \alpha_n(f_1, \cdots, f_{n-1})\big) \end{array}$$

where the α_i's are as in (3.1).

The LL map is a proper, finite morphism and its ramification and monodromy are intimately related to the combinatorics of the noncrossing lattice (see [3, Chapter 7] and [1, Theorem 7.20]). However, in what follows we will mainly take advantage of just two statements about LL.

To set them up, we introduce the notation E_n^{reg} for the subset of E_n that contains those centered configurations that have n **distinct** points. Furthermore, we let Y^{reg} be the preimage $LL^{-1}(E_n^{\text{reg}})$ (that is, the set of those points $y \in Y$ for which the slice L_y intersects the discriminant $\mathcal{H}$ in n-many distinct points).

Proposition 4.2 ([1, Lemma 5.7])**.** *The restriction $LL : Y^{\text{reg}} \to E_n^{\text{reg}}$ is a topological covering map. In particular, any path in E_n^{reg} can be lifted to a path in Y^{reg}.*

Theorem 4.3 ([1, Theorem 7.5])**.** *The map $LL \times \text{rlbl} : Y^{\text{reg}} \to E_n^{\text{reg}} \times \text{Red}_W(c)$ is a bijection. The same is true for the map* clbl*.*

4.1 Interpretation of the Hurwitz action via the labeling maps

The driving (combinatorial) force behind Theorem 4.3 is the transitivity of the Hurwitz action on the set $\text{Red}_W(c)$. Its connection to such a geometric statement comes from the labeling map. Let us, however, first recall the definition:

Definition 4.4. For *any* group G, there is a natural action of the (usual) braid group B_n on the set of n-tuples of elements t_i of G. The generator s_i acts via:

$$s_i * (t_1, \cdots, t_i, t_{i+1}, \cdots, t_n) = (t_1, \cdots, t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, t_{i+2}, \cdots, t_n). \tag{4.1}$$

We call this the (left) *Hurwitz action* of B_n on G_n. It is clear that the Hurwitz action respects the product of the t_i's. Therefore, we may restrict it on tuples that encode fixed length factorizations of elements of G. In our context, we consider the Hurwitz action on the set of reduced reflection factorizations $\text{Red}_W(c)$.

The following Lemma 4.5, which describes how the label $\text{rlbl}(y)$ is affected as y varies in the base space Y, is the main technical support of this note. It is based on the observation, already by Fox and Neuwirth [4], that we may identify braids $g \in B_n$

with loops γ in E_n^{reg}. This of course demands that we first choose a basepoint $\mathbf{e} \in E_n^{\text{reg}}$ *and* order its elements ($\mathbf{e}$ is a point configuration), just so that we may even define the isomorphism.

On the other hand, these loops γ can be lifted to paths in Y^{reg} via the covering map $LL : Y^{\text{reg}} \to E_n^{\text{reg}}$ (see Proposition 4.2). If y is a lift of the basepoint $\mathbf{e}$, we write $\gamma \cdot y$ for the Galois action (that is $\gamma \cdot y$ is the endpoint when we lift the loop γ starting at y). Then we have:

Lemma 4.5 ([1, Corollary 6.20])**.** *The labeling maps are equivariant with respect to the Hurwitz action and the Galois action. That is,*

$$\text{rlbl}(\gamma \cdot y) = g * \text{rlbl}(y),$$

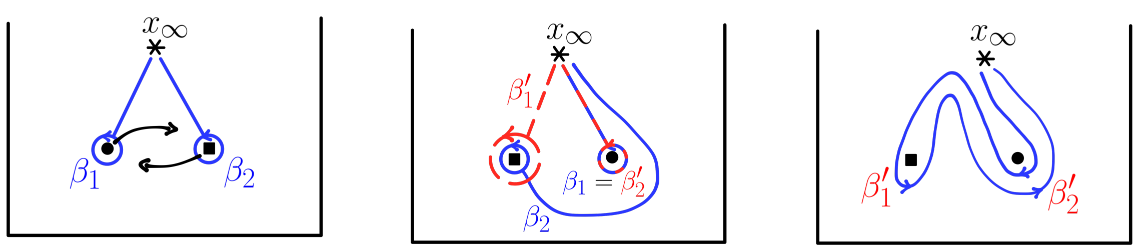
where γ and g are as above. The same is true for clbl.

