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Cluster fan of **z**-vectors and toric degenerations

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Abstract. For a cluster algebra, we introduce and study a notion of **z**-vectors. We prove that **z**-vectors determine cluster variables. We associate to a cluster seed a simplicial cone being span by **z**-vectors of the cluster variables of this seed. These cones form a fan, a polyhedral complex, such that the dual graph of this complex is the Fomin–Zelevinsky exchange graph of the cluster algebra.

Résumé. Étant donnée une algébre amassée, nous introduisons et étudions une notion de **z**-vecteurs. Nous montrons que les **z**-vecteurs déterminént les variables d'amas. Nous associons á une graine d'amas un cone simplicial qui est engendré par les **z**-vecteurs des variables d'amas. Ces cones forment un drapeau, un complexe polyédral, tel que le graphe dual de ce complexe est le graphe d'échange de Fomin–Zelevinsky de l'algébre amassée.

Keywords: Cluster algebras, Laurent polynomials, Polyhedral complexes

1 Introduction

For a cluster algebra $\mathcal{A}$, we introduce a notion of **z**-vectors. These vectors are associated to the cluster variables and defined by choosing an initial seed and an ample total order on cluster variables of the seed. Namely, for an initial seed which possesses an ample total order on the set of its variables, we define corresponding term order on Laurent polynomials in variables of this set. For each cluster variable, there is a Laurent polynomial expressing this cluster variable in the variables of the initial seed. Such a polynomial exists due to the Laurent phenomenon [4]. Then the **z**-vector of a cluster variable is the exponent of the leading monomial of the corresponding Laurent polynomial. For an initial seed which does not possess an ample order, we expand the exchange matrix by adding frozen variables in order to get a seed with an ample order. Such an ample extension exists. In such a case, we define **z**-vectors as projection along added frozen variables of **z**-vectors of the extended cluster algebra. A priori such defined **z**-vectors depend on extensions, a conjecture is that this is up a unimodular transformation.

Our main result, Theorem 3.4, says that, if the Positivity conjecture holds true for a cluster algebra $\mathcal{A}$ (that is the case for example, for geometric cluster algebras [7, 10]), then **z**-vectors uniquely determine the cluster variables and the cluster monomials of $\mathcal{A}$,

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and the collection of cones which are generated by $\mathbf{z}$-vectors of each seed of $\mathcal{A}$, form a fan, a polyhedral complex, which is dual to the Fomin–Zelevinsky exchange graph. Note that from this follows a generalization of the linear independence in [8].

For a cluster algebra of the coordinate ring of the basis affine space $\mathbb{C}[N_-\backslash GL_n]$, we show that the cluster seeds corresponding to lexicographically minimal and maximal reduced decompositions of the longest element w_0 of the Weyl group W are ample, and the corresponding fans, of cones are generated by $\mathbf{z}$-vectors, are unimodular equivalent to the fan defined in [9], in particular, they are unimodular equivalent to the Gelfand–Tsetlin cone.

2 Ample orders and quivers

A quiver $Q = (V, E)$ is a directed graph on the vertex set V and edges set $E \subset V \times V$ without loops and 2-cycles. As usual $(u, v) \in E$ means that $u = h(e)$ is the head of the edge e and $v = t(e)$ is the tail.

An ice quiver is a quiver Q such that some non-empty set of vertices $V_m \subseteq V$ is distinguished as a set of mutable vertices, and the complement $V_f := V \setminus V_m$ as frozen vertices.

We assign to each vertex $v \in V$, a formal variable x_v, such that $(x_v)_{v \in V}$ form a transcendental basis of the field $\mathbb{C}((x_v)_{v \in V})$.

A total order $>$ (a complete transitive antisymmetric binary relation) on the set of variables x_v, $v \in V$ union an extra element $\mathbf{0}$ (corresponding to constants), is *ample* if, for every $v \in V_m$, it holds

$$\prod_{u:(v,u)\in E} x_u \gg \prod_{w:(w,v)\in E} x_w, \tag{2.1}$$

where $\gg$ is the lexicographical extension $>$ to Laurent monomials. We denote also by $\gg$ the corresponding term order on $\mathbb{C}[(x_v^{\pm 1})_{v \in V}]$. We set the product over the empty set to be $\mathbf{0}$.

Example 2.1. *a) For the line quiver with even number of vertices, the ample order is unique, for example, for the quiver $u \longrightarrow v$, $u, v \in V_m$, the order $x_u < \mathbf{0} < x_v$ is ample.*

b) For the line quiver with odd number of variables an ample order does not exist. For example, for the quiver $u \longrightarrow v \longrightarrow w$, $u, v, w \in V_m$, we have $\mathbf{0} < v < \mathbf{0}$, that contradicts transitivity.

c) For any quiver, it suffices to add one more frozen vertex in order to get an ample order on the extended quiver.

