Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #20, 12 pp.

Proceedings of the 31^{st} Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Refining the bijections among ascent sequences, (2+2)-free posets, integer matrices and pattern-avoiding permutations

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Abstract. The combined work of Bousquet-Mélou, Claesson, Dukes, Jelínek, Kitaev, Kubitzke and Parviainen has resulted in non-trivial bijections among ascent sequences, (2+2)-free posets, upper-triangular integer matrices, and pattern-avoiding permutations. To probe the finer behavior of these bijections, we study two types of restrictions on ascent sequences. These restrictions are motivated by our results that their images under the bijections are natural and combinatorially significant. In addition, for one restriction, we are able to determine the effect of poset duality on the corresponding ascent sequences, matrices and permutations, thereby answering a question of the first author and Parviainen in this case. The second restriction should appeal to Catalaniacs.

Keywords: ascent sequence, (2+2)-free poset, interval order, upper-diagonal matrix, pattern avoidance, series-parallel poset

1 Introduction

In the last decade, an interesting collection of results has emerged from the study of (2+2)-free posets, or interval orders as they are also known, and their connection to permutations avoiding a non-standard permutation pattern of length three. The starting point for this story was the introduction in Bousquet-Mélou et al. [3] of a new type of permutation pattern that the authors termed a *bivincular pattern*. In that paper it was proven that length-n permutations avoiding the bivincular pattern $2|3\overline{1}$ were in one-to-one correspondence with unlabelled (2+2)-free posets on n elements. This was shown by encoding both structures as an integer sequence of length n that has come to be known as an *ascent sequence*. Via ascent sequences, Bousquet-Mélou et al. were able to solve the long-standing open problem of enumerating unlabelled (2+2)-free posets.

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In [10], these ascent sequences were shown to uniquely encode another set of objects: all square upper-triangular matrices of non-negative integers whose entries sum to n which have neither rows nor columns consisting of only zeros.

These initial two papers linking (bijectively) four different discrete objects led to a series of papers that studied these bijections and built upon the correspondences. From the enumerative viewpoint, Dukes et al. [9] considered these objects according to several statistics and presented multivariate generating functions for these statistics. Two additional papers [5, 8] have studied analogues of these bijections on *labeled* posets.

In another direction, and more recently, it has emerged that refinements of these correspondences have equally compelling stories to tell. Duncan & Steingrímsson [11] studied pattern avoidance in ascent sequences and established bijections between pattern avoiding ascent sequences and other combinatorial objects such as set partitions and objects enumerated by the Catalan and Narayana numbers. Jelínek [13] presented a new method to derive formulas for the generating functions of interval orders. The method generalised the results of [9] and also allowed the enumeration of self-dual interval orders with respect to several statistics. Using his newly derived generating function formulas, Jelínek proved a bijective relationship between self-dual interval orders and upper-triangular matrices having no zero rows [13]. Andrews & Jelínek [1] built on Jelínek's work and proved several power series identities involving the refined generating functions for interval orders and self-dual interval orders. Keller and Young [15] considered the difficult question of determining which ascent sequences map to semiorders; also known as unit interval orders, semiorders are posets that are both (2+2) and (3+1)-free. Most recently, [4] uses a slightly different bijection to resolve conjectures of Jelínek about the equistribution of certain statistics in $2|3\bar{1}$-avoiding permutations and the class of matrices described above.

This extended abstract summarizes results in our recent paper [7]. It adds to this body of work by analyzing two types of restrictions on ascent sequences. One motivation for these restrictions is that their images through the bijections of [3, 8, 10] are combinatorially significant in the subsets they identify, e.g. series-parallel posets and 231-avoiding permutations. Moreover, the analysis of the images of these ascent sequences allows us to prove results about duals of each of the structures, thus going some way in answering an open problem of Dukes & Parviainen [10]. In our restricted settings, the bijections of [3, 8, 10] coincide with those of [4].

The first type of restriction we study (in Section 3) begins with a restriction on the types of ascents one may have in an ascent sequence. In particular, when the bijection of [3] recursively builds a (2+2)-free poset from an ascent sequence, there are some ascents that cause complicated and unnatural modifications to the poset, while the bijection treats all other ascents in a very natural way. Our first restriction is to those ascent sequences that contain only these ascents that result in this latter natural behaviour. A motivation for this restriction is that this good behaviour carries through to the general framework of bijections. Indeed, the images of these new *restricted ascent sequences* RAsc through the bijections given in [3, 8, 10] are proven to be simple restrictions: the subset RMatrices of the matrices from [10] having only positive diagonal entries, the subset RPosets of (2+2)-free posets which have a chain of the maximal possible length, and the set RPerms of permutations avoiding the barred pattern $3\bar{1}52\bar{4}$. This set RPerms was already identified in [3] in the context of modified ascent sequences. See Figure 1 for a diagram outlining our sets and maps of interest.

