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Hopf dreams

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Abstract. We introduce a Hopf algebra structure on a family of reduced pipe dreams with a natural surjection onto a commutative Hopf algebra of permutations. We then study three Hopf subalgebras of permutations whose preimages by the surjection yield three relevant Hopf subalgebras of pipe dreams. The first is the Loday–Ronco Hopf algebra on binary trees, the second is related to a special family of lattice walks on the quarter plane, and the third is a Hopf algebra on ν-trees related to ν-Tamari lattices. The latter motivates a new notion of Hopf chains in the Tamari lattice with applications in the theory of multivariate diagonal harmonics.

Résumé. Nous introduisons une structure d'algèbre de Hopf sur une famille d'arrangements de tuyaux, avec une surjection naturelle sur une algèbre de Hopf commutative de permutations. Nous étudions ensuite trois sous-algèbres de Hopf de permutations dont les préimages par la surjection donnent trois sous-algèbres de Hopf d'arrangements de tuyaux intéressantes. La première est l'algèbre de Hopf de Loday–Ronco sur les arbres binaires, la seconde est reliée à une famille spéciale de chemins dans le quart de plan, et la troisième est une algèbre de Hopf sur des ν-arbres reliée aux treillis de ν-Tamari. Cette dernière motive une nouvelle notion de chaînes de Hopf dans le treillis de Tamari, avec des applications à la théorie des harmoniques diagonaux multivariés.

Keywords: Pipe dream, Hopf algebra, Tamari lattice, Multivariate diagonal harmonics

1 Introduction

Pipe dreams are combinatorial objects related to reduced expressions of permutations in terms of simple transpositions. They were introduced by N. Bergeron and S. Billey in [3] to compute Schubert polynomials and later revisited in the context of Gröbner geometry by A. Knutson and E. Miller [11], who coined the name *pipe dreams* in reference to a

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game involving pipe connections. In brief, a pipe dream is an arrangement of pipes, each connecting an entry on the vertical axis to an exit on the horizontal axis, and remaining in a triangular shape of the grid. Pipe dreams are grouped according to their exiting permutation, given by the order in which the pipes appear along the horizontal axis.

This paper introduces a Hopf algebra structure on pipe dreams. Hopf algebras are rather rigid structures which often reveal deep combinatorial properties and connections. This paper contributes to this general philosophy: the Hopf algebra of pipe dreams will give us insight on a special family of lattice walks on the quarter plane studied in [5], as well as applications to the still emerging theory of multivariate diagonal harmonics [1].

Our compass for this construction is the Hopf algebra of J.-L. Loday and M. Ronco on complete binary trees [12]. There is a strong correspondence [17, 13, 15] between the complete binary trees with n internal nodes and the reduced pipe dreams with exiting permutation $0n \dots 1$. This correspondence allows to interpret the product and the coproduct of the Loday–Ronco Hopf algebra in terms of pipe dreams. This interpretation yields to an extension of the Loday–Ronco Hopf algebra on a bigger family Π consisting of reduced pipe dreams with an elbow in the top left corner. We show that it results in a free and cofree Hopf algebra structure $(\mathbf{k}\Pi, \cdot, \triangle)$, and that mapping a pipe dream to its exiting permutation defines a surjective morphism from the Hopf algebra $(\mathbf{k}\Pi, \cdot, \triangle)$ of pipe dreams to a commutative Hopf algebra $(\mathbf{k}\mathfrak{S}, \sqcup_\bullet, \triangle_\bullet)$ of permutations.

We then study relevant Hopf subalgebras of $\mathbf{k}\Pi$ obtained from pipe dreams whose exiting permutations belong to a given Hopf subalgebra of $\mathbf{k}\mathfrak{S}$. We obtain this way:

1. the Loday–Ronco Hopf algebra on complete binary trees [12],
2. a Hopf algebra related to a special family of lattice walks on the quarter plane [5],
3. a Hopf algebra on ν-trees connected to the ν-Tamari lattices [14].

Finally, one of our most important contributions is the application of the Hopf algebra of dominant pipe dreams to the theory of multivariate diagonal harmonics. The space of diagonal harmonics is an $\mathfrak{S}_n$-module of polynomials in two sets of variables that satisfy some harmonic properties. The dimensions of these spaces led to numerous important conjectures, including the $(n+1)^{n-1}$-conjecture by A. Garsia and M. Haiman proved by M. Haiman using properties of the Hilbert scheme in algebraic geometry [10], and the shuffle conjecture by J. Haglund et al. [9], recently proved by E. Carlsson and A. Mellit in [6]. The module of diagonal harmonics has natural generalizations in three or more sets of variables. Computational experiments by M. Haiman from the early 1990's suggest explicit simple dimension formulas for the space of diagonal harmonics and its alternating component in the trivariate case. F. Bergeron noticed that these formulas coincide with formulas counting labeled and unlabeled intervals in the classical Tamari lattice, and opened the door to a more systematic study of the r-variate case in terms of certain suitable $(r-1)$-chains in the Tamari lattice [2, 1]. In this paper we present a milestone towards this understanding by introducing a new class of chains in the Tamari lattice that was motivated by our Hopf algebra construction, see Theorem 20.

2 The pipe dream Hopf algebra

2.1 A Hopf algebra on permutations

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \dots, n\}$, and $\mathfrak{S} := \bigsqcup_{n \geq 0} \mathfrak{S}_n$. We consider the graded vector space $\mathbf{k}\mathfrak{S} := \bigoplus_{n \geq 0} \mathbf{k}\mathfrak{S}_n$, where $\mathbf{k}\mathfrak{S}_n$ is the $\mathbf{k}$-span of $\mathfrak{S}_n$.

