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From generalized permutahedra to Grothendieck polynomials via flow polytopes (extended abstract)

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Abstract. We prove that for permutations $1\pi'$ where π' is dominant, the Grothendieck polynomial $\mathfrak{G}_{1\pi'}(\mathbf{x})$ has saturated Newton polytope and that the Newton polytope of each homogeneous component of $\mathfrak{G}_{1\pi'}(\mathbf{x})$ is a generalized permutahedron. We connect these Grothendieck polynomials to generalized permutahedra via a family of dissections of flow polytopes. We naturally label each simplex in a dissection by an integer sequence, called a left-degree sequence, and show that the sequences arising from simplices of a fixed dimension in our dissections of flow polytopes are exactly the integer points of generalized permutahedra. This connection of left-degree sequences and generalized permutahedra together with the connection of left-degree sequences and Grothendieck polynomials established in earlier work of Escobar and the first author reveals a beautiful relation between generalized permutahedra and Grothendieck polynomials.

Keywords: generalized permtuahedra, Grothendieck polynomials, flow polytopes, saturated Newton polytope

1 Introduction

This extended abstract is based on the paper [12] by the authors. We uncover novel relationships between flow polytopes, generalized permutahedra, and Grothendieck polynomials. The flow polytope $\mathscr{F}_G$ associated to a directed acyclic graph G is the set of all flows $f : E(G) \to \mathbb{R}_{\geq 0}$ of size one. Flow polytopes are fundamental objects in combinatorial optimization [15], and in the past decade they were also uncovered in representation theory [1, 10], the study of the space of diagonal harmonics [6, 11], and the study of Schubert and Grothendieck polynomials [2, 3]. In this abstract, we summarize the connection between flow polytopes and generalized permutahedra, and we explain how this connection can be used to prove that for certain permutations, the supports of Schubert

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polynomials as well as the homogeneous components of Grothendieck polynomials are integer points of generalized permutahedra.

A natural way to analyze a convex polytope is to dissect it into simplices. The relations of the subdivision algebra, developed in a series of papers [8, 9, 7], encode dissections of a family of flow (and root) polytopes (see Section 3 for details). The key to connecting flow polytopes and generalized permutahedra lies in the study of the dissections of flow polytopes obtained via the subdivision algebra:

(1) How are the dissections of a flow polytope obtained via the subdivision algebra related to each other?

In Theorem 3.2, we show that while the dissections themselves are different, their multiset of left-degree sequences (Definition 3.1) are the same. That the left-degree sequences do not depend on the particular dissection was previously proved in special cases by Escobar and the first author [2], and independently from the authors, Grinberg [5] recently showed it in slightly higher generality for arbitrary graphs in his study of the subdivision algebra.

Since the left-degree sequences are an invariant of the underlying flow polytope and do not depend on the choice of dissection, it is natural to ask:

(2) What is the significance of the left-degree sequences associated to a flow polytope $\mathcal{F}_G$?

The answer to this question is both inspiring and revealing. In Theorem 4.2, we prove that left-degree sequences of $\mathcal{F}_G$ with fixed sums are exactly lattice points of generalized permutahedra, which were introduced by Postnikov in his beautiful paper [14]. Moreover, we show that the left-degree polynomial $L_G(\boldsymbol{t})$ (Section 3) has saturated Newton polytope (Section 2.3).

In earlier work of Escobar and the first author [2], it was shown that some left-degree polynomials are Grothendieck polynomials. This brings us to:

(3) What does the answer to (2) imply about Schubert and Grothendieck polynomials?

In Theorem 4.3, we conclude that for all permutations $1\pi'$ where π' is dominant, the Grothendieck polynomial $\mathfrak{G}_{1\pi'}(\boldsymbol{x})$ is a weighted integer-point transform of its Newton polytope, with all weights nonzero. Moreover, the Newton polytopes of the homogeneous components of $\mathfrak{G}_{1\pi'}(\boldsymbol{x})$ are all generalized permutahedra. Theorem 4.3 implies in particular that the recent conjectures of Monical, Tokcan, and Yong [13, Conjectures 5.1 & 5.5] are true for permutations $1\pi'$, where π' is a dominant permutation.