Proof. Any loop $\gamma \in E_n^{\text{reg}}$ can be decomposed as a sequence of moves that transpose two neighboring points (we call these *Hurwitz moves*). In this case, the corresponding braid $g \in B_n$ is just one of the canonical generators s_i whose action is given by (4.1).

The following Figure 2 describes the effect of such a Hurwitz move on the labeling maps. The loops β_i in Figure 2a are the ones we used in Section 3 to define the labels. We write $rlbl(y) = (\beta_1, \beta_2)$ forgetting the surjection $\pi : B(W) \twoheadrightarrow W$.

The next two Figures 2b and 2c show the slice $L_{\gamma \cdot y}$ and on it are drawn two pairs of loops. The blue loops (β_1, β_2) are homotopic (notice that the homotopy must happen inside $B(W) = \pi_1(W \setminus V - \mathcal{H})$) to those in Figure 2a. The red ones (β_1', β_2'), on the other hand, are those assigned by the labeling map. As we can see, we have

$$\text{rlbl}(\gamma \cdot y) = (\beta_1', \beta_2') = (\beta_1\beta_2\beta_1^{-1}, \beta_1) = s_1 * (\beta_1, \beta_2) = s_1 * \text{rlbl}(y).$$



(a) As the two points in $LL(y)$ move around each other...

(b) ...the loop β_2 stretches to avoid β_1.

(c) Our previous loop β_2 is now homotopic to $\beta_2'^{-1}\beta_1'\beta_2'$.

Figure 2: The Hurwitz action.

□

Remark 4.6. This is indeed a different theorem for the two labeling maps. The reason is that the way we associate the braid g to the loop γ depends on a choice of ordering of the points in the configuration $\mathbf{e}$. This is different for the various labeling maps as we saw in Definition 3.2. This nuance will become clearer in Section 5, where in Propositions 5.2 and 5.3 we apply this Lemma to both maps clbl and rlbl.

5 Cyclic actions via the labeling maps

In this section we give an interpretation of (all three) cyclic actions Φ, Ψ, and $\mathfrak{Twist}$, from the introduction, as Hurwitz actions. They are in fact induced by particular loops in the configuration space E_n^{reg} via Lemma 4.5. We begin by introducing the latter ones:

Definition 5.1. We consider the following centered configurations of E_n^{reg}:

$$\mathbf{e}_{(n)} := \{e^{\epsilon+2\pi ik/n} \mid k = 1, \cdots n\} \quad \text{and} \quad \mathbf{e}_{(n-1,1)} := \{0\} \cup \{e^{\epsilon+2\pi ik/(n-1)} \mid k = 1, \cdots, n-1\},$$

for small ϵ, and the following loops (also in E_n^{reg}) based on them:

$$\gamma_{(n)}(t) := e^{2\pi it/n} \cdot \mathbf{e}_{(n)},\ t \in [0,1], \quad \text{and} \quad \gamma_{(n-1,1)}(t) := e^{2\pi it/(n-1)} \cdot \mathbf{e}_{(n-1,1)},\ t \in [0,1].$$

Notice that the configurations are cyclically symmetric and the loops γ correspond to the minimal rotations that respect them. We will further consider two *twisting* loops that rotate the two configuration by 180°:

$$\gamma^{\mathfrak{T}}_{(n)}(t) := e^{\pi it} \cdot \mathbf{e}_{(n)},\ t \in [0,1], \quad \text{and} \quad \gamma^{\mathfrak{T}}_{(n-1,1)}(t) := e^{\pi it} \cdot \mathbf{e}_{(n-1,1)},\ t \in [0,1],$$

defined when n is even or odd respectively.

Proposition 5.2. *The labeling map* clbl : $Y^{\text{reg}} \to \text{Red}_W(c)$ *is equivariant with respect to the Galois actions of* $\gamma_{(n)}$ *and* $\gamma_{(n-1,1)}$*, and the actions of* Φ_{cyc} *and* Ψ_{cyc}*, respectively. That is,*