Global descents and atomic permutations. Consider $\omega \in \mathfrak{S}_n$. Index by $\{0, \dots, n\}$ from left to right the *gaps* before the first position, between two consecutive positions, or after the last position of ω. A gap γ is a *global descent* if $\omega([\gamma]) = [n] \smallsetminus [n - \gamma]$. In other words, the first γ positions are sent to the last γ values. For example, the global descents in the permutation $\omega = 635421$ are $0, 1, 4, 5, 6$. Note that the gaps 0 and n are always global descents. A permutation with no other global descent is called *atomic*.

For two permutations $\mu \in \mathfrak{S}_m$ and $\nu \in \mathfrak{S}_n$ we define the permutation $\mu \bullet \nu \in \mathfrak{S}_{m+n}$ by $\mu \bullet \nu(i) := \mu(i) + n$ if $1 \leq i \leq m$, and $\mu \bullet \nu(i) := \nu(i - m)$ if $m + 1 \leq i \leq m + n$. Observe that $\mu \bullet \nu$ has a global descent in position m. Conversely, given a permutation $\omega \in \mathfrak{S}_{m+n}$ with a global descent in position m, there exist a unique pair of permutations $\mu \in \mathfrak{S}_m$ and $\nu \in \mathfrak{S}_n$ such that $\omega = \mu \bullet \nu$. Therefore any permutation $\omega \in \mathfrak{S}$ factorizes in a unique way as a product $\omega = \omega_1 \bullet \omega_2 \bullet \dots \bullet \omega_\ell$ of atomic permutations ω_i. For example, we have $635421 = 1 \bullet 132 \bullet 1 \bullet 1$. For such a factorization, we denote by $\omega^\bullet := \{\omega_1, \omega_2, \dots, \omega_\ell\}$ the set of its atomic factors.

Coproduct on permutations. Consider a permutation $\omega \in \mathfrak{S}$, and let $\omega = \omega_1 \bullet \dots \bullet \omega_\ell$ be its unique factorization into atomic permutations. We define the coproduct $\triangle_\bullet(\omega)$ by $\triangle_\bullet(\omega) := \sum_{i=0}^{\ell} (\omega_1 \bullet \dots \bullet \omega_i) \otimes (\omega_{i+1} \bullet \dots \bullet \omega_\ell)$, where an empty $\bullet$-product is the neutral element ϵ for $\bullet$. This coproduct extends to $\mathbf{k}\mathfrak{S}$ by linearity and is clearly coassociative.

Example 1. *For the permutation* $635421 \in \mathfrak{S}_6$*, we have* $635421 = 1 \bullet 132 \bullet 1 \bullet 1$*. Therefore,* $\triangle_\bullet(635421) = \epsilon \otimes 635421 + 1 \otimes 35421 + 4132 \otimes 21 + 52431 \otimes 1 + 635421 \otimes \epsilon$.

Product on permutations. Consider two permutations $\pi, \omega \in \mathfrak{S}$. Define $\pi \sqcup\!\sqcup_\bullet \epsilon := \pi$ and $\epsilon \sqcup\!\sqcup_\bullet \omega := \omega$. Assume now that $\pi = \mu \bullet \nu$ and $\omega = \sigma \bullet \tau$ where μ and σ are non-trivial atomic permutations, and define $\pi \sqcup\!\sqcup_\bullet \omega := \mu \bullet (\nu \sqcup\!\sqcup_\bullet \omega) + \sigma \bullet (\pi \sqcup\!\sqcup_\bullet \tau)$. This product extends to $\mathbf{k}\mathfrak{S}$ by linearity and is clearly associative. This is the standard shuffle product, but performed on the atomic factorizations of the factors.

Example 2. *For the permutations* $2431 = 132 \bullet 1 \in \mathfrak{S}_4$ *and* $312 = 1 \bullet 12 \in \mathfrak{S}_3$*, we have* $2431 \sqcup\!\sqcup_\bullet 312 = 5764312 + 5764312 + 5764231 + 7465312 + 7465231 + 7562431$

The product $\bullet$ and the coproduct $\triangle_\bullet$ are compatible: $\triangle_\bullet(\pi \sqcup\!\sqcup_\bullet \omega) = \triangle_\bullet(\pi) \sqcup\!\sqcup_\bullet \triangle_\bullet(\omega)$, where the right hand side product has to be understood componentwise. This structure is thus a graded and connected commutative Hopf algebra $(\mathbf{k}\mathfrak{S}, \sqcup\!\sqcup_\bullet, \triangle_\bullet)$. See [16].

2.2 A Hopf algebra on pipe dreams

A *pipe dream* P is a filling of a triangular shape with crosses $+$ and elbows $\not\!\!\!\!\frown$ so that all pipes entering on the left side exit on the top side [3, 11]. We only consider *reduced* pipe dreams, where two pipes have at most one intersection. We also restrict to pipe dreams with an elbow in the top left corner. In particular the pipe entering in the topmost row always exits in the leftmost column. We label this pipe with 0 and the other pipes with $1, 2, \dots, n$ in the order of their entry points from top to bottom. We also label accordingly the rows and the columns of the pipe dream P from 0 to n. We denote by $\omega_P \in \mathfrak{S}_n$ the order of the exit points of the non-zero pipes of P from left to right. In other words, the pipe entering at row $i > 0$ exits at column $\omega_P^{-1}(i) > 0$. See Figure 1. For a permutation $\omega \in \mathfrak{S}_n$, we denote by $\Pi(\omega)$ the set of reduced pipe dreams P with an elbow in the top left corner and such that $\omega_P = \omega$. We let $\Pi_n := \bigsqcup_{\omega \in \mathfrak{S}_n} \Pi(\omega)$ and $\Pi := \bigsqcup_{n \in \mathbb{N}} \Pi_n$. We consider the graded vector space $\mathbf{k}\Pi = \bigoplus_{n \ge 0} \mathbf{k}\Pi_n$, where $\mathbf{k}\Pi_n$ is the $\mathbf{k}$-span of Π_n.