The outline of this paper is as follows: Section 2 covers the necessary background on flow polytopes, Grothendieck polynomials, Newton polytopes, and generalized permutahedra; Section 3 covers the dissection procedure for flow polytopes and the resulting left-degree sequences; Section 4 describes the relation between left-degree sequences and Grothendieck polynomials and the consequences.

2 Preliminaries

In this section we summarize notation and give brief introductions to some of the topics relevant to the main results.

By a **graph**, we mean a loopless directed graph where multiple edges are allowed, as described below. Although we sometimes refer to edges by their endpoints, we keep in mind that $E(G)$ is a multiset. We also adopt the convention of viewing each element of a multiset as being distinct, so that we may speak of subsets. For any integers m and n, we will frequently use the notation $[m, n]$ to refer to the set $\{m, m+1, \ldots, n\}$ and $[n]$ to refer to the set $[1, n]$.

2.1 Flow Polytopes

Let G be a loopless graph on vertex set $[0, n]$ with edges directed from smaller to larger vertices. For each edge e, let $\text{in}(e)$ denote the smaller (initial) vertex of e and $\text{fin}(e)$ the larger (final) vertex of e. Imagine fluid moving along the edges of G. At vertex i let there be an external inflow of fluid a_i (outflow of $-a_i$ if $a_i < 0$), and call $\boldsymbol{a} = (a_0, \ldots, a_n)$ the **netflow vector**. Formally, a **flow** on G with netflow vector $\boldsymbol{a}$ is an assignment $f : E(G) \to \mathbb{R}_{\geq 0}$ of nonnegative values to each edge such that fluid is conserved at each vertex. That is, for each vertex i

$$\sum_{\text{in}(e)=i} f(e) - \sum_{\text{fin}(e)=i} f(e) = a_i.$$

The **flow polytope** $\mathcal{F}_G(\boldsymbol{a})$ is the collection of all flows on G with netflow vector $\boldsymbol{a}$. Alternatively, let M_G denote the incidence matrix of G, that is let the columns of M_G be the vectors $e_i - e_j$ for $(i, j) \in E(G)$, $i < j$, where e_i is the $(i+1)$-th standard basis vector in $\mathbb{R}^{n+1}$. Then,

$$\mathcal{F}_G(\boldsymbol{a}) = \{f \in \mathbb{R}^n_{\geq 0} : M_G f = \boldsymbol{a}\}.$$

From this perspective, note that the number of integer points in $\mathcal{F}_G(\boldsymbol{a})$ is exactly the number of ways to write $\boldsymbol{a}$ as a nonnegative integral combination of the vectors $e_i - e_j$ for edges (i, j) in G, $i < j$, that is the **Kostant partition function** $K_G(\boldsymbol{a})$ from representation theory. For brevity, we write $\mathcal{F}_G := \mathcal{F}_G(1, 0, \ldots, 0, -1)$, and we refer to $\mathcal{F}_G$ as the flow polytope of G.

2.2 Grothendieck Polynomials

Theorem 4.1 provides a beautiful relationship between certain Grothendieck polynomials and degree sequences of graphs. Grothendieck polynomials are an inhomogeneous analogue of Schubert polynomials that arise in the K-theory of the flag manifold. For w_0 the longest permutation in S_n, $\mathfrak{G}_{w_0}$ is defined to be

$$\mathfrak{G}_{w_0}(x_1, \ldots, x_{n-1}) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For any permutation $w \in S_n$, the **Grothendieck polynomial** $\mathfrak{G}_w$ is defined by

$$\mathfrak{G}_w(x_1, \ldots, x_{n-1}) = \partial_i (1 - x_{i+1}) \mathfrak{G}_{ws_i}$$

whenever $w(i) < w(i+1)$, where s_i is the ith adjacent transposition and ∂_i is the ith divided difference operator

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}}$$

The lowest homogeneous component of the Grothendieck polynomial is the **Schubert polynomial** $\mathcal{S}_w$.

