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K-theoretic polynomials

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Abstract. We contribute to modern symmetric function theory, introducing two new bases of the polynomial ring and studying their relations to known bases. The *quasiLascoux* basis is a K-theoretic deformation of the quasikey basis that is also a nonsymmetric lift of quasiGrothendieck polynomials. We give positive expansions of this new basis into the glide and Lascoux atom bases, as well as of the Lascoux basis into this new basis. These results include the first proof that quasiGrothendieck polynomials expand positively in the multifundamental quasisymmetric polynomials of Lam–Pylyavskyy. The *kaons* are K-theoretic deformations of fundamental particles. We give positive expansions of the glide and Lascoux atom bases into this new kaon basis. Throughout, we explore parallels between these K-analogues and their cohomological counterparts.

1 Introduction

Considerations in representation theory and algebraic geometry yield a number of interesting and important bases of $\mathrm{Poly}_n := \mathbb{Z}[x_1, \ldots, x_n]$. This extended abstract summarizes [18], contributing two new bases and studying their relations to known bases; we find that our new bases exhibit well-behaved structure and fill natural holes in the previously developed theory. This study is part of a general program to develop a combinatorial theory of Poly_n that mirrors the rich classical theory of symmetric functions.

Foremost among known bases of Poly_n are the celebrated *Schubert polynomials* $\{\mathfrak{S}_a\}$ of Lascoux–Schützenberger [13]. Let $X = \mathsf{Flags}_m(\mathbb{C})$ be the parameter space of complete flags of nested vector subspaces of $\mathbb{C}^m$. The *Schubert varieties* of X are naturally indexed by weak compositions $a = (a_1, \ldots, a_m)$ (i.e., sequences of nonnegative integers) and the corresponding *Schubert classes* $\{\sigma_a\}$ form a $\mathbb{Z}$-linear basis for the Chow ring $A^\star(X)$. The Schubert polynomials are polynomial representatives for Schubert classes in the sense that one has (up to truncation)

$$\mathfrak{S}_a \cdot \mathfrak{S}_b = \sum_c C^c_{a,b}\, \mathfrak{S}_c \qquad \text{if and only if} \qquad \sigma_a \cdot \sigma_b = \sum_c C^c_{a,b}\, \sigma_c.$$

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Despite explicit formulas for Schubert polynomials, it remains a major open problem to give a positive combinatorial formula for the *Schubert structure constants* $C_{a,b}^c \in \mathbb{Z}_{\geq 0}$.

The (type A) *Demazure characters* $\{\mathfrak{D}_a\}$ form another basis of Poly_n, important in representation theory [3]. Remarkably, Demazure characters *refine* Schubert polynomials:

$$\mathfrak{S}_a = \sum_b E_b^a \mathfrak{D}_b, \text{ for some } \textit{nonnegative} \text{ integers } E_b^a \in \mathbb{Z}_{\geq 0} \text{ [14, 20]}.$$

Letting the symmetric group S_n act on Poly_n by permuting variables, the S_n-invariants are the *symmetric polynomials* $\mathrm{Sym}_n \subset \mathrm{Poly}_n$. Remarkably, $\{\mathfrak{S}_a\}$ and $\{\mathfrak{D}_a\}$ each contains (as a subset) the *Schur basis* $\{s_\lambda\}$ of Sym_n; in fact, $\{\mathfrak{S}_a\} \cap \mathrm{Sym}_n = \{\mathfrak{D}_a\} \cap \mathrm{Sym}_n = \{s_\lambda\}$. Thus, both $\{\mathfrak{S}_a\}$ and $\{\mathfrak{D}_a\}$ are *lifts* of $\{s_\lambda\}$ to Poly_n. The Schur basis, moreover, has important refinements into the *quasiSchur polynomials* $\{S_\alpha\}$ of [6, 7] and further into the *fundamental quasisymmetric polynomials* $\{F_\alpha\}$ of [5], two bases of the quasisymmetric polynomials $\mathrm{QSym}_n \subset \mathrm{Poly}_n$. (A polynomial $f \in \mathrm{Poly}_n$ is *quasisymmetric* if the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ equals the coefficient of $x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$ for every sequence $j_1 < j_2 < \cdots < j_k$.)

