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The equivariant volumes of the permutahedron

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Abstract. We consider the action of the symmetric group S_n on the permutahedron Π_n. We prove that if σ is a permutation of S_n which has m cycles of lengths $l_1, \ldots, l_m$, then the subset of Π_n fixed by σ is a polytope with normalized volume $n^{m-2}\gcd(l_1, \ldots, l_m)$.

Resumen. Consideramos la acción del grupo simétrico S_n sobre el permutaedro Π_n. Demostramos que si σ es una permutación de S_n que tiene m ciclos de longitudes $l_1, \ldots, l_m$, entonces el subconjunto de Π_n que permanece fijo bajo la acción de σ es un politopo cuyo volumen normalizado es igual a $n^{m-2}\text{mcd}(l_1, \cdots, l_m)$.

Keywords: permutahedron, volume, symmetric group, tree

1 Introduction

The n-permutahedron is the polytope in $\mathbb{R}^n$ whose vertices are the permutations of $[n]$:

$$\Pi_n := \text{conv}\left\{(\pi(1), \pi(2), \ldots, \pi(n)) : \pi \in S_n\right\}.$$

The symmetric group S_n acts on $\Pi_n \subset \mathbb{R}^n$ by permuting coordinates; more precisely, a permutation $\sigma \in S_n$ acts on a point $x = (x_1, x_2, \ldots, x_n) \in \Pi_n$, by

$$\sigma \cdot x := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).$$

Definition 1.1. *The* fixed polytope of the permutahedron Π_n under a permutation σ of $[n]$ *is*

$$\Pi_n^\sigma = \{x \in \Pi_n \,:\, \sigma \cdot x = x\}.$$

Our main result is a generalization of the fact, due to Stanley [4], that $\text{Vol}\,\Pi_n = n^{n-2}$; see Theorem 3.1.

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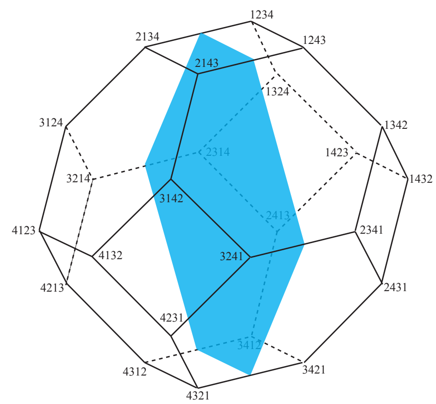

Figure 1: The fixed polytope $\Pi_4^{(12)}$ of the permutahedron Π_4 under $(12) \in S_4$ is a hexagon.

Theorem 1.2. *If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the fixed polytope of Π_n under σ is*

$$\operatorname{Vol} \Pi_n^\sigma = n^{m-2} \gcd(l_1, \ldots, l_m).$$

This is the first step towards describing the equivariant Ehrhart theory of the permutahedron, a question posed by Stapledon [6].

1.1 Normalizing the volume

The permutahedron and its fixed polytopes are not full-dimensional. We normalize volumes so that every primitive parallelotope has volume 1. This is the normalization under which the volume of Π_n equals n^{n-2}.

More precisely, let P be a d-dimensional polytope on an affine d-plane $L \subset \mathbb{Z}^n$. Assume L is integral, in the sense that $L \cap \mathbb{Z}^n$ is a lattice translate of a d-dimensional lattice Λ. We call a lattice d-parallelotope in L *primitive* if its edges generate the lattice Λ; all primitive parallelotopes have the same volume. Then we define the volume of a d-polytope P in L to be $\operatorname{Vol}(P) := \operatorname{EVol}(P)/\operatorname{EVol}(\square)$ for any primitive parallelotope $\square$ in L, where EVol denotes Euclidean volume.

The definition of $\operatorname{Vol}(P)$ makes sense even when P is not an integral polytope. This is important because the fixed polytopes of the permutahedron are not necessarily integral.