2.3 Newton Polytopes and SNP

If f is a multivariable polynomial with variables indexed by some finite set I, the **support** of f is the set of integral points in $\mathbb{R}^I$ consisting of the exponent vectors of the monomials appearing in f with nonzero coefficient. The **Newton polytope** $\mathrm{Newton}(f) \subseteq \mathbb{R}^I$ is the convex hull of the support of f. Following the definition of [13], we say that a polynomial f has **saturated Newton polytope (SNP)** if every integral point in $\mathrm{Newton}(f)$ is a vector in the support of f. In other words, f has SNP if f is a positively weighted integer-point transform of its Newton polytope.

In their recent paper [13], Monical, Tokcan, and Yong introduced the idea of SNP and gave a survey of polynomials with SNP in algebraic combinatorics. They showed many examples of polynomials with SNP and conjectured numerous others. In Theorem 4.3 and Conjecture 4.4, we refine their conjecture, Conjecture 5.5 in [13], regarding the Grothendieck polynomials and prove our conjecture in a special case.

2.4 Generalized Permutahedra

Generalized permutahedra are a class of polytopes that tie together left-degree sequences and Grothendieck polynomials. It was conjectured in [13] that Grothendieck polynomials have SNP, and that the Newton polytope of the Schubert polynomial is a generalized permutahedron.

The standard permutahedron is the polytope in $\mathbb{R}^n$ whose vertices consist of all permutations of the entries of the vector $(1, 2, \ldots, n)$. A **generalized permutahedron** is a deformation of the standard permutahedron obtained by translating the vertices in such a way that all edge directions and orientations are preserved (edges are allowed to degenerate to points). Generalized permutahedra are parametrized by certain collections of real numbers $\{z_I\}$ indexed by nonempty subsets $I \subseteq [n]$, and have the presentation

$$P_n^z(\{z_I\}) = \left\{ \mathbf{t} \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \text{ and } \sum_{i=1}^{n} t_i = z_{[n]} \right\}.$$

Postnikov initiated the study of these fascinating polytopes in [14], and they have since been studied extensively.

3 Dissections of Flow Polytopes and Degree Sequences

For graphs with a special source and sink, there is a systematic way to dissect the flow polytope $\mathcal{F}_{\widetilde{G}}$, studied in [7]. Let G be a graph on $[0,n]$, and define $\widetilde{G}$ on $[0,n] \cup \{s,t\}$ with s being the smallest vertex and t the biggest vertex by setting $E(\widetilde{G}) = E(G) \cup \{(s,i),(i,t) : i \in [0,n]\}$. The systematic dissections can be expressed algebraically in the language of the subdivision algebra [8, 9] or combinatorially in terms of reduction trees [7, 12]. We use the language of reduction trees in this abstract.

Let G_0 be a graph on $[0,n]$ with edges (i,j) and (j,k) for some $i < j < k$. By a **reduction** on G, we mean the construction of three new graphs G_1, G_2 and G_3 on $[0,n]$ given by:

$$\begin{aligned} E(G_1) &= E(G)\backslash\{(j,k)\} \cup \{(i,k)\} \\ E(G_2) &= E(G)\backslash\{(i,j)\} \cup \{(i,k)\} \\ E(G_3) &= E(G)\backslash\{(i,j),\,(j,k)\} \cup \{(i,k)\} \end{aligned} \tag{3.1}$$

We say that the above reduction is at vertex j, on the edges (i,j) and (j,k). Up to integral equivalence, the flow polytopes $\mathcal{F}_{\widetilde{G}_1}$ and $\mathcal{F}_{\widetilde{G}_2}$ subdivide $\mathcal{F}_{\widetilde{G}_0}$ and intersect in $\mathcal{F}_{\widetilde{G}_3}$, which is a facet of both.