While a rich combinatorial theory of Sym_n and QSym_n exists, the analogous theory for Poly_n is more sparse. For example, unlike for $\{\mathfrak{S}_a\}$, many positive combinatorial formulas are known for the structure constants of $\{s_\lambda\}$. A natural program, championed by Lascoux [12], is to develop analogous combinatorial theory for Poly_n by lifting known bases and relationships to Poly_n from the better-understood subrings Sym_n and QSym_n, and by developing uniform combinatorial models for these lifted bases and for the relations among them, extending models from Sym_n and QSym_n. This "modern symmetric function theory" aims to eventually bear dividends on major problems involving polynomials, such as the Schubert problem mentioned above.

Recent work in this area has provided lifts to Poly_n of the quasiSchur and fundamental bases of QSym_n: respectively, the *quasikey polynomials* $\{\mathfrak{Q}_a\}$ of [2] and the *(fundamental) slide polynomials* $\{\mathfrak{F}_a\}$ of [1]. (Note that quasikey polynomials are not quasisymmetric, despite the name.) These families provide further refinements of Schubert polynomials: Each Demazure character is a positive sum of quasikeys, each of which is a positive sum of slides [2]. The slide basis even has positive structure constants; in fact, one even has a Littlewood-Richardson-type rule for multiplying slide polynomials [1].

A more classical approach to Demazure characters is to study their refinement, not into slides, but into *Demazure atoms* $\{\mathfrak{A}_a\}$ [14, 15]. While Demazure atoms refine quasikeys [22], as do slides, Demazure atoms have no known direct relation to slides. The *(fundamental) particles* $\{\mathfrak{P}_a\}$ are a common refinement of $\{\mathfrak{A}_a\}$ and $\{\mathfrak{F}_a\}$ [22]. Figure 1 illustrates the relations among these nine families of 'cohomological' polynomials.

We study *K-theoretic analogues* of these bases. A theme of modern Schubert calculus is the study of Flags_n (and generalized flag varieties) through complex oriented cohomology theories. To avoid difficulty in defining appropriate Schubert classes, we restrict to (specializations of) *connective K-theory*. Complex oriented cohomology theories are

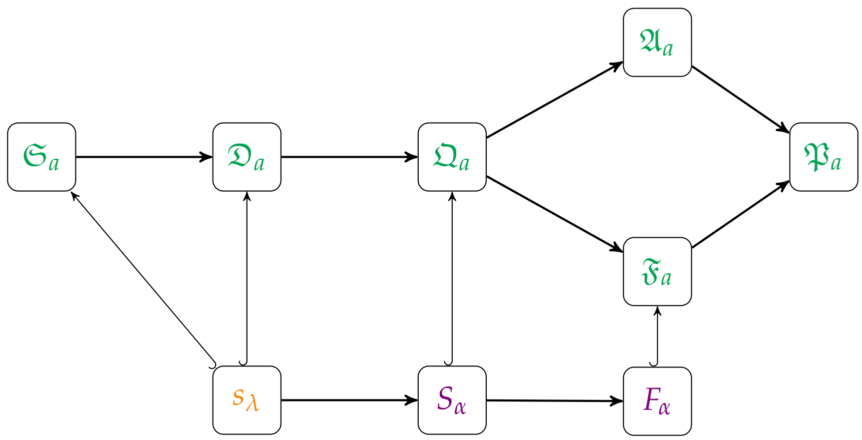

Figure 1: The nine cohomological families of polynomials considered here. Bases of Sym_n are shown in orange, bases of QSym_n in purple, and bases of Poly_n in green. The $\hookrightarrow$ arrows denote that the tail basis is a subset of the head basis. The thick $\rightarrow$ arrows denote that the head basis refines the tail basis.

determined by their formal group laws. For connective K-theory, this law is

$$c_1(L \otimes M) = c_1(L) + c_1(M) + \beta c_1(L)c_1(M), \tag{1.1}$$

where β is a formal parameter and L, M are complex line bundles. Cohomology is recovered at $\beta = 0$, and ordinary K-theory is recovered by fixing $\beta \in \mathbb{C}^\star$. In connective K-theory, the *β-Grothendieck polynomials* $\{\overline{\mathfrak{S}}_a\}$ [4] represent a Schubert basis [8]. These form an inhomogeneous basis of $\mathrm{Poly}_n[\beta]$, where β is the parameter from (1.1). At $\beta = 0$, one recovers the Schubert basis $\{\mathfrak{S}_a\}$. The usual Grothendieck polynomials [13] are realized at $\beta = -1$. (For clarity, we denote the connective K-analogue of each basis of Figure 1 by attaching an 'overbar' to its notation.)