3 Cluster algebras and z-vectors

Let us briefly recall the formalism of cluster algebras (for more details see [5]) needed here. For a positive integer r, an r-regular tree, denoted by $\mathbb{T}_r$, whose edges are labeled by $1, \dots, r$, so that the r edges emanating from each vertex receive different labels. We denote by t_0 the root of $\mathbb{T}_r$. Then an edge of $\mathbb{T}_r$ is denoted by $t \to|_k t'$, indicating that vertices $t, t' \in \mathbb{T}_r$ form an edge (t, t') of $\mathbb{T}_r$ and $k \in [r]$ is the color of this edge.

For a case of geometric cluster algebras, a cluster seed is a pair: an ice quiver, $Q = (V, E)$, and a tuple of variables $\mathbf{x} = (x_j,\ j \in V)$, such that the collection $\{x_j, j \in V\}$ generates a field $\mathbf{C}[x_k,\ k \in V_f](x_j,\ j \in V_m)$.

Let us assign a cluster seed to a root t_0 of the tree T_r with $r = |V_m|$, and denote by $(\mathbf{x}_{t_0}, Q_{t_0})$ this seed. A seed pattern is an assignment of a cluster seed $(\mathbf{x}_t = (x_{j;t})_{j \in V(Q_t)}, Q_t)$ to every vertex $t \in \mathbb{T}_m$, such that the seeds assigned to the endpoints of any edge $t \to_k t'$ are obtained from each other by the seed mutation μ_k, $k \in V_m$. The mutation μ_k transforms the quiver and variables. Namely the mutation sends Q_t into a new quiver $Q_{t'} = \mu_k(Q_t)$ via a sequence of three steps. Firstly, for each oriented two-arrow path $u \to k \to w$, $u, w \in V(Q_t)$, add a new arrow $u \to w$. Secondly, reverse the direction of all arrows incident to the vertex v. Finally, repeatedly remove oriented 2-cycles until unable to do so.

The mutation μ_k assigns the variables to the vertices of $\mu_k(Q_t)$ by the following mutation rule: $\mu_k(x_{j;t}) = x_{j;t}$ if $j \neq k$, and

$$\mu_k(x_{k;t}) = \frac{\prod_{(i,k) \in E(Q_t)} x_{i;t} + \prod_{(k,j) \in E(Q_t)} x_{j;t}}{x_{k;t}}. \tag{3.1}$$

A **d**-vector of a cluster variable ([5]) is the exponent of the denominators of the Laurent polynomial expressing this variables in cluster variables of the initial seed. For a finite cluster algebra, **d**-vectors uniquely determine cluster variables and cluster monomials, and, moreover, the collection of simplicial cones, such that a cone of the collection is generated by **d**-vectors of cluster variables of a seed, forms a fan, while seeds run over the set of cluster seeds. In this case, the union of the cones of this fan covers the whole space. For general cluster algebras **d**-vectors may not determine cluster variables.

Fomin and Zelevinsky ([5]) introduced a notion of **g**-vectors, and Derksen et al ([2]) proved that **g**-vectors uniquely determine cluster variables, and the cones, generated by **g**-vectors of cluster variables of seeds, form a fan. Note that, for a cluster algebra which is not finite, the union of the cones of the fan does not coincide with the linear span of **g**-vectors.

We introduce a notion of **z**-vectors, and show that for every cluster algebra for which the Positivity Conjecture holds true, **z**-vectors share the same properties as **g**-vectors have.

For a geometric cluster algebra, we say that an initial seed is *ample* if its quiver is *ample*. For a general case, we say that an initial seed with an exchange matrix B_0 of a cluster algebra $\mathcal{A}$ is *ample*, if there exists a total order on the set of the variables of the seed, such that (2.1) holds true for any mutable variable v, where the left hand side of (2.1) is replaced by the product $\prod_{u: b_{u,v} \geq 0} u$ and the right hand side is replaced by

$\prod_{w:b_{u,w}\leq 0} w$.

For a non ample quiver $Q = (V, E)$, we call a quiver $\hat{Q} = (V', E')$ such that $V \subset V'$, $V_m = V'_m$, $\hat{Q}|_V = Q$, and $\hat{Q}$ is ample, an *ample extension* of Q. Obviously, any quiver possesses an ample extension. Similarly is defined an ample extension of an exchange matrix.

In what follows we consider geometric cluster algebras. Let $(\mathbf{x}_0, Q_0)$ be an initial cluster seed, without loss of generality we assume that Q_0 is an ample quiver. Let $>$ be an ample order, and denote by $\gg$ the corresponding term order on $\mathcal{F} := \mathbf{C}[x_k,\, k \in V_f](x_j,\, j \in V_m)$ defined with respect to the lexicographical extension $>$ to Laurent monomials.

For a variable $x_{j;0}$, we set the jth basis vector $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^V$ (with 1 at the jth place) to be its $\mathbf{z}$-vector.