Horizontal and vertical packing. Let $P \in \Pi_n$ and $k \in \{0, \dots, n\}$. We color plain red the pipes entering in the rows $1, 2, \dots, k$ and dashed blue the pipes entering in the rows $k+1, k+2, \dots, n$. The *horizontal packing* $\neg_k(P)$ is obtained by removing all blue pipes and contracting the horizontal parts of the red pipes that were contained in a cell $+$ (*i.e.* red horizontal steps that are crossed vertically by a blue pipe). The *vertical packing* $\lrcorner_k(P)$ is defined symmetrically: it keeps the pipes exiting through the columns $1, 2, \dots, k$ and contracts the vertical steps that are crossed horizontally by the pipes exiting through the columns $k+1, k+2, \dots, n$. The remaining pipes need to be relabeled from 1 to k.

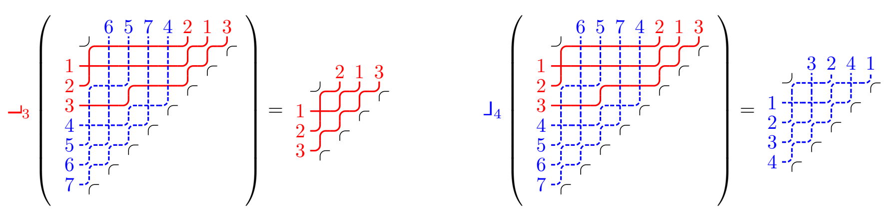


Figure 1: Horizontal packing at $k = 3$ (left) and vertical packing at $k = 4$ (right).

Coproduct on pipe dreams. Consider a reduced pipe dream $P \in \Pi_n$, and a global descent γ of the permutation ω_P. Since γ is a global descent of ω_P, the relevant pipes of P are split into two disjoint tangled sets of pipes: those entering in the first $n - \gamma$ rows (plain red) and those exiting in the first γ columns (dashed blue). The horizontal and vertical packings $\neg_{n-\gamma}(P)$ and $\lrcorner_\gamma(P)$ should thus be regarded as a way to untangle these two disjoint sets of pipes. We denote by $\triangle_{\gamma, n-\gamma}(P)$ the tensor product $\lrcorner_\gamma(P) \otimes \neg_{n-\gamma}(P)$ and we say that $\triangle_{\gamma, n-\gamma}$ *untangles* P. This operation is illustrated on Figure 2 (left). If γ is not a global descent of ω_P, then the pipes of P are not split by γ, and we therefore

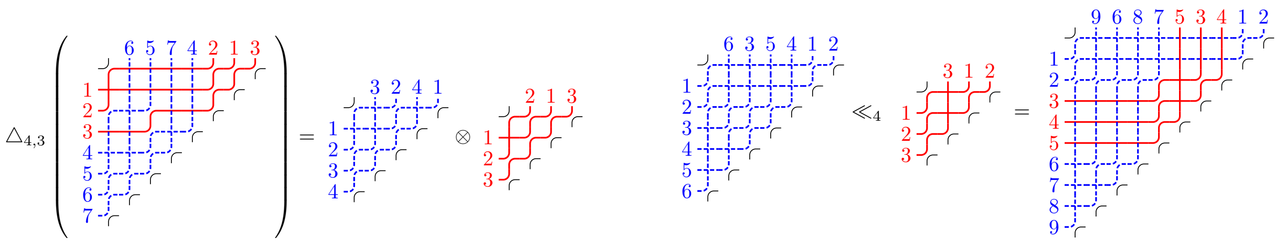


Figure 2: Untangling a pipe dream (left). Inserting a pipe dream (right).

define $\triangle_{\gamma,n-\gamma}(P) = 0$. Finally, we define the coproduct on Π as $\triangle := \sum_{m,n\in\mathbb{N}} \triangle_{m,n}$. See Figure 3. Extended by linearity, the map $\triangle : \mathbf{k}\Pi \to \mathbf{k}\Pi \otimes \mathbf{k}\Pi$ defines a comultiplication on $\mathbf{k}\Pi$.

Proposition 3 ([4, Props. 1.2.2 & 1.2.3])**.** *The coproduct $\triangle$ defines a coassociative graded coalgebra structure on $\mathbf{k}\Pi$. The map $\omega\colon (\mathbf{k}\Pi, \triangle) \to (\mathbf{k}\mathfrak{S}, \triangle_\bullet)$ is a graded morphism of coalgebras.*

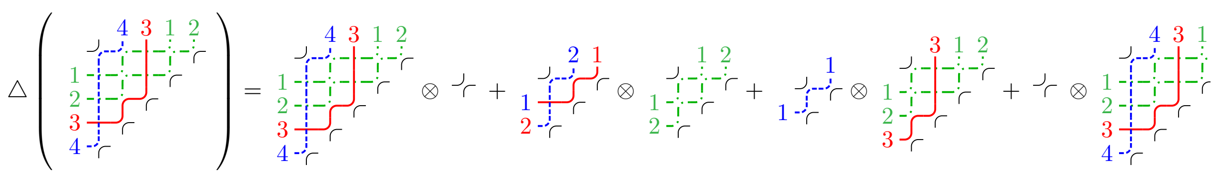


Figure 3: Coproduct of a pipe dream.