Intersecting $\{\overline{\mathfrak{S}}_a\}$ with $\mathrm{Sym}_n[\beta]$ yields the basis $\{\overline{s}_\lambda\}$ of *symmetric Grothendieck polynomials*, representing connective K-theory Schubert classes on Grassmannians. Like Schur polynomials, these have quasisymmetric refinements; each $\overline{s}_\lambda$ expands positively in the *quasiGrothendieck* basis $\{\overline{S}_\alpha\}$ of $\mathrm{QSym}_n[\beta]$, the K-analogue of $\{S_\alpha\}$ [16].

Our first new result is that quasiGrothendiecks refine further into the multifundamental quasisymmetric polynomials $\{\overline{F}_\alpha\}$ of [10, 19], the K-analogues of $\{F_\alpha\}$.

Theorem 1.1. *Each quasiGrothendieck polynomial $\overline{S}_\alpha \in \mathrm{QSym}[\beta]$ is a positive sum of multifundamental quasisymmetric polynomials. That is,*

$$\overline{S}_\alpha = \sum_\gamma J^\alpha_\gamma \, \overline{F}_\gamma, \text{ where } J^\alpha_\gamma \in \mathbb{Z}_{\geq 0}[\beta].$$

The basis $\{\overline{F}_\alpha\}$ of $\mathrm{QSym}_n[\beta]$ lifts to the *glide basis* $\{\overline{\mathfrak{F}}_a\}$ of $\mathrm{Poly}_n[\beta]$ [19], a β-deformation of $\{\mathfrak{F}_a\}$. An analogous deformation of Demazure characters has been studied in [9, 11, 16, 17, 21]. We call these *Lascoux polynomials* $\{\overline{\mathfrak{D}}_a\}$. They can be approached via the *Lascoux atom* basis $\{\overline{\mathfrak{A}}_a\}$ of $\mathrm{Poly}_n[\beta]$ [16], deformations of Demazure atoms.

We introduce the *kaons* $\{\overline{\mathfrak{P}}_a\}$, as a β-deformation of particles, yielding a common refinement of $\{\overline{\mathfrak{F}}_a\}$ and $\{\overline{\mathfrak{A}}_a\}$, for which we give explicit positive formulas.

Theorem 1.2. *The kaons $\{\overline{\mathfrak{P}}_a\}$ are a basis of $\mathrm{Poly}_n[\beta]$. They deform the fundamental particles, in that $\overline{\mathfrak{P}}_a$ recovers $\mathfrak{P}_a$ at $\beta = 0$. Kaons refine both glides and Lascoux atoms; that is,*

$$\overline{\mathfrak{F}}_a = \sum_b P_b^a\, \overline{\mathfrak{P}}_b \quad \textit{and} \quad \overline{\mathfrak{A}}_a = \sum_b Q_b^a\, \overline{\mathfrak{P}}_b, \quad \textit{where } P_b^a, Q_b^a \in \mathbb{Z}_{\geq 0}[b].$$

Finally, we introduce the *quasiLascoux polynomials* $\{\overline{\mathfrak{Q}}_a\}$, both deforming quasikeys and lifting quasiGrothendiecks to $\mathrm{Poly}_n[\beta]$. QuasiLascoux polynomials are a coarsening of glides and of Lascoux atoms. We give explicit positive formulas for these refinements.