1.2 Notation

We identify each permutation $\pi \in S_n$ with the point $(\pi(1), \ldots, \pi(n))$ in $\mathbb{R}^n$. When we write permutations in cycle notation, we do not use commas to separate the entries of each cycle. For example, we identify the permutation 246513 in S_6 with the point $(2,4,6,5,1,3) \in \mathbb{R}^6$, and write it as $(1245)(36)$ in cycle notation.

Our main goal is to find the volume of the fixed polytope Π_n^σ for a permutation $\sigma \in S_n$. We assume that σ has m cycles of lengths $l_1 \geq \cdots \geq l_m$. In fact, for the goals of this paper, it suffices to assume

$$\sigma = (1\ 2\ \ldots\ l_1)(l_1+1\ \ l_1+2\ \ldots\ l_1+l_2)\cdots(l_1+\cdots+l_{m-1}+1\ \ldots\ n-1\ \ n).$$

We let $\{e_1,\ldots,e_n\}$ be the standard basis of $\mathbb{R}^n$, and $e_S := e_{s_1}+\cdots+e_{s_k}$ for $S = \{s_1,\ldots,s_k\} \subseteq [n]$. Recall that the Minkowski sum of polytopes P, $Q \subset \mathbb{R}^n$ is the polytope $P+Q := \{p+q\ :\ p \in P, q \in Q\} \subset \mathbb{R}^n$. [3]

1.3 Organization

Section 2 presents Theorem 2.11, which describes the fixed polytope Π_n^σ in terms of its vertices, its defining inequalities, and a Minkowski sum decomposition. Section 3 uses this to prove our main result, Theorem 1.2, on the normalized volume of Π_n^σ. This is an extended abstract; for complete statements and proofs, see [1].

2 Describing the fixed polytopes of the permutahedron

Proposition 2.1 ([7]). *The permutahedron Π_n can be described in the following three ways:*

1. *(Inequalities) It is the set of points $x \in \mathbb{R}^n$ satisfying*
   (a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, *and*
   (b) *for any proper subset* $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,
   $$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1 + 2 + \cdots + k.$$
2. *(Vertices) It is the convex hull of the points $(\pi(1), \ldots, \pi(n))$ as π ranges over the permutations of $[n]$.*
3. *(Minkowski sum) It is the Minkowski sum: $\sum_{1 \leq j < k \leq n} [e_k, e_j] + \sum_{1 \leq k \leq n} e_k$.*

The n-permutahedron is $(n-1)$-dimensional and every permutation of $[n]$ is indeed a vertex.

Our first goal is to prove the analogous result for the fixed polytopes of Π_n; we do so in Theorem 2.11.

2.1 Standardizing the permutation

We define the *cycle type* of a permutation σ to be the partition of n consisting of the lengths $l_1 \geq \cdots \geq l_m$ of the cycles of σ.

Lemma 2.2. *The volume of Π_n^σ only depends on the cycle type of σ.*

We wish to measure the various fixed polytopes of Π_n, and by Lemma 2.2 we can focus our attention on the polytopes Π_n^σ fixed by a permutation of the form

$$\sigma = (1\ 2\ \ldots\ l_1)(l_1+1\ \ l_1+2\ \ \ldots\ \ l_1+l_2)\cdots(l_1+\cdots+l_{m-1}+1\ \ \ldots\ \ n-1\ \ n) \quad (2.1)$$

for a partition $l_1 \geq l_2 \geq \cdots \geq l_m$ with $l_1 + \cdots + l_m = n$. We do so from now on.

2.2 Towards the inequality description

Lemma 2.3. *For a permutation $\sigma \in S_n$, the fixed polytope Π_n^σ consists of the points $x \in \Pi_n$ satisfying $x_j = x_k$ for any j and k in the same cycle of σ.*

Corollary 2.4. *If a permutation σ of $[n]$ has m cycles then Π_n^σ has dimension $m-1$.*

2.3 Towards a vertex description

In this section we describe a set $\mathrm{Vert}(\sigma)$ of $m!$ points associated to a permutation σ of S_n. We will show in Theorem 2.11 that this is the set of vertices of the fixed polytope Π_n^σ. For a point $w \in \mathbb{R}^n$, let $\overline{w}$ be the average of the σ-orbit of w, that is,

$$\overline{w} := \frac{1}{|\sigma|}\sum_{i=1}^{|\sigma|} \sigma^i \cdot w, \quad (2.2)$$

where $|\sigma|$ is the order of σ as an element of the symmetric group S_n.