Iterating this subdivision process produces a dissection of $\mathcal{F}_{\widetilde{G}_0}$ into simplices. This process can be encoded using a reduction tree. A **reduction tree** of G is constructed as follows. Let the root node of the tree be labeled by G. If a node has any children, then it has three children obtained by performing a reduction on that node and labeling the children with the graphs defined in (3.1). Continue this process until the graphs labeling the leaves of the tree cannot be reduced. See Figure 1 for an example.

Given a reduction tree $\mathcal{R}(G)$ of G, the leaves L with the same number of edges as G label the full-dimensional simplices in the dissection, and the rest of the leaves label intersections of these simplices. Due to the choices inherent in building the reduction tree however, the actual leaves are dependent on the particular reduction tree constructed.

One way to get around this problem is the following: to each leaf L in $\mathcal{R}(G)$ associate a sequence $(a_1, a_2, \ldots, a_n)$ where a_i is the number of edges in L incoming to vertex i, called a **left-degree sequence**.

Definition 3.1. *Denote by* $\mathrm{LD}(G)$ *the multiset of left-degree sequences of leaves in a reduction tree of G.*

This multiset is surprisingly well-defined, as the following theorem states.

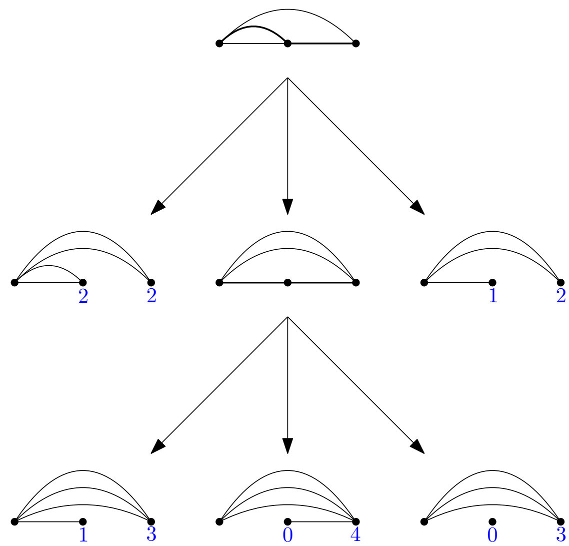

Figure 1: A reduction tree for a graph on three vertices. The edges involved in each reduction are shown in bold. The left-degree sequences of the leaves are shown in blue.

Theorem 3.2 ([12], Theorem A; [5], Theorem 1.7)**.** *For any graph G on $[0,n]$ the multiset of left-degree sequences* $\mathrm{LD}(G)$ *in any reduction tree of G is independent of the choice of reduction tree.*

This result was also found independently of the authors by Grinberg [5] in a more general context using the subdivision algebra. In [12, Theorem A] the authors of this abstract prove a more involved result than Theorem 3.2, which also includes a useful technical characterization of left-degree sequences.

4 Left-Degree Sequences and Grothendieck Polynomials

To relate $LD(G)$ (Definition 3.1) to polynomials, we encode the left-degree sequences of a graph in a polynomial L_G, called the **left-degree polynomial** and defined by

$$L_G(t_1,\ldots,t_n) = \sum_{\boldsymbol{a}\in \mathrm{LD}(G)} (-1)^{\mathrm{codim}(\boldsymbol{a})} t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$$

where $\mathrm{codim}(\boldsymbol{a}) = \#E(G) - a_1 - \cdots a_n$ is the codimension of the simplex indexed by $\boldsymbol{a}$ in any dissection of $\mathcal{F}_{\widetilde{G}}$.