Theorem 1.3. *The quasiLascoux polynomials $\{\overline{\mathfrak{Q}}_a\}$ are a basis of $\mathrm{Poly}_n[\beta]$. They lift the quasi-Grothendieck basis in that $\{\overline{\mathfrak{Q}}_a\} \cap \mathrm{QSym}_n[\beta] = \{\overline{S}_\alpha\}$. Moreover, they deform the quasikeys, in that $\overline{\mathfrak{Q}}_a$ recovers $\mathfrak{Q}_a$ at $\beta = 0$. Finally, the quasiLascoux polynomials refine Lascoux polynomials and are refined by both glides and by Lascoux atoms; that is,*

$$\overline{\mathfrak{D}}_a = \sum_b L_b^a\, \overline{\mathfrak{Q}}_b, \quad \overline{\mathfrak{Q}}_a = \sum_b M_b^a\, \overline{\mathfrak{F}}_b, \quad \textit{and} \quad \overline{\mathfrak{Q}}_a = \sum_b N_b^a\, \overline{\mathfrak{A}}_b, \quad \textit{where } L_b^a, M_b^a, N_b^a \in \mathbb{Z}_{\geq 0}[\beta].$$

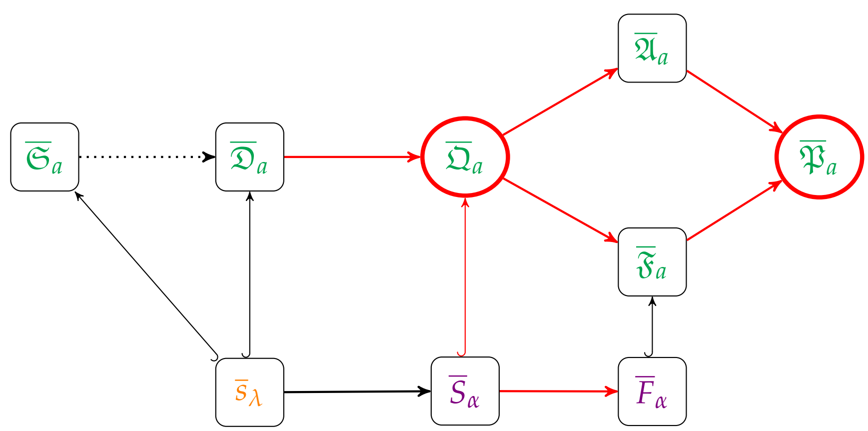

Figure 2: The K-theoretic analogues of the nine polynomials families of Figure 1. The arrows and colors of bases are as in Figure 1. Those polynomials and relations that are new are marked in red. The dotted arrow is conjectural; see [21].

The relations among these nine K-theoretic families are illustrated in Figure 2. Except for $\{\overline{\mathfrak{S}}_a\}$ and its subset $\{\overline{s}_\lambda\}$, the geometric significance of these K-analogues is currently obscure. While, for example, the glide basis has positive structure constants and appears useful in the study of β-Grothendieck polynomials (and thereby of the connective K-theory of Flags_n), it is unknown how to interpret individual glide polynomials geometrically. We conclude with a conjecture that appears to suggest geometric meaning. While it might be proved by combinatorics, the conjecture seems to have the flavor of Euler characteristic calculations. The conjecture is fundamentally K-theoretic, with no cohomological analogue. For weak compositions a and b, let $M_b^a(\beta)$ be the coefficient of $\overline{\mathfrak{F}}_b$ in the glide expansion of $\overline{\mathfrak{Q}}_a$ and let $Q_b^a(\beta)$ be the coefficient of $\overline{\mathfrak{P}}_b$ in the kaon expansion of $\overline{\mathfrak{A}}_a$.

Conjecture 1.4. *For a weak composition a, $\sum_b M_b^a(-1) \in \{0,1\}$ and $\sum_b Q_b^a(-1) \in \{0,1\}$, where both sums are over all weak compositions b.* □

For example, for $a = (0,6,6,2)$, we have $\sum_b M_b^a(\beta) = 16\beta^3 + 75\beta^2 + 94\beta + 36$ and $\sum_b Q_b^a(\beta) = 16\beta^3 + 66\beta^2 + 80\beta + 31$. In both cases, substituting $\beta = -1$ yields 1, as predicted. We have checked Conjecture 1.4 for all a with at most 3 zeros and $|a| \leq 7$.