Suppose that, for a cluster seed $(\mathbf{x}_t, Q_t)$, $t \in \mathbb{T}_{|V_m|}$, the $\mathbf{z}$-vectors of its cluster variables are yet defined. Then, for a cluster $\mu_j(\mathbf{x}_t, Q_t)$, $j \in V_m(Q_t)$, the $\mathbf{z}$-vectors of its variables we define by the following rule: For $k \neq j$, $\mathbf{z}(\mu_j(x_{k;t})) = \mathbf{z}(x_{k;t})$; and

$$\mathbf{z}(\mu_j(x_{j;t})) = \begin{cases} -\mathbf{z}(x_{j;t}) + \sum_{(k,j)\in E(Q_t)} \mathbf{z}(x_{k;t}) & if\ \prod_{(k,j)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{k;t})} \gg \prod_{(j,l)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{l;t})} \\ -\mathbf{z}(x_{j;t}) + \sum_{(j,l)\in E(Q_t)} \mathbf{z}(x_{l;t}) & otherwise \end{cases} \tag{3.2}$$

Here are some useful properties of $\mathbf{z}$-vectors.

Proposition 3.1.
1. For each cluster seed $(\mathbf{x}_t, Q_t)$, the vectors $\mathbf{z}(x_{j;t})$, $j \in V(Q_t)$, form a basis of $\mathbb{Z}^V$.
2. For any cluster seed $(\mathbf{x}_t, Q_t)$, and any vertex $j \in V_m(Q_t)$, it holds

$$\frac{\prod_{(k,j)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{k;t})}}{\prod_{(j,l)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{l;t})}} \text{ is a Laurent monomial in } y_j,\ j \in V_m(Q_0), \text{ where } y_j = \frac{\prod_{(k,j)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{k;0})}}{\prod_{(j,l)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{l;0})}}. \tag{3.3}$$

3. For any cluster seed $(\mathbf{x}_t, Q_t)$, and any vertex $j \in V_m(Q_t)$, we have

$$x_{j;t} = \mathbf{x}_0^{\mathbf{z}(x_{j;t})}(1 + R_{j;t}(y_j,\, j \in V_m(Q_0))), \tag{3.4}$$

where $R_{j;t}(y_j,\, j \in V_m(Q_0))$ is a Laurent polynomial in variables $(y_j,\, j \in V_m(Q_0))$ which is dominated by $\mathbf{0}$ with respect to term order $\gg$.

Remark 3.2. *For general cluster algebras, for validity of analogs of the above proposition and all theorems below, we have to assume that Positivity Conjecture holds true for such cluster algebras. This conjecture says that the Laurent polynomials, expressing cluster variables of one cluster seed in variables of another seed, have positive coefficients. This conjecture was recently proven for geometric cluster algebras.*

Theorem 3.3 ([7, 10]). *For a geometric cluster algebra, Positivity conjecture holds true.*

Proof of Theorem 3.1. The item 1 follows from that the collection of vectors $\mathbf{z}(x_{j;0})$, $j \in V(Q_0)$, is a basis of $\mathbb{Z}^V$, and the mutation rule (3.2) preserves unimodularity.

For proving the item 2, we proceed by induction on distance between $t \in \mathbb{T}_m$ and the root. Namely, let us denote by $P_{u;t} := \frac{\prod_{(k,u)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{k;t})}}{\prod_{(u,l)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{l;t})}}$ the corresponding Laurent polynomial, $u \in V_m(Q_t)$. Then, for the mutation μ_u of $(\mathbf{x}_t, Q_t)$, $u \in V_m(Q_t)$, we have $\mu_u(P_{u;t}) = \frac{1}{P_{u;t}}$, and, for $v \neq u$, it might be

$$\text{either } \mu_u(P_{v;t}) = P_{v;t} \text{ or } \mu_u(P_{v;t}) = P_{v;t}P_{u;t}^{|b_{u,v}|} \text{ or } \mu_u(P_{v;t}) = P_{v;t}(P_{u;t})^{-|b_{u,v}|},$$

where $b_{u,v}$ denotes the multiplicity of edges joining u and v. From this follows the statement of the item 2.

The item 3: by induction on the distance between t and the root, we have

$$\mu_k(x_{k;t}) = \frac{\prod_{(i,k)\in E(Q_t)} x_{i;t} + \prod_{(k,j)\in E(Q_t)} x_{j;t}}{x_{k;t}} =$$

(this is due to the induction assumption)

$$\frac{\prod_{(i,k)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{i;t})}(1+R_{i;t}) + \prod_{(k,j)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{j;t})}(1+R_{j;t})}{\mathbf{x}_0^{\mathbf{z}(x_{k;t})}(1+R_{k;t})} =$$

(here we assume $\prod_{(i,k)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{i;t})} \gg \prod_{(k,j)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{j;t})}$)

$$\frac{\prod_{(i,k)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{i;t})}}{\mathbf{x}_0^{\mathbf{z}(x_{k;t})}} \left(\frac{\prod_{(i,k)\in E(Q_t)}(1+R_{i;t}) + \frac{\prod_{(k,j)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{j;t})}}{\prod_{(i,k)\in E(Q_t)} \mathbf{x}_0^{\mathbf{z}(x_{i;t})}} \prod_{(k,j)\in E(Q_t)}(1+R_{j;t})}{(1+R_{k;t})} \right) =$$

$$\mathbf{x}_0^{\mathbf{z}(\mu_k(x_{k;t}))} \frac{\prod_{(i,k)\in E(Q_t)}(1+R_{i;t}) + P_{k;t}^{-1} \prod_{(k,j)\in E(Q_t)}(1+R_{j;t})}{(1+R_{k;t})}$$