Definition 2.5. *Given $\sigma \in S_n$, we say a permutation $v = (v_1, \ldots, v_n)$ of $[n]$ is* σ-standard *if it satisfies the following property: for each cycle $(j_1\ j_2\ \cdots\ j_r)$ of σ, $(v_{j_1}, v_{j_2}, \ldots, v_{j_r})$ is a sequence of consecutive integers in increasing order. We define the set of* σ-vertices *to be*

$$\mathrm{Vert}(\sigma) := \{\overline{w} \ :\ w \text{ is a } \sigma\text{-standard permutation of } [n]\}.$$

These points should not be confused with the vertices of the ambient permutahedron Π_n. Let us illustrate this definition in an example and prove some preliminary results.

Example 2.6. For $\sigma = (1234)(567)(89)$, the σ-standard permutations in S_9 are

$$\begin{array}{ll} (1,2,3,4,\ 5,6,7,\ 8,9), & (1,2,3,4,\ 7,8,9,\ 5,6), \\ (4,5,6,7,\ 1,2,3,\ 8,9), & (3,4,5,6,\ 7,8,9,\ 1,2), \\ (6,7,8,9,\ 1,2,3,\ 4,5), & (6,7,8,9,\ 3,4,5,\ 1,2), \end{array}$$

and the corresponding σ-vertices are

$$\begin{array}{ll} \frac{1+2+3+4}{4} e_{1234} + \frac{5+6+7}{3} e_{567} + \frac{8+9}{2} e_{89}, & \frac{1+2+3+4}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{5+6}{2} e_{89}, \\ \frac{4+5+6+7}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{8+9}{2} e_{89}, & \frac{3+4+5+6}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{1+2}{2} e_{89}, \\ \frac{6+7+8+9}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{4+5}{2} e_{89}, & \frac{6+7+8+9}{4} e_{1234} + \frac{3+4+5}{3} e_{567} + \frac{1+2}{2} e_{89}. \end{array}$$

Let us give a more explicit description of $\overline{w}$ in general, and of the σ-vertices in particular, which will be important in the proof of Theorem 2.11.

Lemma 2.7. *For any $w \in \mathbb{R}^n$, the average of the σ-orbit of w is*

$$\overline{w} = \sum_{k=1}^{m} \frac{\sum_{j \in \sigma_k} w_j}{l_k} e_{\sigma_k}.$$

Notice that the entries of $\overline{w}$ within each cycle σ_k are constant, bearing witness to the fact that $\overline{w}$, being the average of a σ-orbit, must be in the fixed polytope Π_n^σ.

Corollary 2.8. *The set* $\mathrm{Vert}(\sigma)$ *of σ-vertices consists of the $m!$ points*

$$\overline{v_\prec} := \sum_{k=1}^{m} \left(\frac{l_k + 1}{2} + \sum_{j\,:\,\sigma_j \prec \sigma_k} l_j \right) e_{\sigma_k}$$

as $\prec$ ranges over the $m!$ possible linear orderings of $\sigma_1, \sigma_2, \ldots, \sigma_m$.

2.4 Towards a zonotope description

We will show in Theorem 2.11 that the fixed polytope Π_n^σ is the zonotope given by the following Minkowski sum.

