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Queer supercrystals in SageMath

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Abstract. We describe the implementation of queer supercrystals. Our code is documented with test suites and has been integrated into SageMath. Through computer explorations using this implementation, we provide a counterexample to Assaf's and Oguz' conjecture that their local queer axioms uniquely characterize the queer supercrystal.

Keywords: Crystal graphs, queer Lie superalgebras, Stembridge axioms, SageMath

1 Introduction: SageMath and queer supercrystals

SageMath [7, 20] is a free open-source mathematical software licensed under GPL. It is based on Python, supports object-oriented programming, and has interfaces to many open source packages such as GAP, matplotlib, Numpy, R, and SciPy. Since its first release in 2005, SageMath has an active pool of developers worldwide and supports computations in diverse mathematical areas such as algebra, combinatorics, number theory, numerical analysis, graph theory, and statistics.

Lie superalgebras [16] arose in physics in theories that unify bosons and fermions. They are essential in modern string theories [8] and appear in other areas of mathematics, such as the projective representations of the symmetric group. The crystal basis theory has been developed for various quantum superalgebras [3, 10, 11, 12, 13, 17, 18]. In this paper, we are interested in the queer superalgebra $\mathfrak{q}(n)$ (see for example [5]). A theory of highest weight crystals for the queer superalgebra $\mathfrak{q}(n)$ was recently developed by Grantcharov et al. [10, 11, 12]. They provide an explicit combinatorial realization of the highest weight crystal bases in terms of semistandard decomposition tableaux and show how these crystals can be derived from a tensor product rule and the vector representation. They also use the tensor product rule to derive a Littlewood–Richardson rule. Choi and Kwon [6] provide a new characterization of Littlewood–Richardson–Stembridge tableaux for Schur P-functions by using the theory of $\mathfrak{q}(n)$-crystals. Independently, Hiroshima [15] and Assaf and Oguz [2, 1] defined a queer crystal structure on semistandard shifted tableaux, extending the type A crystal structure of [14] on these tableaux.

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The authors, in joint work with Gillespie and Hawkes [9], provided a characterization of the queer supercrystals in analogy to Stembridge's characterization of crystals associated to classical simply-laced root systems [21]. In the initial stages of this work, there was no implementation of queer supercrystals. As there is an extensive infrastructure to perform computations with other crystals within SageMath, it was a natural choice to use the existing infrastructure in SageMath to code the queer supercrystals. The implementation of queer supercrystals was achieved in the SageMath ticket [19].

Assaf and Oguz [2, 1] conjecture a local characterization of queer crystals in the spirit of Stembridge [21], which involves local relations between the odd crystal operator f_{-1} with the type A_{n-1} crystal operators f_i for $1 \leqslant i < n$. However, through computer explorations using the code in [19], we provide a counterexample to [1, Conjecture 4.16], which conjectures that these local axioms uniquely characterize the queer supercrystals.

The remainder of this paper is organized as follows. In Section 2, we give the definition of queer supercrystals. In Section 3, we describe the main features and methods that the user can access from our implementation of queer supercrystals. As an application, we also outline how a counterexample to Assaf's and Oguz's conjecture is discovered using this implementation in the same section. Finally, in Section 4, we elaborate how our implementation of queer supercrystals was integrated and tested by the SageMath community as well as provide references for documentation of our implementation.

2 Definition of queer supercrystals

An *(abstract) crystal* of type A_n is a nonempty set B together with the maps

$$e_i, f_i \colon B \to B \sqcup \{0\} \qquad \text{for } i \in I \text{ and} \qquad \mathrm{wt} \colon B \to \Lambda, \tag{2.1}$$

where $\Lambda = \mathbb{Z}_{\geqslant 0}^{n+1}$ is the weight lattice of the root of type A_n and $I = \{1, 2, \ldots, n\}$ is the index set, subject to several conditions. Denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I$ the simple roots of type A_n, where ϵ_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$. Then we require:

A1. For $b, b' \in B$, we have $f_i b = b'$ if and only if $b = e_i b'$. In this case $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.