Product on pipe dreams. Let $P \in \Pi_m$ and $Q \in \Pi_n$ be two pipe dreams, and let γ be a global descent of ω_P. We denote by $P \ll_\gamma Q$ the pipe dream of Π_{m+n} obtained from P by

- inserting n columns after column γ and n rows after row $m-\gamma$,
- filling with Q the triangle of boxes located both in one of the rows $\gamma, \dots, n+\gamma$ and in one of the columns $m-\gamma, \dots, m+n-\gamma$,
- filling with crosses the remaining boxes located in a new column,
- filling with crosses the remaining boxes located in a new row.

We say that $\ll_\gamma$ *inserts* Q at gap γ in P. This operation is illustrated in Figure 2 (right).

Consider now a word s on the alphabet $\{p, q\}$ with m letters p and n letters q. We call *p-blocks* (resp. *q-blocks*) the blocks of consecutive letters p (resp. q). We say that s is a P/Q-*shuffle* if all p-blocks appear at global descents of ω_Q while all q-blocks appear at global descents of ω_P. *E.g.* , for $\omega_P = 53412 = 1 \bullet 12 \bullet 12$ and $\omega_Q = 635421 = 1 \bullet 132 \bullet 1 \bullet 1$, then $qpqqqppqppq$ and $qppppqqqqpp$ are P/Q-shuffles, while $qqqppqqqppp$ is not.

For a P/Q-shuffle s with ℓ q-blocks, we denote by $P \star_s Q$ the pipe dream obtained by

- untangling Q at all gaps of ω_Q marked by p-blocks in s, resulting to $Q_1, \dots, Q_\ell$,
- inserting successively the pipe dreams $Q_1, \dots, Q_\ell$ in P at the positions of the q-blocks in s (it does not matter in which order we insert the pipe dreams $Q_1, \dots, Q_\ell$).

We say that $\star_s$ *tangles* the pipe dreams P and Q according to the P/Q-shuffle s. We define the product of $P \in \Pi_m$ and $Q \in \Pi_n$ by $P \cdot Q = \sum_s P \star_s Q$, where s ranges over all possible P/Q-shuffles. See Figure 4. Note that it always contains $P \ll_0 Q$ and $P \ll_m Q$ corresponding to the P/Q-shuffles $q^n p^m$ and $p^m q^n$.

Proposition 4 ([4, Props. 1.2.6 & 1.2.7]). *The product $\cdot$ defines an associative graded algebra structure on $\mathbf{k}\Pi$. The map $\omega\colon (\mathbf{k}\Pi, \cdot) \to (\mathbf{k}\mathfrak{S}, \sqcup\!\sqcup_\bullet)$ is a graded morphism of algebras.*

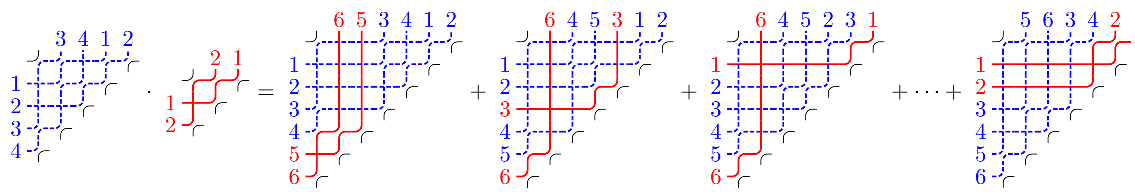


Figure 4: Product of two pipe dreams.

Hopf structure. We now describe the properties of the product $\cdot$ and coproduct $\triangle$.

Proposition 5 ([4, Props. 1.2.8 & 1.2.9]). *The product $\cdot$ and coproduct $\triangle$ endow the family Π of all pipe dreams with a graded Hopf algebra structure. The map $\omega\colon (\mathbf{k}\Pi, \cdot, \triangle) \to (\mathbf{k}\mathfrak{S}, \sqcup\!\sqcup_\bullet, \triangle_\bullet)$ is a morphism of graded Hopf algebras.*

A pipe dream $P \in \Pi_n$ is *$\ll_0$-decomposable* if it can be written as $P = Q \ll_0 R$ for some $Q \in \Pi_r$ and $R \in \Pi_{n-r}$ where $0 < r < n$. Otherwise, we say P is *$\ll_0$-indecomposable*. Define $\Lambda_n := \{P \in \Pi_n \mid P \text{ is } \ll_0\text{-indecomposable}\}$ and $\Lambda := \bigcup_{n \in \mathbb{N}} \Lambda_n$.

Theorem 6 ([4, Thms. 1.2.11 & 1.3.15]). *The algebra $\mathbf{k}\Pi$ is free with generators Λ. The dual Hopf algebra $\mathbf{k}\Pi^*$ is free with generators $\Lambda^* := \{P^* \mid P \in \Lambda\}$.*

2.3 Some relevant subalgebras

Recall from Section 2.1 that a permutation $\omega \in \mathfrak{S}$ has a unique factorization into atomic permutations $\omega = \nu_1 \bullet \nu_2 \bullet \cdots \bullet \nu_\ell$ and that we denote by $\omega^\bullet := \{\nu_1, \nu_2, \dots, \nu_\ell\}$ the set of atomic permutations that appear in its factorization. Given a subset S of atomic permutations, we define $\mathfrak{S}_n\langle S\rangle := \{\omega \in \mathfrak{S}_n \mid \omega^\bullet \subseteq S\}$ and $\mathfrak{S}\langle S\rangle := \bigsqcup_{n\in\mathbb{N}} \mathfrak{S}_n\langle S\rangle$, from which we derive $\Pi_n\langle S\rangle := \{P \in \Pi_n \mid \omega_P \in \mathfrak{S}_n\langle S\rangle\}$ and $\Pi\langle S\rangle := \bigsqcup_{n\in\mathbb{N}} \Pi_n\langle S\rangle$.