Since $\mathbf{0} \gg P_{k;t}^{-1}$, and $\mu_k(x_{k;t})$ is a Laurent polynomial in $x_{j;0}$, $j \in V(Q_0)$, we get that, due to Theorem 3.3,

$$\frac{\prod_{(i,k)\in E(Q_t)}(1+R_{i;t}) + P_{k;t}^{-1} \prod_{(k,j)\in E(Q_t)}(1+R_{j;t})}{(1+R_{k;t})}$$

is a Laurent polynomial of the required form. □

For a cluster seed $(\mathbf{x}_t, Q_t)$ with an ample initial seed, we denote by $\mathbf{Z}_t$ the cone generated by the vectors $\mathbf{z}(x_{j;t})$, $j \in V(Q_t)$.

Theorem 3.4.

1. *For geometric cluster algebra $\mathcal{A}$ with an ample initial seed $(\mathbf{x}_0, Q_0)$, the cones $\mathbf{Z}_t$, $t \in \mathbb{T}_{|V_m|}$, form a conical polyhedral complex, a fan $\mathcal{Z}(\mathcal{A})$, and two cones coincide, $\mathbf{Z}_t = \mathbf{Z}_{t'}$, if and only if the cluster seed $(\mathbf{x}_t, Q_t)$ is isomorphic to $(\mathbf{x}_{t'}, Q_{t'})$.*

2. *For a non ample cluster seed $(\mathbf{x}_0, Q_0)$ of $\mathcal{A}$, let $(\hat{\mathbf{x}}_0, \hat{Q}_0)$ be an ample extension, and $\hat{\mathcal{A}}$ be the corresponding extension of $\mathcal{A}$ obtained by adding coefficients (frozen variables). Then the cones $\hat{\mathbf{Z}}_t$, $t \in \mathbb{T}_{|V_m|}$, form a fan $\hat{\mathcal{Z}}(\mathcal{A})$, and $\hat{\mathbf{Z}}_t = \hat{\mathbf{Z}}_{t'}$ if and only if the cluster seed $(\mathbf{x}_t, Q_t)$ is isomorphic to $(\mathbf{x}_{t'}, Q_{t'})$.*

For a case of non-ample initial quiver, the projection of the fan $\mathcal{Z}(\hat{\mathcal{A}})$ along the added frozen coordinates turns out to be a fan, and the projections of any two different cones of $\mathcal{Z}(\hat{\mathcal{A}})$ are different cones.

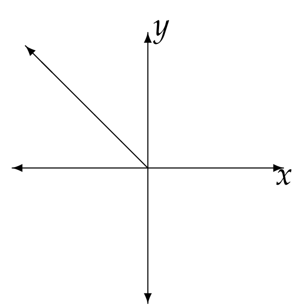
Denote by $\mathcal{Z}_{\hat{Q}_0 \to Q_0}(\mathcal{A})$ such a projection of the fan $\mathcal{Z}(\hat{\mathcal{A}})$ and call the generators of one-dimensional rays of the former fan **z**-*vectors of $\mathcal{A}$ conditionally on $\hat{Q}_0$*.

Our conjecture is that, for any two ample extensions of Q_0, $\hat{Q}_0$ and Q_0', the fans $\mathcal{Z}_{\hat{Q}_0 \to Q_0}(\mathcal{A})$ and $\mathcal{Z}_{Q_0' \to Q_0}(\mathcal{A})$ are unimodular equivalent.

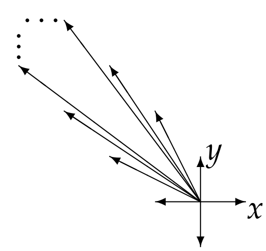
Before going to prove Theorem 3.4, we pause for examples.

Example 3.5. *a) Consider the cluster algebra with the initial seed $x \longrightarrow y$. This quiver is ample with the ample order $y > \mathbf{0} > x$ and the corresponding* **z**-*vectors are depicted below. Note that the cluster cone is the normal fan to the pentagon.*

b) Consider the cluster algebra with the initial seed $x \rightrightarrows y$. This quiver is also ample, $y > \mathbf{0} > x$ and corresponding **z**-*vectors are depicted below on Picture b).*



a



b

Theorem 3.6. *Let $\mathcal{A}$ be a geometric cluster algebra with an ample $(\mathbf{x}_0, Q_0)$. Then*

$$\mathbf{z}(x_{j;t}) = \mathbf{z}(x_{j';t'}) \text{ if and only if } x_{j;t} = x_{j';t'}.$$

Proof of Theorem 3.6. Suppose there are two seeds $\mathbf{S}_t = (\mathbf{x}_t, Q_t)$ and $\mathbf{S}_{t'} = (\mathbf{x}_{t'}, Q_{t'})$ and cluster variables $x_{j;t}$ and $x_{j';t'}$ in $\mathbf{S}_t$ and $\mathbf{S}_{t'}$, respectively, such that $\mathbf{z}(x_{j;t}) = \mathbf{z}(x_{j';t'})$. Then $x_{j;t}$ and $x_{j';t'}$ have the same leading monomials with respect to $\gg$, and from (3.4), we have $x_{j;t} = \mathbf{x}_0^{\mathbf{z}(x_{j;t})}(1 + R_{j;t}(\mathbf{y}))$, and $x_{j';t'} = \mathbf{x}_0^{\mathbf{z}(x_{j';t'})}(1 + R_{j';t'}(\mathbf{y}'_1))$.