Definition 2.9. *Let M_σ denote the Minkowski sum*

$$\begin{aligned} M_\sigma &:= \sum_{1 \le j < k \le m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k + 1}{2} e_{\sigma_k} \\ &= \sum_{1 \le j < k \le m} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^{m} \left(\frac{l_k + 1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}. \end{aligned} \tag{2.3}$$

Proposition 2.10. *The zonotope M_σ is combinatorially equivalent to the standard permutahedron Π_m, where m is the number of cycles of σ.*

2.5 Three descriptions of the fixed polytope of the permutahedron

Theorem 2.11. *Let σ be a permutation of $[n]$ whose cycles $\sigma_1, \ldots, \sigma_m$ have respective lengths $l_1, \ldots, l_m$. The fixed polytope Π_n^σ can be described in the following ways:*

0. *It is the set of points x in the permutahedron Π_n such that $\sigma \cdot x = x$.*
1. *It is the set of points $x \in \mathbb{R}^n$ satisfying*
   (a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$,
   (b) *for any proper subset* $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,
   $$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1 + 2 + \cdots + k, \text{ and}$$
   (c) *for any i and j which are in the same cycle of σ, $x_i = x_j$.*
2. *It is the convex hull of the set* $\mathrm{Vert}(\sigma)$ *of σ-vertices, as described in Corollary 2.8.*
3. *It is the Minkowski sum M_σ of Definition 2.9*

Consequently, the fixed polytope Π_n^σ is a zonotope that is combinatorially isomorphic to the permutahedron Π_m. It is $(m-1)$-dimensional and every σ-vertex is indeed a vertex of Π_n^σ.

Proof. Description 0. is the definition of the fixed polytope Π_n^σ, and we already observed in Lemma 2.3 that description 1. is accurate. Recall that we denoted the polytopes described in 2. and 3. by $\mathrm{conv}(\mathrm{Vert}(\sigma))$ and M_σ, respectively. It remains to prove that

$$\Pi_n^\sigma = \mathrm{conv}(\mathrm{Vert}(\sigma)) = M_\sigma.$$

We proceed in three steps as follows:

A. $\mathrm{conv}(\mathrm{Vert}(\sigma)) \subseteq \Pi_n^\sigma$ *B.* $M_\sigma \subseteq \mathrm{conv}(\mathrm{Vert}(\sigma))$ *C.* $\Pi_n^\sigma \subseteq M_\sigma$

A. $\mathrm{conv}(\mathrm{Vert}(\sigma)) \subseteq \Pi_n^\sigma$: It suffices to show that Π_n^σ contains any point in $\mathrm{Vert}(\sigma)$, say

$$\overline{v_\prec} = \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot v_\prec,$$

where $\prec$ is a total order of $\sigma_1, \ldots, \sigma_m$ and $v_\prec$ is the associated σ-standard permutation. Since $v_\prec$ is a vertex of Π_n, we conclude that $\sigma^i \cdot v_\prec$ is a vertex of Π_n for all i, and hence their average $\overline{v_\prec}$ is in Π_n. Also, since $\sigma^{|\sigma|} = 1$, we have that $\sigma \cdot \overline{v_\prec} = \overline{v_\prec}$. Therefore, $\overline{v_\prec}$ is in Π_n^σ by 0., as desired.

B. $M_\sigma \subseteq \mathrm{conv}(\mathrm{Vert}(\sigma))$: It suffices to show that any vertex of M_σ is in $\mathrm{Vert}(\sigma)$.

For a polytope $P \subset \mathbb{R}^n$ and a linear functional $c \in (\mathbb{R}^n)^*$, we let P_c denote the face of P where c is maximized. In particular, for any given vertex v of M_σ, consider a linear functional $c = (c_1, c_2, \ldots, c_n) \in (\mathbb{R}^n)^*$ such that $v = (M_\sigma)_c$ is the unique point in M_σ maximizing c. For $k = 1, \ldots, m$, let $c_{\sigma_k} := \frac{1}{l_k} \sum_{i \in \sigma_k} c_i$. One can verify that

(a) $c_{\sigma_j} \neq c_{\sigma_k}$ for $j \neq k$, and

(b) $v = \overline{v_\prec}$ for the linear order $\prec$ on $\sigma_1, \sigma_2, \ldots, \sigma_m$ where $\sigma_j \prec \sigma_k$ if and only if $c_{\sigma_j} < c_{\sigma_k}$. This shows that every vertex of M_σ is a σ-vertex, as desired.