For $b \in B$, we also define $\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid f_i^k(b) \neq 0\}$ and $\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geqslant 0} \mid e_i^k(b) \neq 0\}$. For further details, see for example [4, Definition 2.13].

There is an action of the symmetric group S_n on a type A_n crystal B given by the operators

$$s_i(b) = \begin{cases} f_i^k(b) & \text{if } k \geqslant 0, \\ e_i^{-k}(b) & \text{if } k < 0, \end{cases} \tag{2.2}$$

for $b \in B$, where $k = \varphi_i(b) - \varepsilon_i(b)$. An element $b \in B$ is called *highest weight* if $e_i(b) = 0$ for all $i \in I$. Similarly, b is called *lowest weight* if $f_i(b) = 0$ for all $i \in I$. For a subset

$$\begin{tikzcd} \boxed{1} \arrow[r, "1", blue] \arrow[r, dashed, swap, "-1", blue] & \boxed{2} \arrow[r, "2", red] & \boxed{3} \arrow[r, "3", green] & \cdots \arrow[r, "n"] & \boxed{n+1} \end{tikzcd}$$

Figure 1: $\mathfrak{q}(n+1)$-queer crystal of letters $\mathcal{B}$

$J \subseteq I$, we say that b is J-highest weight if $e_i(b) = 0$ for all $i \in J$ and similarly b is J-lowest weight if $f_i(b) = 0$ for all $i \in J$. We are now ready to define an abstract queer crystal.

Definition 2.1. [11, Definition 1.9] An *abstract $\mathfrak{q}(n+1)$-crystal* is a type A_n crystal B together with the maps $e_{-1}, f_{-1} \colon B \to B \sqcup \{0\}$ satisfying the following conditions:

Q1. $\mathrm{wt}(B) \subset \Lambda$;

Q2. $\mathrm{wt}(e_{-1}b) = \mathrm{wt}(b) + \alpha_1$ and $\mathrm{wt}(f_{-1}b) = \mathrm{wt}(b) - \alpha_1$;

Q3. for all $b, b' \in B$, $f_{-1}b = b'$ if and only if $b = e_{-1}b'$;

Q4. if $3 \leqslant i \leqslant n$, we have

(a) the crystal operators e_{-1} and f_{-1} commute with e_i and f_i;

(b) if $e_{-1}b \in B$, then $\varepsilon_i(e_{-1}b) = \varepsilon_i(b)$ and $\varphi_i(e_{-1}b) = \varphi_i(b)$.

Given two $\mathfrak{q}(n+1)$-crystals B_1 and B_2, Grantcharov et al. [11, Theorem 1.8] provide a crystal on the tensor product $B_1 \otimes B_2$, which we state here in reverse convention. It consists of the type A_n tensor product rule (see for example [4, Section 2.3]) and the *tensor product rule* for $b_1 \otimes b_2 \in B_1 \otimes B_2$

$$e_{-1}(b_1 \otimes b_2) = \begin{cases} b_1 \otimes e_{-1}b_2 & \text{if } \mathrm{wt}(b_1)_1 = \mathrm{wt}(b_1)_2 = 0, \\ e_{-1}b_1 \otimes b_2 & \text{otherwise,} \end{cases} \tag{2.3}$$

and similarly for f_{-1}. The crystals of interest are the *crystals of words* $\mathcal{B}^{\otimes \ell}$, where $\mathcal{B}$ is the $\mathfrak{q}(n+1)$-queer crystal of letters depicted in Figure 1.

In addition to the queer crystal operators $f_{-1}, f_1, \ldots, f_n$ and $e_{-1}, e_1, \ldots, e_n$, we define the crystal operators for $1 < i \leqslant n$

$$f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i} \qquad \text{and} \qquad e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i}, \tag{2.4}$$

where $s_{w_i} = s_2 \cdots s_i s_1 \cdots s_{i-1}$ and s_i is the reflection along the i-string in the crystal defined in (2.2).