For each j, $x_{j,;t} = P'_{j,t}(\mathbf{x}_{t'})$, where $P'_{j,t}$ is a Laurent polynomial of the cluster variables $\mathbf{x}_{t'}$. We claim that such a Laurent polynomial is of the form

$$x_{j;t} = P'_{j,t}(\mathbf{x}_{t'}) = x_{j';t'} + \overline{P}'(\mathbf{x}_{t'} \setminus x_{j';t'}), \tag{3.5}$$

where $\overline{P}'(\mathbf{x}_{t'} \setminus x_{j';t'})$ is a Laurent polynomial which does not depend on the variable $x_{j';t'}$.

In fact, the leading monomial of $x_{j;t}$ with respect to $\gg$ is equal to the leading monomial of $x_{j';t'}$. Hence, if the leading monomial of $P_{j,t}(\mathbf{x}_{t'})$ would be different of $x_{j';t'}$, we will get linear dependence among the vectors $\mathbf{z}(x_{k,t'})$, $k \in V(Q_{t'})$. This is not the case because of the item 1 of Proposition 3.3.

By analogy, we have

$$x_{j';t'} = x_{j;t} + \overline{P}(\mathbf{x}_t \setminus x_{j;t}), \tag{3.6}$$

where $\overline{P}(\mathbf{x}_t \setminus x_{j;t})$ is a Laurent polynomial which does not depend on the variable $x_{j;t}$. Substituting (3.5) in (3.6), we get

$$\overline{P}'(\mathbf{x}_{t'} \setminus x_{j';t'}) + \overline{P}(\mathbf{x}_t \setminus x_{j;t}) = 0. \tag{3.7}$$

Because of positivity the coefficients of Laurent polynomials in (3.7), we get $\overline{P'} = \overline{P} = 0$. □

Proof of Theorem 3.4. Suppose there are two seeds, $\mathbf{S}_t$ and $\mathbf{S}_{t'}$, such that the intersection of cones $C := \mathbf{Z}_t$ and $C' := \mathbf{Z}_{t'}$ is not a common face. Let us firstly consider a case $int(C) \cap int(C') \neq \emptyset$, where $int(C')$ denotes interior of a cone. Because the cones C and C' are unimodular simplicial cones (item 1 of Proposition 3.3), there is an integer vector $\mathbf{z} \in C \cap C'$. Hence there is a monomial X of the cluster variables $\mathbf{x}_t$ such that it has the form, due to (3.4), $X = \mathbf{x}_0^{\mathbf{z}}(1 + R(\mathbf{y}))$ with the leading monomial $\mathbf{x}_0^{\mathbf{z}}$.

Analogously, there is a monomial X' of the cluster variables $\mathbf{x}_{t'}$ such that $X' = \mathbf{x}_0^{\mathbf{z}}(1 + R'(\mathbf{y}))$ with the same leading monomial $\mathbf{x}_0^{\mathbf{z}}$.

Follow the same reasoning as in the proof of Theorem 3.6, we get $X = X'$.

Claim. Each cluster variable $x_{j;t}$ of the seed $\mathbf{S}_t$ is a Laurent monomial in the cluster variables $\mathbf{x}_{t'}$, and vice versa, each cluster variable $x_{j';t'}$ of the seed $\mathbf{S}_{t'}$ is a Laurent monomial in cluster variables $\mathbf{x}_t$.

In fact, $X = \prod_j x_{j;t}^{a_{j;t}}$ is a cluster monomial in variables of $\mathbf{S}_t$. Each $x_{j;t}$, $j \in V(Q_t)$ is a Laurent polynomial in the cluster variables $\mathbf{x}_{t'}$. Hence X is a monomial of the Laurent polynomials in variables of $\mathbf{x}_{t'}$. Since $X = X'$ and X' is a monomial in cluster variables of $\mathbf{x}_{t'}$, we conclude that each $x_{j;t}$ of the seed $\mathbf{S}_t$ is a Laurent monomial in the cluster variables $\mathbf{x}_{t'}$. That proves the claim.