In Section 4 we give a partial solution to an open problem of Dukes & Parviainen [10] by addressing the topic of structural duality. The dual P^* of a (2+2)-free poset P is also a (2+2)-free poset. This observation prompts the question as to whether one can derive the (ascent sequence/matrix/permutation) corresponding to P^* from the (ascent sequence/matrix/permutation) corresponding to P. In the matrix case, [8] provides the answer; see Observation 4.2 below. In the ascent sequence and permutation cases, this question seems intractable in general because of the complicated map between some ascent sequences and posets as mentioned in the previous paragraph. However, consistent with our motivation for restricting to better behaved sets, we can answer this duality question completely for all posets in RPosets, which we do in Section 4.

In Section 5, we consider the Catalan family CAsc of 101-avoiding ascent sequences studied in [11], and investigate their images under the bijections of [3, 8, 10]. The results are perhaps even nicer than the R-families and are shown in Figure 1. The posets that arise are the series-parallel interval orders, i.e., those that are both (2+2)-free and N-free. This class of posets appears in [6], while series-parallel posets in general are widespread in the literature, partially because their recursive structure permits many polynomial-time algorithms (see, for example, [12] and the references therein). The matrices and permutations which correspond to these ascent sequences are those matrices from RMatrices that are termed *SE-free* in [14], and 231-avoiding permutations, respectively.

2 Preliminaries

While the intimate connections between four different types of objects are certainly a strength of this area of study, the drawback for our present purposes is that there is a considerable amount of background that needs to be introduced, including all four classical sets and many of the bijections among them. Our use of the words "classical" refers to the full sets considered by most of the papers mentioned in the Introduction, and as shown by the largest boxes in Figure 1: ascent sequences, (2+2)-free posets, upper-triangular matrices with non-negative integer entries having neither rows nor columns of all zeros, and permutations avoiding $2|3\bar{1}$.

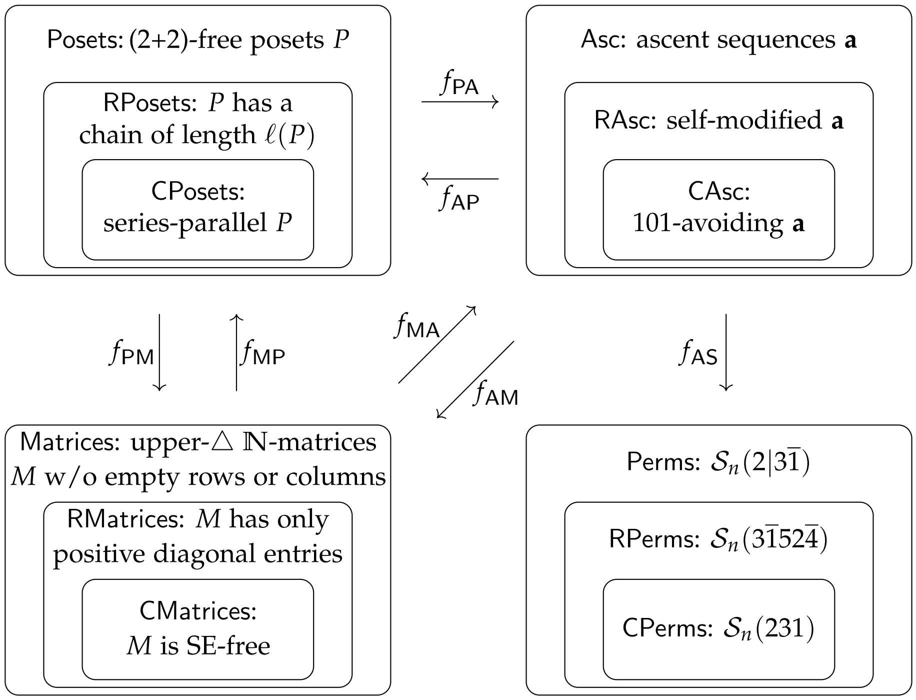

Figure 1: A diagrammatic summary of the sets and bijections of interest.

2.1 The classical sets

An *ascent sequence* is a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of non-negative integers such that $a_1 = 0$, and for all i with $1 < i \leq n$ we have $a_i \leq \mathsf{asc}(a_1, \ldots, a_{i-1}) + 1$, where $\mathsf{asc}(a_1, \ldots, a_k)$ denotes the number of ascents in the sequence $(a_1, \ldots, a_k)$. For example $(0, 1, 0, 1, 3)$ is an ascent sequence whereas $(0, 1, 0, 2, 4)$ is not. Let Asc_n be the set of all ascent sequences of length n, and let Asc denote the union of these sets over all n, with the same convention applying to all the notation below when the subscript n is dropped.