Conjecture 1.5. *For a, b weak compositions, $\overline{\mathfrak{D}}_a^{(\beta)} \cdot \overline{\mathfrak{D}}_b^{(\beta)}$ is a positive sum of Lascoux atoms.*

We have checked Conjecture 1.5 for all a, b such that $|a|, |b| \leq 5$ and a and b have at most three 0s. Specializing at $\beta = 0$ recovers an old conjecture of Reiner–Shimozono.

2 Combinatorial models for K-theoretic bases

For a weak composition a, the **positive part** of a is the (strong) composition a^+ obtained by deleting all 0s from a. For a, b weak compositions, say b **dominates** a, denoted $b \geq a$, if $b_1 + \cdots + b_i \geq a_1 + \cdots + a_i$ for all i. As in [19], a **weak komposition** is a weak composition whose positive integers are colored arbitrarily black or red. The **excess** $\mathrm{ex}(b)$ of a weak komposition b is its number of red entries. A weak komposition b is a **glide** of a if b can be obtained from a by the following local moves on the colored word:

(m.1) $0p \Rightarrow p0$, (for $p \in \mathbb{Z}_{>0}$);

(m.2) $0p \Rightarrow qr$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p$);

(m.3) $0p \Rightarrow q\color{red}{r}$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p + 1$).

For a weak composition a of length n, the **glide (polynomial)** $\overline{\mathfrak{F}}_a^{(\beta)} = \overline{\mathfrak{F}}_a^{(\beta)}(x_1, \ldots, x_n)$ is

$$\overline{\mathfrak{F}}_a^{(\beta)} = \sum_b \beta^{\mathrm{ex}(b)} x_1^{b_1} \cdots x_n^{b_n},$$

where the sum is over all glides of a [19]. The **fundamental slide polynomials** $\mathfrak{F}_a$ of [1] are given by $\mathfrak{F}_a = \overline{\mathfrak{F}}_a^{(0)}$ [19]; we take this as definitional for fundamental slides.

The **skyline diagram** $D(a)$ of a weak composition a has a_i left-justified boxes in row i (for us, row 1 is the lowest). A **triple** of $D(a)$ is a set of 3 boxes with 2 adjacent in a row and either the third box above the right box and the lower row weakly longer, or the third box below the left box and the higher row strictly longer. Given a numerical filling of the skyline diagram, a triple is called a **coinversion triple** if $\alpha \leq \gamma \leq \beta$ (where γ is the label of the third box); otherwise, it is an **inversion triple**. Following [16], a **set-valued filling** of a skyline diagram is an assignment of a non-empty set of positive integers to each box. The greatest entry in each box is the **anchor**; other entries are **free**. A set-valued filling is **semistandard** if (S.1) entries do not repeat in a column, (S.2) rows are weakly decreasing (where sets $A \geq B$ if $\min A \geq \max B$), (S.3) every triple of anchors is an inversion triple, (S.4) each free entry appears with the least anchor in its column such that (S.2) is not violated, and (S.5) anchors in column 1 equal their row indices.

Given a set-valued filling F of shape a, the **weight** of F is the weak composition $\mathrm{wt}(F) = (c_1, \ldots, c_n)$ where c_i is the number of i's in F. The **excess** $\mathrm{ex}(F)$ of F is its number of free entries. For a weak composition a, let $\overline{\mathfrak{A}}\mathsf{SSF}(a)$ be the set of semistandard set-valued skyline diagrams of shape a.; see Figure 3. Then, the **Lascoux atom** $\overline{\mathfrak{A}}_a^{(\beta)}$ is

$$\overline{\mathfrak{A}}_a^{(\beta)} = \sum_{F \in \overline{\mathfrak{A}}\mathsf{SSF}(a)} \beta^{\mathrm{ex}(F)} \mathbf{x}^{\mathrm{wt}(F)},$$

while the **Demazure atom** $\mathfrak{A}_a$ is the $\beta = 0$ specialization: $\mathfrak{A}_a = \overline{\mathfrak{A}}_a^{(0)}$ [14, 15, 16]. Lascoux atoms form a basis of $\mathrm{Poly}[\beta]$ [16].

Figure 3: Two elements of $\overline{\mathfrak{A}}\mathsf{SSF}(1,0,3,2)$ with the anchors drawn in bold.