By [11, Theorem 1.14], with all operators e_i, f_i for $i \in \{\pm 1, \pm 2, \ldots, \pm n\}$ each connected component of $\mathcal{B}^{\otimes \ell}$ has a unique highest weight vector.

The operators f_i for $i \in I_0 = \{1, 2, \ldots, n\}$ have an easy combinatorial description on $b \in \mathcal{B}^{\otimes \ell}$ given by the *signature rule*, which can be directly derived from the tensor product rule (see for example [4, Section 2.4]). One can consider b as a word in the alphabet $\{1, 2, \ldots, n+1\}$. Consider the subword of b consisting only of the letters i and $i+1$. Pair (or bracket) any consecutive letters $i+1, i$ in this order, remove this pair, and repeat. Then f_i changes the rightmost unpaired i to $i+1$; if there is no such letter $f_i(b) = 0$. Similarly, e_i changes the leftmost unpaired $i+1$ to i; if there is no such letter $e_i(b) = 0$.

From (2.3), one may also derive a simple combinatorial rule for f_{-1} and e_{-1}. Consider the subword v of $b \in \mathcal{B}^{\otimes \ell}$ consisting of the letters 1 and 2. The crystal operator f_{-1} on b is defined if the leftmost letter of v is a 1, in which case it turns it into a 2. Otherwise $f_{-1}(b) = 0$. Similarly, e_{-1} on b is defined if the leftmost letter of v is a 2, in which case it turns it into a 1. Otherwise $e_{-1}(b) = 0$.

3 Main Functionalities and Applications

3.1 Description of Features

The user can construct the crystal of words $\mathcal{B}^{\otimes \ell}$ via tensors of the standard queer supercrystal $\mathcal{B}$ of type $\mathfrak{q}(n+1)$ in SageMath [19]. As with the case of classical crystals, for every $b \in \mathcal{B}^{\otimes \ell}$ the user can query $\mathrm{wt}(b)$, $e_i(b)$, $f_i(b)$, $\varepsilon_i(b)$ and $\varphi_i(b)$ for each $i \in \{\pm 1, \pm 2, \ldots, \pm n\}$.

Additionally, the user may specify a subcrystal either by restricting to a certain subset of indices in the index set or by specifying a set of generators. This is useful when one wants to focus on say, J-highest weight elements, for $J \subset \{\pm 1, \pm 2, \ldots, \pm n\}$ for computer explorations. Moreover, if dot2tex is installed, the user may also view the crystal graph and import the LaTeX commands for rendering the crystal graph.

Example 3.1. One can retrieve a list of $\{1, 2, -1, -2\}$-highest weight elements in $\mathcal{B}^{\otimes 6}$, where $\mathcal{B}$ is a standard crystal of type $\mathfrak{q}(3)$, using the following commands:

```
sage: Q = crystals.Letters(['Q',3]); Q
The queer crystal of letters for q(3)
sage: T = tensor([Q]*6)
sage: L = [t for t in T if all(t.epsilon(i)==0 for i in Q.index_set())]
sage: L
[[1, 1, 1, 1, 1, 1],
 [1, 1, 1, 1, 2, 1],
 [1, 1, 1, 2, 1, 1],
 [1, 1, 2, 1, 1, 1],
```

```
[1, 1, 2, 1, 2, 1],
[1, 1, 2, 2, 1, 1],
[1, 2, 1, 1, 1, 1],
[1, 2, 1, 1, 2, 1],
[1, 2, 1, 2, 1, 1],
[1, 2, 1, 3, 2, 1],
[1, 2, 2, 1, 1, 1],
[1, 2, 3, 1, 2, 1]]
```

Example 3.2. To view a crystal or to obtain the latex code of the crystal graph, one may type:

```
sage: Q = crystals.Letters(['Q',3])
sage: T = tensor([Q]*2)
sage: view(T)
sage: latex(T)
```

The output is given in Figure 2. Note that the crystal graph contains the additional arrows $f_{-i'} := s_{w_0} e_{-(n+1-i)} s_{w_0}$ for $i \in I_0$, where w_0 is the long word in the symmetric group S_{n+1}. The crystal operators $f_{-i'}$ are labeled by $\overline{i+n}$, whereas f_{-i} are labeled by $\overline{i}$ in SAGEMATH.