Since the cones C and C' have a common interior point, there is a facet of one of these cones such that this facet has a common interior point with another cone. Without loss of generality, suppose that such a facet F is a facet of C. It is easy to see that the vertices of C corresponding to $\mathbf{z}$-vectors of the frozen variables belong F. Let a vertex $\{u;t\} \in V(Q_t)$ be such that $\mathbf{z}(x_{u;t})$ spans an extreme ray of C which does not belong to F. Then denote by $\mathbf{S}_{t''}$ the seed $\mu_{\{u;t\}}(\mathbf{S}_t)$. Then cones $C'' := \mathbf{Z}_{t''}$ and C' have a common interior point (for example, such a point exists in a small neighbor of F). Thus, we are in the situation of Claim, and, hence, the new cluster variable $\mu_u(x_{u;t})$ of the seed $\mathbf{S}_{t''}$ is a Laurent monomial in cluster variable of $\mathbf{S}_{t'}$. But this can not be the case. In fact, all cluster variables in $\mathbf{S}_t$ are Laurent monomials in cluster variables $\mathbf{S}_{t'}$, and hence, due to the mutation rule (3.1), $\mu_u(x_{u;t})$ has to be the sum of two Laurent monomials in cluster variables $\mathbf{S}_{t'}$. Thus, we get the statement under the assumption $int(C) \cap int(C') \neq \emptyset$.

Now, let F_1 and F_2 be minimal faces of C and C', respectively, such that $C \cap C' \subset F_i$, $i = 1, 2$, and $F_1 \neq F_2$.

Let the cluster variables $x_{f;t}$, $f \in L_1$, be such that the vectors $\mathbf{z}(x_{f;t})$, $f \in L_1$, span the cone F_1, and let the cluster variables $x_{f';t'}$, $f' \in L_2$, be such that the vectors $\mathbf{z}(x_{f';t'})$, $f' \in L_2$, span the cone F_2.

We claim that each cluster variable $x_{f;t}$, $f \in L_1$, is a Laurent monomial in the variables $x_{f';t'}$, $f' \in L_2$, and vice versa each cluster variable $x_{f';t'}$, $f' \in L_2$, is a Laurent monomial in the variables $x_{f;t}$, $f \in L_1$.

From the contrary, suppose there is a cluster variable $x_{f;t}$ such that the corresponding Laurent monomials in variables of the cluster $\mathbf{S}_{t'}$, contains a variable $x_{u;t'}$, which does not belong to the set $x_{f';t'}$, $f' \in L_2$. Then we consider a mutation $\mu_{\{u;t'\}}(\mathbf{S}_{t'})$. Denote by $\mathbf{S}_{t''} := \mu_{\{u;t'\}}(\mathbf{S}_{t'})$ the new seed.

Because $\mathbf{z}(x_{u;t'}) \notin F_2$, the cone $C'' = \mathbf{Z}_{t''}$ contains the intersection $F_1 \cap F_2$. Hence each of the cluster variables $x_{f;t}$, $f \in L_1$, is a Laurent monomial in the variables $x_{j;t''}$ of the cluster $\mathbf{S}_{t''}$.

If the expression of $x_{f;t}$ as a Laurent monomial in variables of $\mathbf{S}_{t''}$ contains $\mu_u(x_{u;t'})$, then $\mu_u(x_{u;t'})$ can be expressed as a Laurent monomial in variables of the cluster $\mathbf{S}_{t'}$, that is not the case.

Hence the expression of $x_{f;t}$ as a Laurent monomial in variables of $\mathbf{S}_{t''}$ does not contain $\mu_u(x_{u;t'})$, and since $x_{u;t'}$ is not a variable of $\mathbf{S}_{t''}$, this expression of $x_{f;t}$ does not contain $x_{u;t'}$. Thus we get the claim.

Because of this claim, F_1 belong to the linear span of F_2, and F_2 belong to the linear span of F_1. This implies that $F_1 \cap F_2$ is full-dimensional in each of the linear spans of F_1

and F_2.

Then by the same reasoning as for the full-dimensional intersection $C_1 \cap C_2$, we can find a cluster variable $x_{w;t}$ of $\mathbf{S}_t$, such that $\mathbf{z}(x_{w;t})$ is a vertex of F_1, such that, for the cone F_1' being span by $F_1 \setminus \mathbf{z}(x_{w;t})$ and $\mathbf{z}(\mu_w(x_{w;t}))$, we have the full-dimensional (of dimension F_2) intersection $F_1' \cap int(F_2)$ (here $int(F_2)$ denotes the relative interior of F_2). Due to minimality of F_1, F_1' belongs to the linear span of F_2, otherwise F_2 would belong to the linear span of $F_1 \setminus \mathbf{z}(x_{w;t})$. This implies that the variable $\mu_w(x_{w;t})$ can be expressed as a Laurent monomial in the variables $x_{f';t'}$, $f' \in L_2$, that is not the case due to (3.1). □

4 Base affine space $N_-\backslash GL_n$ and the simplicial fan in the cone of semistandard Young tableaux

For $GL_n(\mathbb{C})$ and its unipotent subgroup N_-, consider an action N_- on $GL_n(\mathbb{C})$ by multiplication on the left. The set of regular functions on $GL_n(\mathbb{C})$ which are invariant under such an action of N_-, $\mathbb{C}[GL_n]^{N_-} = \mathbb{C}[x_{ij}]^{N_-}$, is the coordinate ring of the *base affine space*, the GIT quotient $N_-\backslash\backslash GL_n$. The ring $\mathbb{C}[N_-\backslash GL_n]$ plays an important role in the representation theory of $GL_n(\mathbb{C})$: it naturally carries all its irreducible representations, each with multiplicity 1.