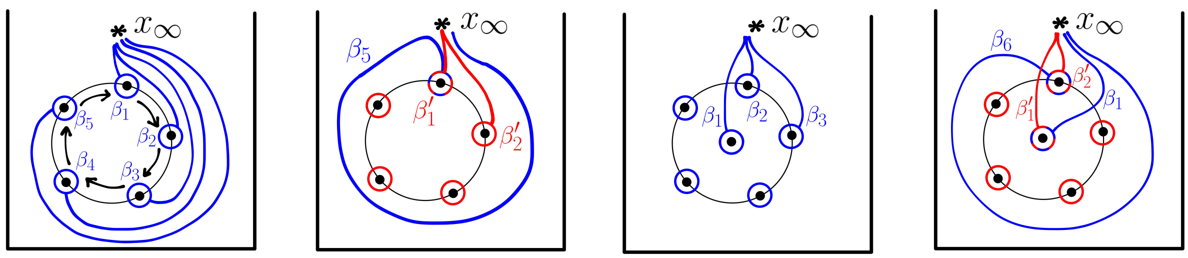
$$\Phi_{\text{cyc}} * \text{clbl}(y_1) = \text{clbl}(\gamma_{(n)} \cdot y_1) \quad \textit{and} \quad \Psi_{\text{cyc}} * \text{clbl}(y_2) = \text{clbl}(\gamma_{(n-1,1)} \cdot y_2),$$

for any $y_1 \in LL^{-1}(\mathbf{e}_{(n)})$ *and any* $y_2 \in LL^{-1}(\mathbf{e}_{(n-1,1)})$.

Proof. The proof is encoded in Figure 3. We write $\text{clbl}(y_1) = (\beta_1, \cdots, \beta_n)$ and similarly $\text{clbl}(\gamma_{(n)} \cdot y_1) = (\beta'_1, \cdots, \beta'_n)$ forgetting the fixed surjection $\pi : B(W) \twoheadrightarrow W$. As the Galois action of $\gamma_{(n)}$ rotates the configuration in Figure 3a, the blue loop β_i ends up encircling the $(i+1)^{\text{th}}$ point in Figure 3b. That is, it is homotopic to its label there ($\beta_i = \beta'_{i+1}$).

The situation is different only for the last loop β_n. As we can see in Figure 3b, after the rotation β_n is no longer a label. To express it in terms of the β'_i's, notice that it surrounds the configuration clockwise, then follows β'_1, and then it unwinds again. That is $\beta_n = c^{-1}\beta'_1 c$, which we reverse to obtain $\beta'_1 = c\beta_n c^{-1}$ and complete the proof for Φ_{cyc}.

The proof is similar for Ψ_{cyc}. As before, we have $\beta_i = \beta'_{i+1}$, $i = 2 \cdots n-1$. But now, the last loop β_n ends up encircling the *second* point (after the rotation). That is, $\beta'_2 = c\beta_n c^{-1}$. Moreover, the loop β_1 is not a label anymore as it is now on the right of β'_2. We compute the Hurwitz move $\beta'_1 = \beta'^{-1}_2 \beta_1 \beta'_2$, which after replacing β'_2 with $c\beta_n c^{-1}$ gives us precisely Ψ_{cyc}. □



(a) As the configuration rotates, the loop (b) ... β_5 can no longer be part of the label. (c) As **0** is labeled first (Definition 3.2) in (plo), (d) ... in this case, β_1 cannot be a label either.

Figure 3: A Hurwitz interpretation of Φ_{cyc} and Ψ_{cyc}.

As we discussed in Remark 4.6, the labeling maps rlbl and clbl are affected in a different way as we follow the paths γ in the configuration space E_n^{reg}.

Proposition 5.3. *The labeling map* rlbl : $Y^{\text{reg}} \to \text{Red}_W(c)$ *is equivariant with respect to the Galois actions of* $\gamma^{\mathfrak{T}}_{(n)}(t)$ *and* $\gamma^{\mathfrak{T}}_{(n-1,1)}(t)$*, and the action of* $\mathfrak{Twist}$*, when* n *is even or odd respectively. That is,*

$$\mathfrak{Twist} * \text{rlbl}(y_1) = \text{rlbl}(\gamma^{\mathfrak{T}}_{(n)}(t)\cdot y_1) \quad \textit{and} \quad \mathfrak{Twist} * \text{rlbl}(y_2) = \text{rlbl}(\gamma^{\mathfrak{T}}_{(n-1,1)}(t)\cdot y_2),$$

for n *even and odd respectively, and for any* $y_1 \in LL^{-1}(\mathbf{e}_{(n)})$ *and any* $y_2 \in LL^{-1}(\mathbf{e}_{(n-1,1)})$*.*

Proof. The same method applies here as in Proposition 5.2. The difference is that the labeling map rlbl orders the points differently, as we can see in Figure 4. We write $\text{rlbl}(y_1) = (t_1, \cdots, t_n)$ and $\text{rlbl}(\gamma^{\mathfrak{T}}_{(n)}\cdot y_1) = (t'_1, \cdots, t'_n)$. The Galois action is clockwise rotation by $180°$.