Theorem 7 ([4, Thm. 2.1.1]). *For any set S of atomic permutations,*

- *the subspace $\mathbf{k}\mathfrak{S}\langle S\rangle$ defines a Hopf subalgebra of $(\mathbf{k}\mathfrak{S}, \sqcup\!\sqcup_\bullet, \triangle_\bullet)$,*
- *the subspace $\mathbf{k}\Pi\langle S\rangle$ defines a Hopf subalgebra of $(\mathbf{k}\Pi, \cdot, \triangle)$.*

Theorem 8 ([4, Thm. 2.1.2]). *The Hopf subalgebra $\mathbf{k}\Pi\langle S\rangle$ is free and cofree. The generators and cogenerators of $\mathbf{k}\Pi\langle S\rangle$ are exactly the $\ll_0$-indecomposable pipe dreams in $\Pi\langle S\rangle$.*

In the following, we exploit Theorem 7 to construct relevant subalgebras of $(\mathbf{k}\Pi, \cdot, \triangle)$.

Hopf subalgebra $\mathbf{k}\Pi\langle 1\rangle$ and Loday–Ronco algebra. A pipe dream $P \in \Pi_n$ is reversing if it reverses the order of its relevant pipes. That is, $\omega_P = [n, n-1, \dots, 1] = 1 \bullet 1 \bullet \dots \bullet 1$, or equivalently $P \in \mathbf{k}\Pi\langle 1\rangle$. As observed in [17, 13, 15] reversing pipe dreams belong to the Catalan family. In particular, there is a bijection Ψ from pipe dreams in $\mathbf{k}\Pi_n\langle 1\rangle$ to complete binary trees with n internal nodes. Namely, given a reversing pipe dream P, replace each elbow of P by a node in the tree, and connect each node with the next node below it (if any) and with the next node to its right (if any). See Section 3.3 for illustrations.

Proposition 9 ([4, Prop. 2.1.3])**.** *The map Ψ is a Hopf algebra isomorphism between the Hopf subalgebra $\mathbf{k}\Pi\langle 1\rangle$ of reversing pipe dreams and the Loday–Ronco Hopf algebra on binary trees [12].*

Hopf subalgebra $\mathbf{k}\Pi\langle 1, 12, 123, \dots\rangle$ and lattice walks on the quarter plane. We now consider the subalgebra $\mathbf{k}\Pi\langle 1, 12, 123, \dots\rangle$ corresponding to permutations whose atoms are identity permutations of arbitrary size. Experimental computations show that the dimensions of the graded components of this Hopf subalgebra are given by the sequence $1, 1, 3, 12, 57, 301, 1707, 10191, 63244, 404503, 2650293, \dots$ which coincides with the sequence determined by the number of walks in a special family of walks in the quarter plane considered by M. Bousquet-Mélou and M. Mishna in [5].

Conjecture 10 ([4, Conj. 2.2.1])**.** *The dimension of $\mathbf{k}\Pi_n\langle 1, 12, 123, \dots\rangle$ is equal the number of walks in the quarter plane (within $\mathbb{N}^2 \subset \mathbb{Z}^2$) starting at $(0,0)$, ending on the horizontal axis, and consisting of $2n$ steps taken from $\{(-1,1), (1,-1), (0,1)\}$.*

Refining by the number of atomics gives the following stronger conjecture.

Conjecture 11 ([4, Conj. 2.2.3])**.** *The following two families have the same cardinality:*

1. *Pipe dreams P such that $\omega_P \in \mathfrak{S}_n$ is a permutation whose factorization into atomics consists of k identity permutations.*
2. *Walks in the quarter plane starting at $(0,0)$ and ending on the horizontal axis, consisting of $2n$ steps taken from $\{(-1,1), (1,-1), (0,1)\}$ from which k of them are $(0,1)$.*

Finally, this conjecture has a natural translation in the world of Dyck paths (see [4, Sects. 2.2.2 & 2.2.3] for details on this translation). A pair of Dyck paths (π_1, π_2) is said to be nested if π_1 is weakly below π_2. We say that a Dyck path π is

- bounce if it is of the form $N^{i_1}E^{i_1}N^{i_2}E^{i_2}\dots N^{i_k}E^{i_k}$ for some positive integers $i_1, \dots, i_k$,
- steep if it contains no consecutive east steps EE, except at the end on top of the grid.

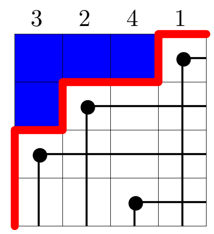
Conjecture 12 ([4, Conj. 2.2.8])**.** *For any $k \leq n$, there is a bijection between:*

1. *Nested pairs of Dyck paths (π_1, π_2) of size n such that π_1 is a bounce path with k parts.*
2. *Nested pairs of Dyck paths (π'_1, π'_2) of size n such that π'_2 is a steep path ending with k east steps on top of the grid.*

Evidences for this conjecture, and a discussion on its connection to the zeta map can be found in [4, Sect. 2.2.4].