3.2 Application: Discovery of a Counterexample

In [1, Definition 4.11], Assaf and Oguz give a definition of regular queer crystals. In essence, their axioms are rephrased in the following definition, where $\tilde{I} := I_0 \cup \{-1\}$.

Definition 3.3 (*Local queer axioms*)**.** Let $\mathcal{C}$ be a graph with labeled directed edges given by f_i for $i \in I_0$ and f_{-1}. If $b' = f_j b$ for $j \in \tilde{I}$, define e_j by $b = e_j b'$.

LQ1. The subgraph with all vertices but only edges labeled by $i \in I_0$ is a type A_n Stembridge crystal [21].

LQ2. $\varphi_{-1}(b), \varepsilon_{-1}(b) \in \{0, 1\}$ for all $b \in \mathcal{C}$.

LQ3. $\varphi_{-1}(b) + \varepsilon_{-1}(b) > 0$ if $\mathrm{wt}(b)_1 + \mathrm{wt}(b)_2 > 0$.

LQ4. Assume $\varphi_{-1}(b) = 1$ for $b \in \mathcal{C}$.

(a) If $\varphi_1(b) > 2$, we have $f_1 f_{-1}(b) = f_{-1} f_1(b)$, $\varphi_1(b) = \varphi_1(f_{-1}(b)) + 2$, and $\varepsilon_1(b) = \varepsilon_1(f_{-1}(b))$.

(b) If $\varphi_1(b) = 1$, we have $f_1(b) = f_{-1}(b)$.

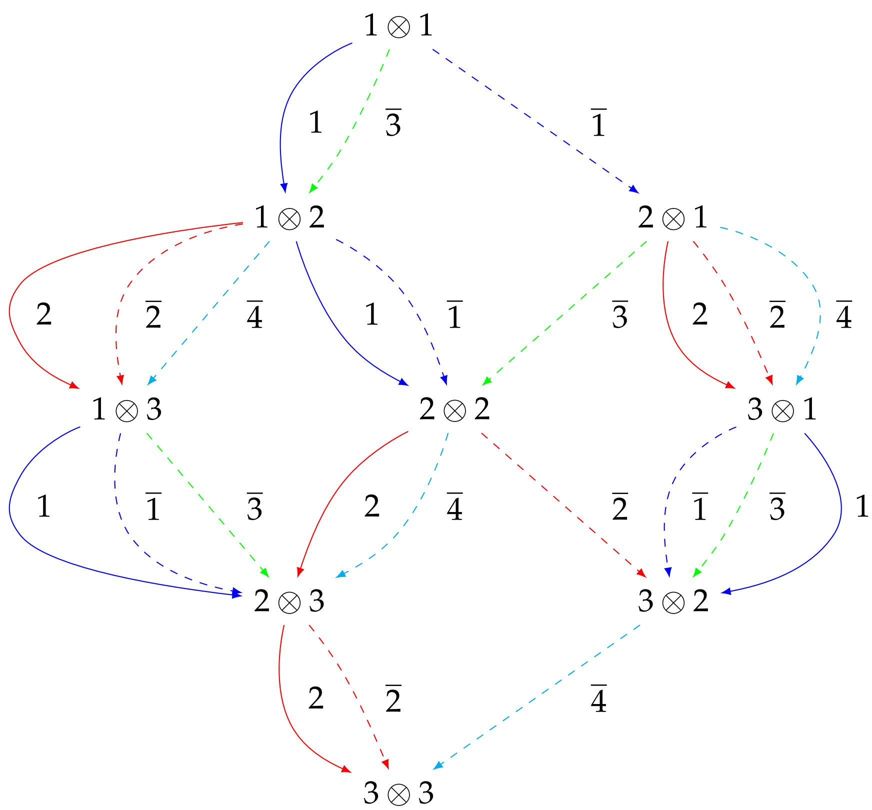

Figure 2: The crystal $\mathcal{B}^{\otimes 2}$ of type $\mathfrak{q}(3)$ of Example 3.2.