According to First Fundamental Theorem of invariant theory, $\mathbb{C}[N_-\backslash GL_n]$ is generated by the *flag minors*. (A flag minor Δ_I of a matrix $x = (x_{ij}) \in GL_n(\mathbb{C})$ is a minor occupying in the first $|I|$ rows and in columns of the set I.) Moreover, there hold (see, for example [6]): (i) The coordinate ring $\mathbb{C}[N_-\backslash GL_n]$ is a unique factorization domain. (ii) For any $I \subset [n]$, the flag minor Δ_I is an irreducible polynomial in $\mathbb{C}[x_{ij}]$ (and hence in $\mathbb{C}[N_-\backslash GL_n]$). (iii) Flag minors Δ_I are non-equivalent among themselves.

Let us recall a construction of a distinguished tableaux basis in $\mathbb{C}[N_-\backslash GL_n]$. For a semistandard Young tableau Y, an element Δ_Y of this basis is the product of the flag minors with column sets corresponding to the fillings of columns of Y. Because of the Désaménien–Kung–Rota algorithm ([3, 12]), the tableaux basis is an additive basis in $\mathbb{C}[N_-\backslash GL_n]$.

Recall (see, for example [12, Theorem 14.11]), that the flag minors form also a sagbi basis.[1] The existence of a finite sagbi basis is a special property for a subalgebra R. Namely, such a sagbi basis defines a flat degeneration of R the algebra of its leading term monomials. The latter is an affine toric variety, and hence a finite sagbi basis provides a flat family connecting $Spec(R)$ and the affine toric variety.

For a cluster algebra on $\mathbb{C}[N_-\backslash GL_n]$ there are seeds labeled by reduced decompositions of w_0. Recall a construction of a cluster quiver corresponding to a reduced decomposition of w_0 ([1]).

[1] The term *sagbi* is an acronym for *subalgebra analogue of Grobner bases for ideals*.

A reduced decomposition of $w \in W$ is a word $\mathbf{i} = (i_1 \cdots i_l)$ in the alphabet $[n-1] = \{1, \ldots, n-1\}$, such that $w = s_{i_1} \cdots s_{i_l}$ gives a factorization of smallest length. Let us denote by $|i_r|$ the letter of $[n-1]$ at the position i_r in $\mathbf{i}$. Denote by $\mathbf{i}_{\min}$ the lexicographical minimal reduced decomposition $1, (2,1), (3,2,1), \ldots, (n-1, n-2, \ldots, 2, 1)$ of w_0, and by $\mathbf{i}_{\max}$ the lexicographical maximal reduced decomposition $n-1, (n-2, n-1), (n-3, n-2, n-1), \ldots, (1, 2, \ldots, n-1)$.

Let us add n additional entries $i_{-n+1}, \ldots, i_{-1}$ at the beginning of a reduced decomposition $\mathbf{i}$ of w_0, by setting $i_{-j} = -j$ for $j \in [n]$. For $k \in -[n-1] \cup [l(w_0)]$, we denote by $k_+ = k_{\mathbf{i}}^+$ the smallest index r such that $k < r$ and $|i_r| = |i_k|$; if $|i_k| \neq |i_r|$ for $k < r$, then we set $k_+ = l(w_0) + 1$. An index $k \in [n] \cup [l(w_0)]$ is $\mathbf{i}$-exchangeable if both k and k_+ belong to $[l(w_0)]$. Denote by $A = (a_{ij})$ the Cartan matrix for SL_n.

Now, the vertices of the quiver $G(\mathbf{i})$ are labeled by elements of $-[n] \cup [l(w_0)]$; vertices i_k and i_r, with $k < r$, k, $r \in -[n] \cup [l(w_0)]$, are connected by an edge in the quiver if and only if either k or r (or both) are $\mathbf{i}$-exchangeable, and one of the following conditions is satisfied: (1) $r = k_+$; (2) $r < k_+ < r_+$, $a_{|i_k|,|i_r|} < 0$, $k, r \in [l(w_0)]$; (3) $r < r_+ < k_+$, $a_{|i_k|,|i_r|} < 0$, $k \in -[n-1]$. The edges of type (1) are called *horizontal*, and of types (2) and (3) are called *inclined*. Horizontal edges are directed from smaller to bigger and inclined are directed vice versa.