3 Dominant pipe dreams, ν-Tamari lattices, and multivariate diagonal harmonics

3.1 Dominant permutations and dominant pipe dreams



Recall that the <u>*Rothe diagram*</u> of a permutation $\omega \in \mathfrak{S}_n$ is the set $R_\omega := \big\{ \big(\omega(i), j\big) \;\big|\; i > j \text{ and } \omega(i) < \omega(j) \big\}$. If we represent this diagram in matrix notation (*i.e.* the box (i, j) appears in row i and column j), then the Rothe diagram of ω is the set of boxes which are not weakly below or weakly to the right of a box $\big(\omega(i), i\big)$ for all $i \in [n]$. A permutation $\omega \in \mathfrak{S}_n$ is <u>*dominant*</u> if its Rothe diagram R_ω is a partition containing the top-left corner. Such a permutation is uniquely determined by its Rothe diagram R_ω, or equivalently by the Dyck path π_ω delimiting the boundary of its Rothe diagram. Define $\mathfrak{S}_n^{\mathrm{dom}} := \{\omega \in \mathfrak{S}_n \mid \omega \text{ is dominant}\}$ and $\mathfrak{S}^{\mathrm{dom}} := \bigsqcup_{n \in \mathbb{N}} \mathfrak{S}_n^{\mathrm{dom}}$.

Proposition 13 ([4, Coro. 3.1.2])**.** *The subspace $\mathbf{k}\mathfrak{S}^{\mathrm{dom}}$ is a Hopf subalgebra of $(\mathbf{k}\mathfrak{S}, \sqcup_\bullet, \triangle_\bullet)$. The dimension of the homogeneous component $\mathbf{k}\mathfrak{S}_n^{\mathrm{dom}}$ is the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.*

Finally, we pull back this Hopf subalgebra of dominant permutations of $\mathbf{k}\mathfrak{S}$ to a Hopf subalgebra of $\mathbf{k}\Pi$ via the Hopf morphism $\omega : \mathbf{k}\Pi \to \mathbf{k}\mathfrak{S}$. A <u>*dominant pipe dream*</u> is a pipe dream P whose permutation ω_P is dominant. We denote the set of dominant pipe dreams by $\Pi_n^{\mathrm{dom}} := \big\{P \in \Pi_n \;\big|\; \omega_P \in \mathfrak{S}_n^{\mathrm{dom}}\big\}$ and $\Pi^{\mathrm{dom}} := \bigsqcup_{n \in \mathbb{N}} \Pi_n^{\mathrm{dom}}$.

Corollary 14 ([4, Coro. 3.1.4])**.** *The subspace $\mathbf{k}\Pi^{\mathrm{dom}}$ defines a Hopf subalgebra of $(\mathbf{k}\Pi, \cdot, \triangle)$.*

3.2 ν-trees

We now consider the following family of combinatorial objects defined by C. Ceballos, A. Padrol and C. Sarmiento in [7]. In the following, we consider a Dyck path drawn on the semi-integer lattice $(1/2, 1/2) + \mathbb{Z}^2$ and points on the lattice $\mathbb{Z}^2$.

Definition 15 ([7])**.** *Let ν be a Dyck path of size n drawn on the semi-integer lattice.*

1. *Two lattice points p, q inside the $n \times n$ grid and weakly above ν are said <u>ν-incompatible</u> if p is located strictly southwest or northeast to q, and the smallest rectangle containing p and q lies above ν. Otherwise, p and q are called <u>ν-compatible</u>.*
2. *A <u>ν-tree</u> is a maximal collection of pairwise ν-compatible lattice points, called <u>nodes</u>.*
3. *Two ν-trees are related by a <u>rotation</u> if they differ by only two nodes.*

We let $\Theta(\nu)$ the set of ν-trees and we let $\Theta_n := \bigsqcup_\nu \Theta(\nu)$ and $\Theta := \bigsqcup_{n \in \mathbb{N}} \Theta_n$.

A ν-tree T can be viewed as a tree in the graph-theoretical sense by connecting each node p of T with the next node of T below it (if any), and with the next node of T to its right (if any). For example, we obtain classical binary trees when $\nu = (NE)^n$ is the staircase Dyck path. See Section 2.3. We consider the graded space $\mathbf{k}\Theta = \bigoplus_{n \geq 0} \mathbf{k}\Theta_n$, where $\mathbf{k}\Theta_n$ is the $\mathbf{k}$-span of Θ_n and define a product and coproduct similar to Section 2.2.

Horizontal and vertical packing. A leaf of a ν-tree T is called a <u>*diagonal leaf*</u> if it belongs to the main diagonal of the $n \times n$ grid. A diagonal leaf b in T divides the path ν into two paths ν_ℓ (on the left) and ν_r (on the right). Cutting the tree T along the path from b to its root gives rise to two trees $\tilde{T}_\ell$ (on the left) and $\tilde{T}_r$ (on the right). We define the <u>*vertical packing*</u> $\lrcorner_b(T)$ as the ν_ℓ-tree obtained by contracting all vertical segments of $\tilde{T}_\ell$ that are above b. Similarly, the <u>*horizontal packing*</u> $\urcorner_b(T)$ is the ν_r-tree obtained by contracting all horizontal segments of $\tilde{T}_r$ that are on the left of b.

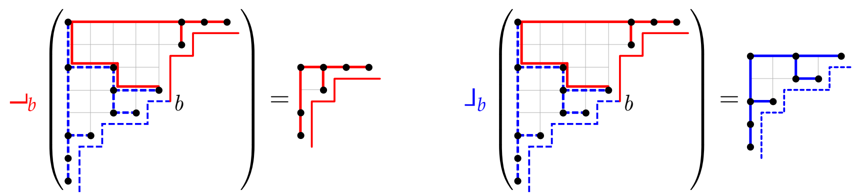


Figure 5: The horizontal and vertical packings of a ν-tree at a diagonal leaf b.