C. $\Pi_n^\sigma \subseteq M_\sigma$: Any point $p \in \Pi_n^\sigma$ can be written as a convex combination $p = \sum_{\tau \in S_n} \lambda_\tau \tau$ of the $n!$ permutations of $[n]$, where $\lambda_\tau \geq 0$ for all τ and $\sum_{\tau \in S_n} \lambda_\tau = 1$. Recall from (2.2) that $\overline{w}$ represents the average of the σ-orbit of $w \in \mathbb{R}^n$. Since p is fixed by σ we have

$$p = \overline{p} = \sum_{\tau \in S_n} \lambda_\tau \overline{\tau}.$$

It follows that $\Pi_n^\sigma \subseteq \operatorname{conv}\{\overline{\tau} : \tau \in S_n\}$. Therefore, to show that $\Pi_n^\sigma \subseteq M_\sigma$, it suffices to show that $\overline{\tau} \in M_\sigma$ for all permutations τ. To do so, let us first derive an alternative expression for $\overline{\tau}$.

Let us begin with the vertex $\mathrm{id} = (1, 2, \ldots, n)$ of Π_n corresponding to the identity permutation. As described in Corollary 2.8, this is the σ-standard permutation corresponding to the order $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_m$, so

$$\overline{\mathrm{id}} = \sum_{k=1}^{m} \left(\frac{l_k + 1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}. \tag{2.4}$$

Notice that this is the translation vector for the Minkowski sum of (2.3).

Now, let us compute $\overline{\tau}$ for any permutation τ. Let

$$l = \operatorname{inv}(\tau) = |\{(a, b) : 1 \leq a < b \leq n, \ \tau(a) > \tau(b)\}|$$

be the number of inversions of τ. Consider a minimal sequence $\mathrm{id} = \tau_0, \tau_1, \ldots, \tau_l = \tau$ of permutations such that τ_{i+1} is obtained from τ_i by exchanging the positions of numbers p and $p+1$, thus introducing a single new inversion without affecting any existing inversions. Such a sequence corresponds to a minimal factorization of τ as a product of simple transpositions $(p \;\; p+1)$ for $1 \leq p \leq n-1$. We have $\operatorname{inv}(\tau_i) = i$ for $1 \leq i \leq l$.

Now we compute $\overline{\tau}$ by analyzing how $\overline{\tau_i}$ changes as we introduce new inversions, using that

$$\overline{\tau} - \overline{\mathrm{id}} = (\overline{\tau_l} - \overline{\tau_{l-1}}) + \cdots + (\overline{\tau_1} - \overline{\tau_0}). \tag{2.5}$$

If $a < b$ are the positions of the numbers p and $p+1$ that we switch as we go from τ_i to τ_{i+1}, then regarding τ_i and τ_{i+1} as vectors in $\mathbb{R}^n$ we have

$$\tau_{i+1} - \tau_i = e_a - e_b.$$

If σ_j and σ_k are the cycles of σ containing a and b, respectively, we have

$$\overline{\tau_{i+1}} - \overline{\tau_i} = \overline{e_a} - \overline{e_b} = \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} = \frac{1}{l_j l_k}(l_k e_{\sigma_j} - l_j e_{\sigma_k}) \tag{2.6}$$

in light of Lemma 2.7. This is the local contribution to (2.5) that we obtain when we introduce a new inversion between a position a in cycle σ_j and a position b in cycle σ_k in our permutation. Notice that this contribution is 0 when $j = k$. Also notice that we will still have an inversion between positions a and b in all subsequent permutations, due to the minimality of the sequence. We conclude that

$$\overline{\tau} - \overline{\mathrm{id}} = \sum_{j<k} \frac{\mathrm{inv}_{j,k}(\tau)}{l_j l_k}(l_k e_{\sigma_j} - l_j e_{\sigma_k}) \tag{2.7}$$

where

$$\mathrm{inv}_{j,k}(\tau) = |\{(a,b) \,:\, 1 \leq a < b \leq n,\ a \in \sigma_j,\ b \in \sigma_k \ \text{ and } \tau(a) > \tau(b)\}|$$

is the number of inversions in τ between a position in σ_j and a position in σ_k for $j < k$.