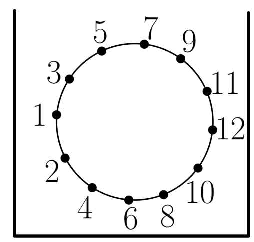


Figure 4: 12 angry points

According to Lemma 4.5 a label t_i is conjugated by a label t_j only when t_j crosses t_i from above and to the right. In particular, as the points move in the upper half circle, the labels are not affected. For instance, t_1 travels to the n^{th} point giving us $t'_n = t_1$.

If we follow t_2 however, using Figure 4 to enhance our imagination, we see that t_1 crosses it to the right; this gives us $t'_{n-1} = {}^{t_1}t_2$. We may proceed by induction: Pick an even number $2k$ (think of 8 in the figure) and start rotating it. It will first be conjugated by the odd numbers $7, 5, \cdots$, or really $2k-1, 2k-3, \cdots$ (in that order), and then by the labels of the even numbers too. Those however will have already reached their final version (since they are now on the upper half circle). That is, the total action on t_{2k} until it becomes t'_{n+1-2k} will be:

$$t'_{n+1-2k} = w t_{2k} w^{-1} \quad \text{for } w = {}^{(t_1\cdots t_{2k-3})}t_{2k-2}\cdots {}^{(t_1t_2t_3)}t_4 \cdot {}^{t_1}t_2 \cdot t_1 \cdot t_3 \cdots t_{2k-1},$$

and direct calculations shows that $w = t_1 t_2 \cdots t_{2k-1}$. The proof is similar for an odd number $2k+1$ and similar for the Galois action of $\gamma^{\mathfrak{T}}_{(n-1,1)}$. □

The following Lemma identifies the previous Galois actions on the symmetric configurations $\mathbf{e}$, essentially as scalar multiplication. Note that since the coordinates f_i of the space Y have weights d_i, the correct way to define a scalar action on it, called *weighted multiplication*, is via:

$$\lambda \in \mathbb{C}^*, y = (f_1, \cdots, f_{n-1}) \Rightarrow \lambda * y := (\lambda^{d_1} f_1, \cdots, \lambda^{d_{n-1}} f_{n-1}). \tag{5.1}$$

Lemma 5.4. *The Galois actions of $\gamma_{(n)}$ on $LL^{-1}(\mathbf{e}_{(n)})$, and of $\gamma_{(n-1,1)}$ on $LL^{-1}(\mathbf{e}_{(n-1,1)})$, are identical to* **weighted** *multiplication by $e^{-2\pi i/nh}$ and $e^{-2\pi i/(n-1)h}$ respectively. Similarly, both twisting loops $\gamma^{\mathfrak{T}}_{(n)}$ and $\gamma^{\mathfrak{T}}_{(n-1,1)}$ act on the corresponding fibers as* **weighted** *multiplication by $e^{-\pi i/h}$.*

Proof. The loops γ simply rotate the roots of the discriminant polynomial (3.1) by some angle θ. To do this is to scale its coefficients $\alpha_k(y)$ by $(e^{i\theta})^k$. Since the α_k are of weighted degree kh, we see that this translates to weighted multiplication on y by $e^{i\theta/h}$. □

Definition 5.5. We record here the cyclic subgroups of the weighted $\mathbb{C}^*$-action considered above. We will view the fibers $LL^{-1}(\mathbf{e})$ as modules over them. We define:

$$C_{nh} := \left\langle e^{\frac{-2\pi i}{nh}} \right\rangle \quad \text{and} \quad C_{(n-1)h} := \left\langle e^{\frac{-2\pi i}{(n-1)h}} \right\rangle \quad \text{and} \quad C_{2h} := \left\langle e^{\frac{-\pi i}{h}} \right\rangle.$$