For a composition α, the **quasiGrothendieck polynomial** $\overline{S}_\alpha^{(\beta)}$ in n variables is

$$\overline{S}_\alpha^{(\beta)}(x_1, \ldots, x_n) = \sum_{a^+ = \alpha} \overline{\mathfrak{A}}_a,$$

where the sum is over weak compositions of length n [16]. The $\beta = 0$ specialization is the **quasiSchur polynomial** S_α [7]. QuasiGrothendieck polynomials are another basis of $\mathrm{QSym}[\beta]$ [16]. The Lascoux atoms refine the symmetric Grothendieck polynomials:

Theorem 2.1 ([16])**.**

$$\overline{s}_\lambda^{(\beta)}(x_1, \ldots x_n) = \sum_{\mathtt{sort}(a) = \lambda} \overline{\mathfrak{A}}_a^{(\beta)},$$

where the sum is over weak compositions of length n and $\mathtt{sort}(a)$ is the partition formed by sorting the parts of a in weakly decreasing order.

Corollary 2.2 ([16]).

$$\bar{s}_\lambda^{(\beta)}(x_1,\ldots,x_n) = \sum_{\mathtt{sort}(\alpha)=\lambda} \overline{\mathfrak{S}}_\alpha^{(\beta)}(x_1,\ldots,x_n).$$

3 New bases and their relations

We introduce a new "kaon" basis of polynomials, simultaneous refinements of both glides and Lascoux atoms. Let a be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. The weak komposition b is a **mesonic glide** of a if b can be obtained from a by a finite sequence of the local moves (m.1), (m.2), and (m.3) that never applies (m.1) at positions $n_j - 1$ and n_j for any j. Note that mesonic glides are special cases of glides.

Definition 3.1. Let a be a weak composition. The **kaon** $\overline{\mathfrak{P}}_a^{(\beta)}$ is the following generating function for mesonic glides:

$$\overline{\mathfrak{P}}_a^{(\beta)} := \sum_b \beta^{\mathrm{ex}(b)}\mathbf{x}^b, \text{ where the sum is over all mesonic glides of } a.$$

Proposition 3.2. *Every glide $\overline{\mathfrak{F}}_a^{(\beta)}$ is a positive sum of kaons. Indeed,*

$$\overline{\mathfrak{F}}_a^{(\beta)} = \sum_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{P}}_b^{(\beta)}.$$

Theorem 3.3. *The set*

$$\{\beta^k \overline{\mathfrak{P}}_a^{(\beta)} : k \in \mathbb{Z}_{\geq 0} \text{ and } a \text{ is a weak composition of length } n\}$$

is an additive basis of the free $\mathbb{Z}$-module $\mathrm{Poly}_n[\beta]$.

The *fundamental particle* basis $\{\mathfrak{P}_a\}$ of Poly_n is a common refinement of fundamental slides and Demazure atoms [22]. We show that kaons play the analogous role for glides and Lascoux atoms.

Proposition 3.4. *The fundamental particles $\mathfrak{P}_a$ are the kaons at $\beta = 0$: $\mathfrak{P}_a = \overline{\mathfrak{P}}_a^{(0)}$.*

Kaons do not have positive structure coefficients. Nonetheless, we conjecture:

Conjecture 3.5. *For any weak compositions a and b, the product $\overline{\mathfrak{P}}_a \cdot \overline{\mathfrak{F}}_b$ of a kaon and a glide polynomial expands positively in the kaon basis.*

We have verified Conjecture 3.5 for all weak compositions a, b with at most 3 zeros and $|a|, |b| \leq 5$. To our knowledge, Conjecture 3.5 is new even in the special $\beta = 0$ case of the fundamental particle expansion of the product of a fundamental particle by a fundamental slide polynomial.

Definition 3.6. Let a be a weak composition and $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$. We say T is **meson-highest** if, for every integer i appearing in T, either

- the leftmost i is in the leftmost column and is an anchor, or
- there is a $i^{\uparrow}$ in some column weakly to the right of the leftmost i and in a different box, where $i^{\uparrow}$ is the smallest label greater than i appearing in T.

We write $\overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)$ for the set of all meson-highest $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$.