Equations (2.4) and (2.7) give us an alternative description for $\overline{\tau}$. This description makes it apparent that $\overline{\tau} \in M_\sigma$: Notice that $|\sigma_j| = l_j$ and $|\sigma_k| = l_k$ imply that $0 \leq \mathrm{inv}_{j,k}(\tau) \leq l_j l_k$, so

$$\overline{\tau} - \overline{\mathrm{id}} \in \sum_{1 \leq j < k \leq n} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}];$$

combining this with (2.3) and (2.4) gives the desired result. □

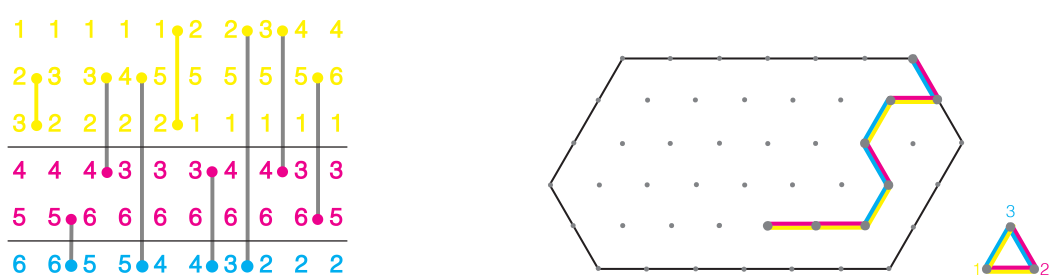

Figure 2: (a) A minimal sequence of permutations $\mathrm{id} = \tau_0, \tau_1, \ldots, \tau_9 = 461352$ adding one inversion at a time and (b) the corresponding path from $\overline{\mathrm{id}}$ to $\overline{\tau}$ in the zonotope M_σ.

Example 2.12. Figure 2 illustrates part C of the proof above for $n = 6$, $\sigma = (123)(45)(6)$, and the permutation $\tau = 461352$. This permutation has $\mathrm{inv}(\tau) = 9$ inversions, and the columns of the left panel show a minimal sequence of permutations $\mathrm{id} = \tau_0, \tau_1, \ldots, \tau_9 =$

τ where each τ_{i+1} is obtained from τ_i by swapping two consecutive numbers, thus introducing a single new inversion.

The rows of the diagram are split into three groups 1, 2, and 3, corresponding to the support of the cycles of σ. Out of the $\text{inv}(\tau) = 9$ inversions of τ, there are $\text{inv}_{1,2}(\tau) = 3$ involving groups 1 and 2, $\text{inv}_{1,3}(\tau) = 2$ involve groups 1 and 3, and $\text{inv}_{2,3}(\tau) = 2$ involving groups 2 and 3.

This sequence of permutations gives rise to a walk from $\overline{\text{id}}$, which is the top right vertex of the zonotope M_σ, to $\overline{\tau}$. In the rightmost triangle, which is not drawn to scale, vertex i represents the point e_{σ_i}/l_i for $1 \leq i \leq 3$. Whenever two numbers in groups $j < k$ are swapped in the left panel, to get from permutation τ_i to τ_{i+1}, we take a step in direction $e_{\sigma_j}/l_j - e_{\sigma_k}/l_k$ in the right panel, to get from point $\overline{\tau_i}$ to $\overline{\tau_{i+1}}$. This is the direction of edge jk in the triangle, and its length is $1/l_j l_k$ of the length of the generator $l_k e_{\sigma_j} - l_j e_{\sigma_k}$ of the zonotope. Then

$$\overline{\tau} - \overline{\text{id}} = \frac{3}{l_1 l_2}(l_2 e_{\sigma_1} - l_1 e_{\sigma_2}) + \frac{2}{l_1 l_3}(l_3 e_{\sigma_1} - l_1 e_{\sigma_3}) + \frac{2}{l_2 l_3}(l_3 e_{\sigma_2} - l_2 e_{\sigma_3}).$$

Since $3 = \text{inv}_{1,2}(\tau) \leq l_1 l_2 = 6$, $2 = \text{inv}_{1,3}(\tau) \leq l_1 l_3 = 3$ and $2 = \text{inv}_{2,3}(\tau) \leq l_2 l_3 = 2$, the resulting point $\overline{\tau}$ is in the zonotope M_σ.