The motivation for defining L_G this way is the following crucial result, which ties together Grothendieck polynomials and degree sequences. Recall that a permutation w is dominant if there do not exist integers $i<j<k$ with $w(i)<w(k)<w(j)$.

Theorem 4.1 ([2], Theorem 5.3). *Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ where π' is a dominant permutation of $\{2, 3, \dots n+1\}$. Then, there is a tree $T(\pi)$ and nonnegative integers $g_i = g_i(\pi)$ such that*

$$\widetilde{R}_{T(\pi)}(\boldsymbol{t}) = \left(\prod_{i=1}^{n} t_i^{g_i}\right) \mathfrak{G}_{\pi^{-1}}(t_1^{-1}, \dots, t_n^{-1}).$$

where $\widetilde{R}_G$ is the reduced right-degree polynomial of a graph G.

A few technical remarks are in order due to the different conventions of [2]. Right-degree sequences of a graph are defined exactly like left-degree sequences except that they count outgoing edges instead of incoming edges. The associated multiset $\mathrm{RD}(G)$ and polynomial R_G are defined analogously to their left-degree counterparts, and enjoy the same properties. In fact, up to a flipping and relabeling operation on graphs, right-degree and left-degree sequences are the same. The reduced right degree polynomial $\widetilde{R}_G$ is a particular relabeling of R_G, see [2] for details.

This result shows that the Newton polytopes of certain Grothendieck polynomials are isomorphic to the Newton polytopes of certain left-degree polynomials. Consequently, any results about the Newton polytopes of left-degree polynomials translate into results about the Newton polytopes of this family of Grothendieck polynomials.

In particular, if $L_G^k(\boldsymbol{t})$ denotes the degree $\#E(G) - k$ homogeneous component of $L_G(\boldsymbol{t})$, then we have the following result.

Theorem 4.2 ([12], Theorem B). *Each integer point point of* $\mathrm{Newton}(L_G(\boldsymbol{t}))$ *is a left-degree sequence, so L_G has SNP. Moreover, for each $k \geq 0$ there exist numbers $\{z_I^{(k)}\}$ such that* $\mathrm{Newton}(L_G^k(\boldsymbol{t}))$ *is the generalized permutahedron*

$$\mathrm{Newton}(L_G^k(\boldsymbol{t})) = P_n^z\{z_I^{(k)}\}_{I \subseteq [n]}.$$

Furthermore, each integer point of $P_n^z\{z_I^{(k)}\}$ is a left-degree sequence, so $\mathrm{Newton}(L_{G,F}(\boldsymbol{t}))$ *has SNP.*

Thus, applying Theorem 4.1 proves:

Theorem 4.3. *Let $\pi \in S_{n+1}$ be of the form $\pi = 1\pi'$ where π' is a dominant permutation of $\{2, 3, \dots n+1\}$. Then the Grothendieck polynomial $\mathfrak{G}_\pi$ has SNP and the Newton polytope of each homogeneous component of $\mathfrak{G}_\pi$ is a generalized permutahedron. In particular, the Schubert polynomial $\mathfrak{S}_\pi$ has SNP and* $\mathrm{Newton}(\mathfrak{S}_\pi)$ *is a generalized permutahedron.*

Theorem 4.3 implies that several recent conjectures of Monical, Tokcan, and Yong [13, Conjecture 5.1 & 5.5] are true for permutations of the form $1\pi'$, where π' is a dominant permutation. The following conjecture, discovered jointly with Alex Fink, is a strengthening of [13, Conjecture 5.5]. We have tested it for all $w \in S_n$, for $n \leq 8$. It has recently been proven in the case of Grassmannian permutations in [4] by Escobar and Yong.

Conjecture 4.4. *For each $w \in S_n$, the Grothendieck polynomial $\mathfrak{G}_w$ has SNP and the Newton polytope of each homogeneous component of $\mathfrak{G}_w$ is a generalized permutahedron.*

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