Theorem 3.7. *For any weak composition a, we have*

$$\overline{\mathfrak{A}}_a^{(\beta)} = \sum_{T \in \overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)} \beta^{|T|-|a|} \overline{\mathfrak{P}}_{\mathrm{wt}(T)}^{(\beta)}. \tag{3.1}$$

In particular, every Lascoux atom $\overline{\mathfrak{A}}_a^{(\beta)}$ is a positive sum of kaons.

The quasikey basis of [2] is a coarsening of the fundamental slides and Demazure atoms, a refinement of the Demazure characters, and a lift of the quasiSchur basis from QSym_n to Poly_n. We introduce a K-analogue of the quasikey basis, a lift of the quasiGrothendiecks from $\mathrm{QSym}[\beta]$ to $\mathrm{Poly}[\beta]$.

Definition 3.8. For a weak composition a, the **quasiLascoux polynomial** $\overline{\mathfrak{Q}}_a$ is

$$\overline{\mathfrak{Q}}_a^{(\beta)} = \sum_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{A}}_b^{(\beta)}.$$

Proposition 3.9. *The set*

$$\{\beta^k \overline{\mathfrak{Q}}_a^{(\beta)} : k \in \mathbb{Z}_{\geq 0} \text{ and } a \text{ is a weak composition of length } n\}$$

is an additive basis of the free $\mathbb{Z}$-module $\mathrm{Poly}_n[\beta]$.

Proposition 3.10. *The quasikey polynomials are the $\beta = 0$ specialization of quasiLascoux polynomials: $\overline{\mathfrak{Q}}_a^{(0)} = \mathfrak{Q}_a$.*

Proposition 3.11. *Suppose that the positions of the nonzero entries in the weak composition a form an interval and that a_k is the last nonzero entry of a. Then,*

$$\overline{\mathfrak{Q}}_a^{(\beta)} = \overline{S}_{a^+}^{(\beta)}(x_1, \ldots, x_k).$$

In particular, every quasiGrothendieck polynomial is a quasiLascoux polynomial.

Moreover, we have

Proposition 3.12. *Let a be a weak composition. Then the stable limit $\lim_{m\to\infty} \overline{\mathfrak{Q}}^{(\beta)}_{0^m\times a}$ of the quasiLascoux polynomial $\overline{\mathfrak{Q}}^{(\beta)}_a$ is the quasiGrothendieck function $\overline{\mathsf{S}}^{(\beta)}_{a^+}(x_1, x_2, \ldots)$.*

To give the monomial expansion of a quasiLascoux polynomial directly, we define $\overline{\mathfrak{Q}}\mathsf{SSF}(a)$ (the **set-valued quasi-skyline fillings** of a) to be all set-valued skyline fillings of shape a satisfying (S.1)–(S.4), as well as

(S.5′) anchors in the first column are at most their row index and decrease from top to bottom.

Proposition 3.13. *Given a weak composition a, we have*

$$\overline{\mathfrak{Q}}^{(\beta)}_a = \sum_{S\in\overline{\mathfrak{Q}}\mathsf{SSF}(a)} \beta^{|S|-|a|}\mathbf{x}^{\mathrm{wt}(S)}.$$

Definition 3.14. Let a be a weak composition and let $S \in \overline{\mathfrak{Q}}\mathsf{SSF}(a)$ be a set-valued quasi-skyline filling. We say S is **quasiYamanouchi** if, for every integer i appearing in S, either

- the leftmost i is an anchor in row i of the leftmost column, or
- there is an $i+1$ in some column weakly right of the leftmost i and in a different box.

We write $\overline{\mathfrak{Q}}\mathfrak{Z}\overline{\mathfrak{F}}(a)$ for the set of all quasiYamanouchi $S \in \overline{\mathfrak{Q}}\mathsf{SSF}(a)$.

Theorem 3.15. *For any weak composition a, we have*

$$\overline{\mathfrak{Q}}^{(\beta)}_a = \sum_{S\in\overline{\mathfrak{Q}}\mathfrak{Z}\overline{\mathfrak{F}}(a)} \beta^{|S|-|a|}\overline{\mathfrak{F}}^{(\beta)}_{\mathrm{wt}(S)}.$$

In particular, every quasiLascoux polynomial $\overline{\mathfrak{Q}}^{(\beta)}_a$ is a positive sum of glide polynomials.