3 The volumes of the fixed polytopes of the permutahedron

To compute the volume of Π_n^σ we use its description as a zonotope, recalling that a zonotope can be tiled by parallelotopes as follows. If A is a set of vectors, then $B \subseteq A$ is called a *basis* for A if B is linearly independent and $\text{rank}(B) = \text{rank}(A)$. We define the parallelotope $\square B$ to be the Minkowski sum of the segments in B, that is,

$$\square B := \left\{ \sum_{b \in B} \lambda_b b \ : \ 0 \leq \lambda_b \leq 1 \text{ for each } b \in B \right\}.$$

Theorem 3.1 ([2, 4, 7])**.** *Let $A \subset \mathbb{Z}^n$ be a set of lattice vectors of rank d.*

1. *The zonotope $Z(A)$ can be tiled using one translate of the parallelotope $\square B$ for each basis B of A. Therefore, the volume of the d-dimensional zonotope $Z(A)$ is*

$$\text{Vol}\,(Z(A)) = \sum_{\substack{B \subseteq A \\ B \text{ basis}}} \text{Vol}\,(\square B).$$

2. *For each $B \subset \mathbb{Z}^n$ of rank d, $\text{Vol}(\square B)$ equals the index of $\mathbb{Z}B$ as a sublattice of $(\text{span}\, B) \cap \mathbb{Z}^n$. Using the vectors in B as the columns of an $n \times d$ matrix, $\text{Vol}(B)$ is the greatest common divisor of the minors of rank d.*

By Theorem 2.11, the fixed polytope Π_n^σ is a translate of the zonotope generated by the set

$$F_\sigma = \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \, ; \, 1 \leq j < k \leq m \right\}.$$

This set of vectors has a nice combinatorial structure, allowing us to describe the bases B and the volumes $\mathrm{Vol}\,(\square B)$ combinatorially. We do this in the next two lemmas. For a tree T whose vertex set is $[m]$, let

$$\begin{aligned} F_T &= \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \; : \; j < k \text{ and } jk \text{ is an edge of } T \right\}, \\ E_T &= \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \; : \; j < k \text{ and } jk \text{ is an edge of } T \right\}. \end{aligned}$$

Lemma 3.2 ([4]). *The vector configuration*

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \; : \; 1 \leq j < k \leq m \right\}$$

has exactly m^{m-2} bases: they are the sets F_T as T ranges over the spanning trees on $[m]$.

Lemma 3.3. *For any tree T on $[m]$ we have*

$$\begin{aligned} 1. \quad \mathrm{Vol}(\square F_T) &= \prod_{i=1}^{m} l_i^{\deg_T(i)} \mathrm{Vol}(E_T), \\ 2. \quad \mathrm{Vol}(\square E_T) &= \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m}, \end{aligned}$$

where $\deg_T(i)$ is the number of edges containing vertex i in T.