Coproduct. We define the coproduct of a ν-tree T as $\triangle(T) = \sum \lrcorner_b(T) \otimes \urcorner_b(T)$, where the sum runs over all diagonal leaves b of T. See Figure 6.

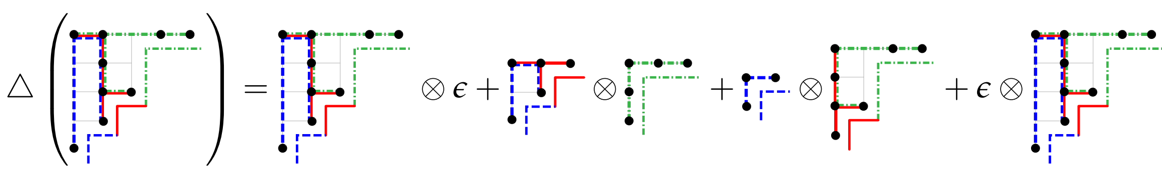


Figure 6: The coproduct of a ν-tree.

Product. Let $\mathbf{b} = (b_1, \ldots, b_{\ell-1})$ be a tuple of $\ell - 1$ diagonal leaves of a ν-tree T which are located in order along the main diagonal with possible repetitions. They partition ν into ℓ Dyck paths $\nu_1, \ldots, \nu_\ell$. The tree T is subdivided into ℓ trees $\tilde{T}_1, \ldots, \tilde{T}_\ell$ by cutting along the paths from the leaves b_i to the root. Define T_i to be the ν_i-tree obtained by contracting segments of $\tilde{T}_i$ that are either horizontal on the left of b_{i-1} or vertical above b_i. By convention, b_0 and b_ℓ denote two extra leaves at coordinates $(0,0)$ and (n,n) respectively.

Given a μ-tree S and a ν-tree T we will define the product $S \cdot T$ as follows. If S has ℓ diagonal leaves, we choose a tuple $\mathbf{b} = (b_1, \ldots, b_{\ell-1})$ of $\ell - 1$ leaves of T and we "cut" T along $\mathbf{b}$ to produce ℓ trees $T_1, \ldots, T_\ell$ as described above. We then "glue" these trees $T_1, \ldots, T_\ell$ on the ℓ diagonal leaves of S. The resulting tree $S \star_{\mathbf{b}} T$ is a λ-tree for some Dyck path λ obtained as a shuffle of μ and ν with cuts at diagonal leaves.

If S has ℓ diagonal leaves, we define the product of S and T by $S \cdot T = \sum_{\mathbf{b}} S \star_{\mathbf{b}} T$, where the sum ranges over all ordered tuples $\mathbf{b} = (b_1, \ldots, b_{\ell-1})$ of $\ell - 1$ diagonal leaves in T with possible repetitions. An example of this product is illustrated in Figure 7.

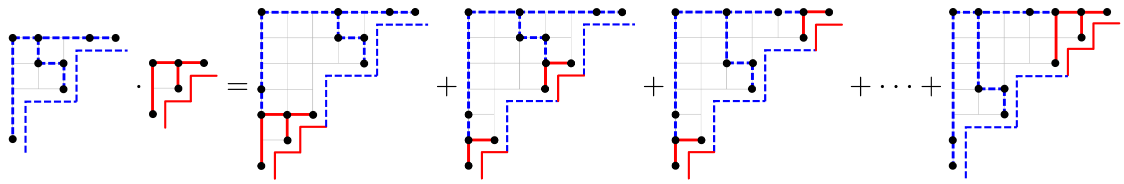
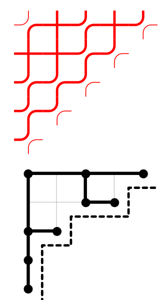


Figure 7: The product of a μ-tree and a ν-tree.

3.3 Dominant pipe dreams versus ν-trees.



We now connect dominant pipe dreams with ν-trees and show that the Hopf algebras considered in the previous two sections are isomorphic. As illustrated on the right, we consider the map Ψ that sends a pipe dream $P \in \Pi(\omega)$ with dominant permutation ω to a ν-tree T where $\nu = \pi_\omega$ is the Dyck path associated to ω. This ν-tree is defined as the set of lattice points $\Psi(P)$ given by the elbows of P located in the topmost row or leftmost column, or inside the Rothe diagram of ω_P.

Proposition 16 ([15, 7]). *For any dominant permutation ω with corresponding Dyck path $\nu = \pi_\omega$, the map Ψ is a bijection between the dominant pipe dreams in $\Pi(\omega)$ and the ν-trees of $\Theta(\nu)$.*

Theorem 17 ([4, Coro. 3.1.9]). *The map Ψ is a Hopf algebra isomorphism between the Hopf algebra $(\mathbf{k}\Pi^{\mathrm{dom}}, \cdot, \triangle)$ of dominant pipe dreams and the Hopf algebra $(\mathbf{k}\Theta, \cdot, \triangle)$ of ν-trees.*