Corollary 5.6. *The following module isomorphisms hold:*

$$\begin{aligned} &\mathrm{Red}_W(c) \cong_{C_{nh}} LL^{-1}(\mathbf{e}_{(n)}), && \mathrm{Red}_W(c) \cong_{C_{(n-1)h}} LL^{-1}(\mathbf{e}_{(n-1,1)}) \\ &\mathrm{Red}_W(c) \cong_{C_{2h}} LL^{-1}(\mathbf{e}_{(n)}),\ n \text{ even}, && \mathrm{Red}_W(c) \cong_{C_{2h}} LL^{-1}(\mathbf{e}_{(n-1,1)}),\ n \text{ odd}, \end{aligned}$$

where the cyclic actions on $\mathrm{Red}_W(c)$ are those of Φ_{cyc}, Ψ_{cyc}, and $\mathfrak{Twist}$ respectively, and the actions on the fibers $LL^{-1}(\mathbf{e})$ are via weighted multiplication and according to Definition 5.5.

Proof. This is immediate after Propositions 5.2 and 5.3, and Lemma 5.4. □

6 CSP's through finite quasi-homogeneous morphisms

Lemma 6.1. *Let $\boldsymbol{f} : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial morphism, finite and quasihomogeneous, which corresponds to the inclusion of algebras $\mathbb{C}[x_1, \cdots, x_n] \hookleftarrow \mathbb{C}[\theta_1, \cdots, \theta_n]$ over weighted parameters x_i and θ_i of degrees a_i and b_i respectively. Consider further a point $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_n)$ in $\mathbb{C}^n$, such that the fiber $\boldsymbol{f}^{-1}(\boldsymbol{\epsilon})$ is stable under* **weighted** *multiplication by $c := e^{-2\pi i/N}$ for some number N. Then, if $C_N := \langle c \rangle$, we have the isomorphism of C_N-modules:*

$$\boldsymbol{f}^{-1}(\boldsymbol{\epsilon}) \cong_{C_N} \mathbb{C}[x_1, \cdots, x_n] \Big/ (\theta_1, \cdots, \theta_n) =: \mathcal{R},$$

where the action on $\mathcal{R}$ is the induced action on functions. That is, $c * x_i = (e^{2\pi i/N})^{a_i} x_i$ (notice the change of sign). In particular, for $X := \boldsymbol{f}^{-1}(\boldsymbol{\epsilon})$ and $X(q) := \mathrm{Hilb}(\mathcal{R}, q) = \prod_{i=1}^n \frac{[b_i]_q}{[a_i]_q}$, the triple $(X, X(q), C_N)$ exhibits the CSP.

Sketch. Since $\mathbb{C}[\boldsymbol{x}]$ is Cohen-Macaulay and finite over $\mathbb{C}[\boldsymbol{\theta}]$, it is a free $\mathbb{C}[\boldsymbol{\theta}]$-module. This allows us to construct a C_N-stable filtration of the (otherwise ungraded) coordinate ring $K[\boldsymbol{f}^{-1}(\boldsymbol{\epsilon})]$ whose associated graded ring is isomorphic to $\mathcal{R}$ as a C_N-module. The final statement about CSP's is [5, Proposition 2.1: $(i) \iff (iii)$]. □

Proof of Theorem 1.2. Our main theorem is an immediate consequence of the previous lemma and Corollary 5.6. Indeed, the LL map is finite, quasi-homogeneous, and the Hilbert series of $LL^{-1}(\mathbf{0})$ is precisely [1, Theorem 5.3] the polynomial $\prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$. □

Remark 6.2. Lemma 6.1 is of a geometric nature and it takes advantage of the special role that is played by the origin $\mathbf{0}$ for quasi-homogeneous morphisms. Intuitively, it says that the scalar action that defines the weights of the morphism can be *transferred* to nearby fibers to the extent that their symmetry allows.

On the other hand, it gives a very pleasant elucidation of CSP's where the *same* polynomial encodes orbital information of different cyclic actions. In our case for instance, the different cyclic groups $C_{(n-1)h}$ and C_{nh} appear as we deform the origin $\mathbf{0} \in E_n$ to the configurations $\mathbf{e}_{(n-1,1)}$ and $\mathbf{e}_{(n)}$, which in turn have different cyclic symmetries.

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