Proof. 1. Since $l_k e_{\sigma_j} - l_j e_{\sigma_k} = l_j l_k \big(\frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \big)$ for each edge jk of T, and volumes scale linearly with respect to each edge length of a parallelotope, we have

$$\mathrm{Vol}(\square F_T) = \Big(\prod_{jk \text{ edge of T}} l_j l_k \Big) \mathrm{Vol}(\square E_T) = \prod_{i=1}^{m} l_i^{\deg_T(i)} \mathrm{Vol}(\square E_T).$$

2. The parallelotopes $\square E_T$ are the images of the parallelotopes $\square A_T$ under the linear bijective map

$$\begin{aligned} \phi : \mathbb{R}^m &\to (\mathbb{R}^n)^\sigma \\ f_i &\mapsto \frac{e_{\sigma_i}}{l_i}, \end{aligned}$$

where

$$A_T := \{ f_j - f_k \; : \; j < k, \, jk \text{ is an edge of } T \}.$$

Since the vector configuration $\{f_j - f_k : 1 \leq j < k \leq m\}$ is unimodular, all parallelotopes $\square A_T$ have unit volume. Therefore, the parallelotopes $\square E_T = \phi(\square A_T)$ have the same normalized volume, so $\mathrm{Vol}(E_T)$ is independent of T.

It follows that we can use any tree T to compute $\mathrm{Vol}(E_T)$ or, equivalently, $\mathrm{Vol}(F_T)$. We choose the tree $T = \mathrm{Claw}_m$ with edges $1m, 2m, \ldots, (m-1)m$. Writing the $m-1$ vectors of

$$F_{\mathrm{Claw}_m} = \{l_m e_{\sigma_i} - l_i e_{\sigma_m} : 1 \leq i \leq m-1\}$$

as the columns of an $n \times (m-1)$ matrix, then $\mathrm{Vol}(F_{\mathrm{Claw}_m})$ is the greatest common divisor of the non-zero maximal minors of this matrix. This quantity does not change when we remove duplicate rows; the result is the $m \times (m-1)$ matrix

$$\begin{bmatrix} l_m & 0 & 0 & \cdots & 0 \\ 0 & l_m & 0 & \cdots & 0 \\ 0 & 0 & l_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & l_m \\ -l_1 & -l_2 & -l_3 & \cdots & -l_{m-1} \end{bmatrix}.$$

This matrix has m maximal minors, whose absolute values equal $l_m^{m-2} l_1, \ldots, l_m^{m-2} l_{m-1}$, l_m^{m-1}. Therefore,

$$\mathrm{Vol}(\square F_{\mathrm{Claw}_m}) = l_m^{m-2} \gcd(l_1, \ldots, l_{m-1}, l_m)$$

and part 1 then implies that

$$\mathrm{Vol}(\square E_{\mathrm{Claw}_m}) = \frac{\mathrm{Vol}(\square F_{\mathrm{Claw}_m})}{l_1 \cdots l_{m-1} l_m^{m-1}} = \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m}$$

as desired. □

Lemma 3.4. *For any positive integer $m \geq 2$ and unknowns $x_1, \ldots, x_m$, we have*

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} = (x_1 + \cdots + x_m)^{m-2}.$$

Sketch of proof. This is a variant of the analogous result for rooted trees [5, Theorem 5.3.4], which states that

$$\sum_{\substack{(T,r) \text{ rooted} \\ \text{tree on } [m]}} \prod_{i=1}^{m} x_i^{\mathrm{children}_{(T,r)}(i)} = (x_1 + \cdots + x_m)^{m-1}$$

where $\mathrm{children}_{(T,r)}(v)$ counts the children of v. It can be proved similarly, or derived directly from it. □

Theorem 1.2. *If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the fixed polytope of Π_n under σ is*

$$\operatorname{Vol} \Pi_n^\sigma = n^{m-2} \gcd(l_1, \ldots, l_m).$$

Proof. Since Π_n^σ is a translate of the zonotope for $F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : 1 \le j < k \le m \right\}$, we invoke Theorem 3.1. Using Lemmas 3.2 to 3.4, it follows that

$$\begin{aligned}
\operatorname{Vol} \Pi_n^\sigma &= \sum_{T \text{ tree on } [m]} \operatorname{Vol}(\square F_T) \\
&= \sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} l_i^{\deg_T(i)-1} \gcd(l_1, \ldots, l_m) \\
&= (l_1 + \cdots + l_m)^{m-2} \gcd(l_1, \ldots, l_m) = n^{m-2} \gcd(l_1, \ldots, l_m).
\end{aligned}$$

□

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