3.4 Hopf chains and multivariate diagonal harmonics

We conclude this extended abstract with a surprising connection to the multivariate diagonal harmonic spaces [1]. Let $X = [x_{ij}]$ be a set of nr variables for $i \in [r]$ and $j \in [n]$. We refer to r as the number of sets of variables and to n as the number of variables in each of the sets. The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{C}[X]$ by permuting the n columns of the matrix X. Let $\mathrm{Sym}_{n,r}$ be the subring of polynomials $f(X)$ invariant under the action of $\mathfrak{S}_n$. The *multivariate diagonal harmonic space* $\mathrm{DH}_{n,r}$ is the space

$$\mathrm{DH}_{n,r} = \big\{ p \in \mathbb{C}[X] \mid f(\partial)p = 0 \text{ for all } f \in \mathrm{Sym}_{n,r} \text{ such that } f(\mathbf{0}) = 0 \big\},$$

where $f(\partial)$ is the partial differential operator obtained by replacing the variables x_{ij} by $\partial/\partial x_{ij}$ in $f(X)$. This space is closed under the action of the symmetric group $\mathfrak{S}_n$, and thus defines a representation of $\mathfrak{S}_n$. For small fixed values of r, the dimension of $\mathrm{DH}_{n,r}$ and the multiplicity of the sign representation $\mathrm{Alt}(\mathrm{DH}_{n,r})$ satisfy beautiful formulas:

r	$r=1$	$r=2$	$r=3$ (conj.)	$r \geq 4$
$\dim\big(\mathrm{DH}_{n,r}\big)$	$n!$	$(n+1)^{n-1}$	$2^n(n+1)^{n-2}$	?
$\dim\big(\mathrm{Alt}(\mathrm{DH}_{n,r})\big)$	1	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	?

When $r = 2$ (resp. $r = 3$), the dimension of $\mathrm{Alt}(\mathrm{DH}_{n,r})$ is the number of elements (resp. intervals) in the Tamari lattice. The dimension of $\mathrm{DH}_{n,r}$ can be interpreted by labeled versions of the latter. In fact, the spaces $\mathrm{DH}_{n,r}$ and $\mathrm{Alt}(\mathrm{DH}_{n,r})$ can be further decomposed into homogeneous components invariant under the action of the symmetric group. Let the degree of a monomial $X^A = \prod x_{ij}^{a_{ij}}$ be $\deg_r(X^A) = \left(\sum_{j=1}^n a_{1j}, \ldots, \sum_{j=1}^n a_{rj}\right)$. The subspaces of $\mathrm{DH}_{n,r}$ and $\mathrm{Alt}(\mathrm{DH}_{n,r})$ of fixed degree are clearly invariant under the action of $\mathfrak{S}_n$. The weighted sum of the dimensions of these subspaces for $\mathrm{DH}_{n,r}$ and $\mathrm{Alt}(\mathrm{DH}_{n,r})$ give two polynomials in r variables $q_1, \ldots, q_r$, whose evaluations at $q_i = 1$ for all i recover their total dimensions. For $r = 2$, an expression of the resulting q,t-polynomial for $\mathrm{DH}_{n,r}$ in terms of Macdonald polynomials was conjectured by A. Garsia and M. Haiman [8] and proved by M. Haiman in [10]. A combinatorial description of this polynomial, involving a pair of statistics area and dinv on parking functions, is described by the former shuffle conjecture of J. Haglund et al. [9] which was recently proved by E. Carlsson and A. Mellit [6]. For $r = 3$, there is no known triple of statistics on labeled and unlabeled intervals in the Tamari lattice that would match the tri-degree of $\mathrm{DH}_{n,3}$ and $\mathrm{Alt}(\mathrm{DH}_{n,3})$. F. Bergeron and L.-F. Préville-Ratelle conjectured in [2] that length of a longest chain in the interval and dinv are two of the statistics. Using combinatorial objects arising from the pipe dream algebra, we now provide an interpretation of the dimensions of $\mathrm{DH}_{n,r}$ and $\mathrm{Alt}(\mathrm{DH}_{n,r})$ and their q-Frobenius characteristic for any $n \leq 4$. The details on the definitions and the proof can be found in [4, Sect. 3.2.2].

Definition 18. *A Hopf chain is a nested tuple $(\pi_1, \pi_2, \ldots, \pi_r)$ of Dyck paths of size n such that:*

1. *π_1 is the bottom diagonal path $(NE)^n$,*
2. *for every $1 \leq i < j < k \leq r$, the subtriple (π_i, π_j, π_k) comes from an interval of dominant pipe dreams. In other words, the pair (π_j, π_k) is an interval in the π_i-Tamari lattice of [14].*

We denote by $\mathrm{HC}_{n,r}$ the set of Hopf chains of length r and size n.

Definition 19. *The collar $\mathrm{col}(\boldsymbol{\pi})$ of a Hopf chain $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_r)$ is one plus the maximal number of distinct Dyck paths that can be inserted in $\boldsymbol{\pi}$ strictly between π_{r-1} and π_r such that the result is a Hopf chain.*

Theorem 20 ([4, Thm. 3.2.5])**.** *For degree $n \leq 4$ and **any number r of sets of variables**, the following two symmetric functions coincide:*

- *the q-Frobenius characteristic $\Phi_{n,r}(q)$ of $\mathrm{DH}_{n,r}$ expanded in the elementary basis, and*
- *the sum over Hopf chains of $\mathrm{HC}_{n,r}$ given by*

$$\Psi_{n,r}(q) := \sum_{\substack{\boldsymbol{\pi}=(\pi_1,\pi_2,\ldots,\pi_r) \\ \text{Hopf chain of } \mathrm{HC}_{n,r}}} q^{\mathrm{col}(\boldsymbol{\pi})} e_{\mathrm{type}(\pi_r)},$$

where $\mathrm{type}(\pi_r)$ is the partition of the connected N steps lengths in π_r.

Corollary 21 ([4, Coro. 3.2.6])**.** *For $n \leq 4$ and any $r \in \mathbb{N}$, the dimension of $\mathrm{Alt}(\mathrm{DH}_{n,r})$ (resp. of $\mathrm{DH}_{n,r}$) is the number of Hopf chains (resp. labelled Hopf chains) of length r and size n.*

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