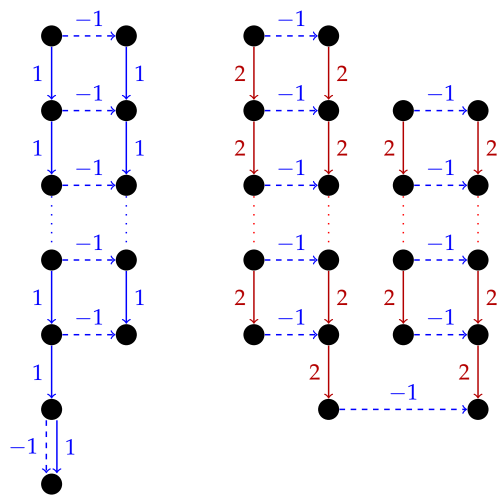

Figure 3: Illustration of axioms **LQ4** (left) and **LQ5** (right). The (-1)-arrow at the bottom of the right figure might or might not be there.

LQ5. Assume $\varphi_{-1}(b) = 1$ for $b \in \mathcal{C}$.

(a) If $\varphi_2(b) > 0$, we have $f_2 f_{-1}(b) = f_{-1} f_2(b)$, $\varphi_2(b) = \varphi_2(f_{-1}(b)) - 1$, and $\varepsilon_2(b) = \varepsilon_2(f_{-1}(b))$.

(b) If $\varphi_2(b) = 0$, we have

$$\begin{aligned} \varphi_2(b) &= \varphi_2(f_{-1}(b)) - 1 = 0, & \text{or} \quad \varphi_2(b) &= \varphi_2(f_{-1}(b)) = 0, \\ \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)), & \varepsilon_2(b) &= \varepsilon_2(f_{-1}(b)) + 1. \end{aligned}$$

LQ6. Assume that $\varphi_{-1}(b) = 1$ and $\varphi_i(b) > 0$ with $i \geqslant 3$ for $b \in \mathcal{C}$. Then $f_i f_{-1}(b) = f_{-1} f_i(b)$, $\varphi_i(b) = \varphi_i(f_{-1}(b))$, and $\varepsilon_i(b) = \varepsilon_i(f_{-1}(b))$.

Axioms **LQ4** and **LQ5** are illustrated in Figure 3.

Proposition 3.4 ([1])**.** *The queer crystal of words $\mathcal{B}^{\otimes \ell}$ satisfies the axioms in Definition 3.3.*

In [1, Conjecture 4.16], Assaf and Oguz conjecture that every regular queer crystal is a normal queer crystal. In other words, every connected graph satisfying the local queer axioms of Definition 3.3 is isomorphic to a connected component in some $\mathcal{B}^{\otimes \ell}$. Using the implementation of crystals of words in SAGEMATH [19], we provide a counterexample to this claim [9]. The $\{1, 2, -1, -2\}$-highest weight elements within $\mathcal{B}^{\otimes 6}$ are listed in Example 3.1. Now one may ask whether there is any choice in the -1 arrows after retaining all edges labeled -1 that are asserted by **LQ2**, but cannot be deduced from the top by axioms **LQ4** to **LQ6**. The answer is that there is choice, which led to the counterexample depicted in Figure 4.

In Figure 4, the I_0-components of the $\mathfrak{q}(3)$-crystal of highest weight $(4,2,0)$ are shown. Some of the f_{-1}-arrows are drawn in green. The remaining arrows can be filled in using the axioms of Figure 3 in a consistent manner. If the dashed green arrow from 331131 to 332131 and the dashed green arrow from 331132 to 332132 are replaced by the dashed purple arrow from 331131 to 331231 and the dashed purple arrow from 331132 to 332231, respectively, all axioms of Definition 3.3 are still satisfied with the remaining f_{-1}-arrows filled in. However, the I_0-component with highest weight element 132121 has become disconnected and hence the two crystals are not isomorphic.