The set of frozen vertices is constituted from nodes labeled by i_{-j}, $j \in [n]$, and nodes labeled by i_k with $k_+ = l(w_0) + 1$, $k \in [n-1]$. So, there are $(n-1)(n-2)/2$ mutable nodes and $2n-1$ frozen ones

Denote by $\mathcal{P}(\mathbf{i})$ a cluster algebra on $\mathbb{C}[N_- \backslash GL_n]$ with the initial quiver $Q_0 := G(\mathbf{i})$, and the following set of initial cluster variables: to a vertex i_r, $r = 1, \ldots, l(w_0)$, is associate the minor $\Delta_{s_{i_1} \ldots s_{i_r}(\omega_{i_r})}$, and to the vertex i_{-j}, is associate the minors Δ_{ω_j}, $j \in [n-1]$, where ω_j denotes jth fundamental weight, $j \in [n-1]$, and we regard ω_j as a subset $[j] \subset [n]$. (For example, $\Delta_{s_1 s_2 \omega_2}$ is nothing but the flag minor $\Delta_{\{2,3\}}$.)

Denote by $\mathcal{P}^{w_0}(\mathbf{i})$ a cluster algebra on $\mathbb{C}[N_- \backslash GL_n]$ with the initial quiver $Q_0 := G(\mathbf{i})$, and the following set of initial cluster variables: to a vertex i_r, $r = 1, \ldots, l(w_0)$, is associate the minor $\Delta_{w_0 s_{i_1} \ldots s_{i_r}(\omega_{i_r})}$, and to the vertex i_{-j}, is associate the flag minors $\Delta_{w_0 \omega_j}$, $j \in [n-1]$.

The cluster algebras $\mathcal{P}(\mathbf{i})$ and $\mathcal{P}^{w_0}(\mathbf{i})$ are isomorphic. This isomorphism takes the form of reversing directions of edges in all quivers.

4.1 The lexicographical minimal and maximal reduced decompositions of w_0

Here we show that the quivers $G(\mathbf{i}_{\min})$ and $G(\mathbf{i}_{\max})$ are ample, and the corresponding fans of $\mathbf{z}$-vectors are unimodular equivalent to the fan constructed in [9].

Consider the reverse row lexicographical order on matrix entries $x_{nn} > x_{n-1\,n} > \ldots x_{1\,n} > \ldots x_{n\,1} > \ldots x_{11}$, and the corresponding term order $\succ^{rr}$ on $\mathbb{C}[x_{ij}]$.

Then we have the following

Lemma 4.1. *The order on vertices of the quiver* $G(\mathbf{i}_{\min})$*, defined by ordering the minors associated to the vertices as above with respect to* $\succ^{rr}$*, is ample, and hence* $G(\mathbf{i}_{\min})$ *is ample.*

Proof. The statement follows from that the vertices of $G(\mathbf{i}_{\min})$ are labeled by interval flag minors. In fact, since product of the elements of the main diagonal of a minor is the leading term with respect to $\succ^{rr}$, we get that nodes of $G(\mathbf{i}_{\min})$ are ordered in descending from left to right and from top to bottom. Because of such descending order, we get the statement. □

Then we have the following

Proposition 4.2. *For the initial ample seed* $G(\mathbf{i}_{\min})$*, the fan of* $\mathbf{z}$*-vectors of* $\mathbf{Z}(\mathcal{P}^{w_0}(\mathbf{i}_{\min}))$ *is unimodular equivalent to the fan constructed in [9].*

From this Proposition, we get that the cone generated by $\mathbf{z}$-vectors is unimodular equivalent to the Gelfand–Tsetlin cone. Moreover, because of this we get that the cluster algebra $\mathcal{P}^{w_0}(G(\mathbf{i}_{\min}))$ has a flat toric degeneration to the affine toric variety defined by the GT-cone. This situation corresponds to the case of sagbi basis for antidiagonal term order in [12, Theorem 14.11].

The sagbi basis for diagonal term order in the loc. cit. corresponds to the initial seed with $G(\mathbf{i}_{\max})$. Then we have the following analog to Lemma 4.1.

Lemma 4.3. *The order on vertices of the quiver* $G(\mathbf{i}_{\max})$ *corresponding to ordering the minors, associated to the vertices as above with respect to* $\succ^{rr}$*, is ample, and hence* $G(\mathbf{i}_{\max})$ *is ample.*

Proof. The statement follows from that the vertices of $\mathbf{G}(\mathbf{i}_{\max})$ are labeled by complements to interval flag minors. In fact, since product of the elements of the main diagonal of a minor is the leading term with respect to $\succ^{rr}$, the nodes of $G(\mathbf{i}_{\max})$ are ordered, follow $\succ^{rr}$, in descending from right to left and from top to bottom. From that we get the statement. □

Then we have the following

Proposition 4.4. *For the initial ample seed* $G(\mathbf{i}_{\max})$*, the fan of* $\mathbf{z}$*-vectors,* $\mathbf{Z}(\mathcal{P}(\mathbf{i}_{\max}))$*, is unimodular equivalent to the fan constructed in [9].*

From this Proposition, we get that the cone generated by $\mathbf{z}$-vectors is unimodular equivalent to the Gelfand–Tsetlin cone. Note that the cone of $\mathbf{g}$-vectors studied in [11] coincides with the cone of $\mathbf{z}$-vectors for the ample initials seed $G(\mathbf{i}_{\max})$.

In view of these results on sagbi bases, we can say that cluster algebras, possessing finitely generated cone of $\mathbf{z}$-vectors, have flat toric degenerations.

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