Remark 3.16. Setting $\beta = 0$ in the statement of Theorem 3.15 yields a positive combinatorial formula for the fundamental slide expansion of a quasikey polynomial in terms of quasiYamanouchi semi-skyline fillings. Such a formula was alluded to in [22], but not stated explicitly.

Corollary 3.17. *The quasiGrothendieck polynomials expand positively in the basis of multi-fundamental quasisymmetric polynomials.*

We now study the Lascoux polynomials of [16] and their relations to the other polynomials in this paper. Given a skyline diagram, we augment it on the left with an additional column 0, called the **basement**. We write b_i for the basement entry in row i. A set-valued skyline filling with basement is **semistandard** if it (including the basement) satisfies (S.1), (S.2), (S.3), and (S.4). Basement entries do not count towards the weight $\mathrm{wt}(F)$ of a filling F with basement. For a weak composition a, let $\overleftarrow{a}$ denote the weak composition formed by reversing the order of the parts of a. Let $\overline{\mathfrak{D}}\mathsf{SSF}(a)$ be the set of semistandard set-valued skyline fillings of shape $\overleftarrow{a}$ with basement $b_i = n - i + 1$.

Definition 3.18 ([7], [16])**.** Let a be a weak composition. The **Lascoux polynomial** $\overline{\mathfrak{D}}_a$ is

$$\overline{\mathfrak{D}}_a^{(\beta)} = \sum_{F \in \overline{\mathfrak{D}}\mathsf{SSF}(a)} \beta^{\mathrm{ex}(F)} \mathbf{x}^{\mathrm{wt}(F)}.$$

The **Demazure character** is the $\beta = 0$ specialization: $\mathfrak{D}_a = \overline{\mathfrak{D}}_a^{(0)}$.

Our last main result is that Lascoux polynomials are positive sums of quasiLascoux polynomials. We first give a positive formula for the Lascoux atom expansion of a Lascoux polynomial. Given a weak composition a, let $w(a)$ be the minimal (Coxeter) length permutation sending a to $\mathtt{sort}(a)$.

Theorem 3.19. *For any weak composition a, we have*

$$\overline{\mathfrak{D}}_a^{(\beta)} = \sum_{\substack{\mathtt{sort}(b) = \mathtt{sort}(a) \\ w(b) \leq w(a)}} \overline{\mathfrak{A}}_b^{(\beta)}, \text{ where } \leq \text{ denotes strong Bruhat order on permutations.}$$

In particular, every Lascoux polynomial $\overline{\mathfrak{D}}_a^{(\beta)}$ is a positive sum of Lascoux atoms.

Specializing Theorem 3.19 at $\beta = 0$ recovers a known formula for the Demazure atom expansion of a Demazure character (see, e.g., [15], [7]).

Remark 3.20. In [11], Lascoux introduced K-theoretic analogues of Demazure characters in terms of divided difference operators. Monical [16] conjectured that these are the Lascoux polynomials $\overline{\mathfrak{D}}_a^{(\beta)}$. Similarly, operators give polynomials that conjecturally equal the Lascoux atoms [16]. By [16], the basis change between these two operator-defined bases is also as in Theorem 3.19. Hence, Theorem 3.19 proves the equivalence of [16, Conjecture 5.2] and [16, Conjecture 5.3].

The Lascoux polynomials expand positively in the basis of quasiLascoux polynomials. Following [2], define $\mathtt{Qlswap}(a)$ to be all $b \in \mathtt{lswap}(a)$ such that if $c \in \mathtt{lswap}(a)$ and $b^+ = c^+$, then $c \geq b$.

Theorem 3.21. *For any weak composition a, we have*

$$\overline{\mathfrak{D}}_a^{(\beta)} = \sum_{b \in \mathtt{Qlswap}(a)} \overline{\mathfrak{Q}}_b^{(\beta)}.$$

In particular, every Lascoux polynomial $\overline{\mathfrak{D}}_a^{(\beta)}$ is a positive sum of quasiLascoux polynomials.

Remark 3.22. The $\beta = 0$ specialization of Theorem 3.21 recovers the expansion of Demazure characters in the quasikey basis [2].

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