4 Ease of use and sustainability

The standard queer supercrystals and their tensor product rule have been implemented and integrated into SageMath within the classes Crystal of Letters and Tensor Product of Crystal Elements, respectively. SageMath is a free open-source mathematical software licensed under GPL and is available for download at `http://www.sagemath.org/` or within the cloud service CoCalc `https://cocalc.com/`.

The trac ticket for this implementation was created during SageDays@ICERM in July 2018 (see [19]), received a positive review by a reviewer and was integrated into SageMath . This implementation is available in the current stable release of SageMath (v8.6) [20].

Documentation of this implementation is available in the SageMath online documentation for combinatorics as well as locally by appending a ? to a particular method or attribute as in the example shown below:

```
sage: Q = crystals.Letters(['Q',3])
sage: t = Q(1)
sage: t.e?
```

Furthermore, within the docstring of the implemented methods, the user can access examples of usage within SageMath . These examples also serve as a test suite and are run on a regular basis by SageMath to ensure that the code continues to work in future version of SageMath .

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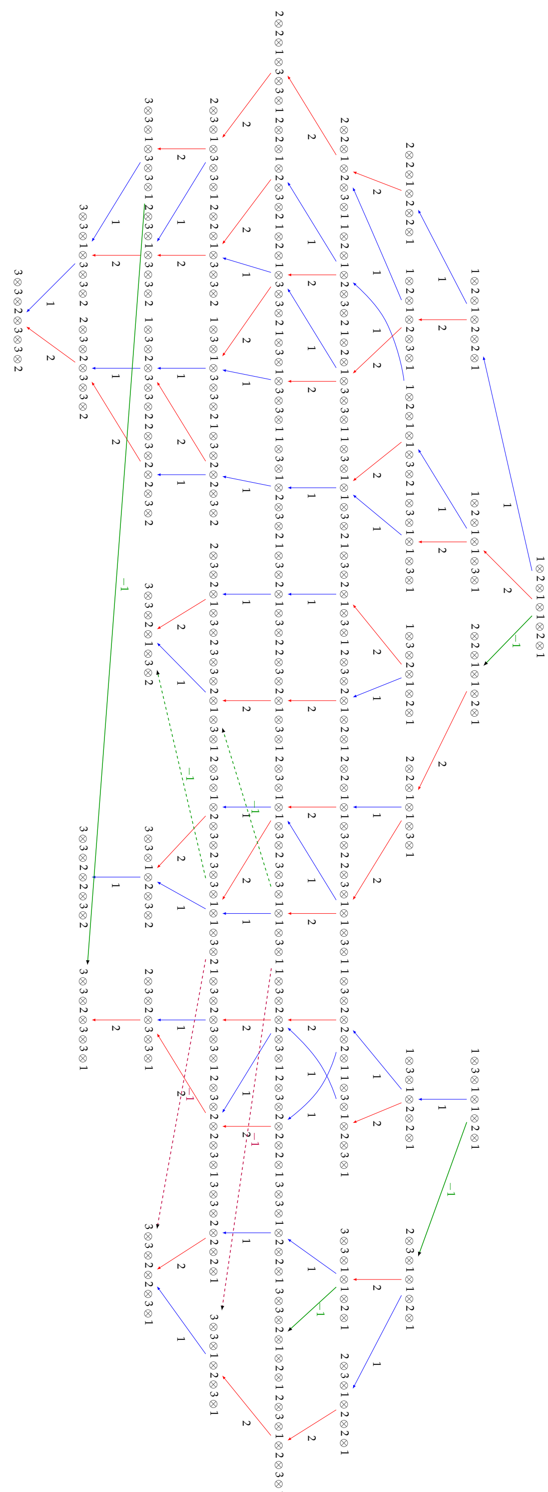

Figure 4: Counterexample to the unique characterization of the local queer axioms of Definition 3.3.

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