Let Posets_n be the set of (2+2)-free posets on n elements, meaning posets that have no induced subposet isomorphic to a disjoint union of two 2-element chains. We will be interested in a different defining property of (2+2)-free posets, as we now describe. Let $P = (P, \preceq)$ be a poset with n elements. Given $x \in P$, the set $D(x) = \{y \in P : y \prec_P x\}$ is called the *strict downset* of x. A fact described as "well-known" in [2] and which is easy to check is that a poset is (2+2)-free if and only if the set of strict downsets of elements of P can be linearly ordered by inclusion. We let $\ell(P)$ denote the number of distinct nonempty such downsets, so that $D(P) = (D_0, \ldots, D_{\ell(P)})$ is the sequence of downsets of P linearly ordered by inclusion. In other words $\emptyset = D_0 \subsetneq D_1 \subsetneq \ldots \subsetneq D_{\ell(P)}$. For example, for the poset P in the top left of Figure 3, we have

$$D(P) = (\emptyset, \{p_1, p_2\}, \{p_1, p_2, p_5\}, \{p_1, p_2, p_3, p_5\}).$$

(Note that while this example P is labelled for the purposes of the explanation, the elements of Posets are unlabelled.) We will call D_i *level* i of P, and an element $x \in P$ with $D(x) = D_i$ for some i will be said to *lie at level* i of P. Let $L_i = L_i(P)$ be the set of elements lying at level i of P and set $L(P) = (L_0, \ldots, L_{\ell(P)})$. Again for P from Figure 3,

$$L(P) = (\{p_1, p_2, p_5\}, \{p_6\}, \{p_3\}, \{p_4\}).$$

Let $\mathsf{Matrices}_n$ be the set of all upper-triangular matrices of indeterminate dimension whose entries are all non-negative integers such that there is neither a row nor a column containing only zeros, and whose sum of all entries is n. Throughout, the entries of the matrix M will be denoted m_{ij} or $m_{i,j}$ but we use M_{ij} when the entries of the matrix are sets.

A sequence $\mathbf{a} = (a_1, \ldots, a_r)$ of non-negative integers is said to *contain* a sequence $\mathbf{b} = (b_1, \ldots, b_s)$ *as a pattern* if there exists a subsequence of $\mathbf{a}$ of length s whose elements are in the same relative order as those of $\mathbf{b}$. We say $\mathbf{a}$ is $\mathbf{b}$-*avoiding* is it does not contain $\mathbf{b}$. For example, $(0, 2, 1, 3, 1, 0, 2)$, which we write as 0213102 for short, contains the pattern 0101 because of its subsequence 0202, but avoids the pattern 1010.

When considering pattern-avoidance in sequences that are permutations, we allow for a more general notion of pattern: a permutation $\pi = \pi_1 \ldots \pi_n$ is said to *contain* the pattern $2|3\overline{1}$ if there exists an occurrence $\pi_i \pi_j \pi_k$ of 231 in π with the additional conditions that $j = i + 1$ and $\pi_i = \pi_k + 1$. For example, 32541 contains $2|3\overline{1}$ because of the occurrence 251 of 231, whereas 31452 avoids $2|3\overline{1}$ even though it has three occurrences of the classical pattern 231. The pattern $2|3\overline{1}$ is an example of a *bivincular pattern* as introduced in [3] since it puts conditions on both the entries and positions of an occurrence. As usual, we let $\mathcal{S}_n(2|3\overline{1})$ denote the set of permutations of length n that avoid $2|3\overline{1}$, and this is exactly our set Perms_n.

2.2 The bijections

The results we present concern a variety of combinatorial objects and several bijections between them. These bijections are the 'classical' bijections that were presented in the papers [3, 8, 10]. Given the size limitation for this extended abstract, we will present just one of these bijections here as a way to give a sense of their non-triviality. An exposition of the full set of bijections can be found in the companion paper [7, Sec. 2.2] where it requires five pages. We will introduce the following convention: to denote the bijections from [3, 8, 10], we will use labels according to their domain and codomain, but rather than use the labels Asc, Posets, Matrices and Perms, we use the single-letter subscripts A, P, M and S ("S" for "symmetric group"). For example, f_{AP} denotes the bijection of [3] from ascent sequences to (2+2)-free posets, which is the bijection we choose to present in full. An example of all the bijections is given in Figure 3.

We use the notation $[a, b]$, if $a < b$ are integers, for the set $\{a, \dots, b\}$ and $[a, b)$ for the set $\{a, \dots, b-1\}$, etc. Given a (2+2)-free poset P, recall that $\ell(P)$ denotes the highest index of a level. Let $\ell^\star(P)$ denote the minimum index of a level that contains a maximal element.

Definition 2.1. Given $\mathbf{a} = (a_1, \dots, a_n) \in \mathsf{Asc}_n$, we define $f_{\mathsf{AP}}(\mathbf{a})$ recursively. Let $\mathbf{a}^{(k)} := (a_1, \dots, a_k)$. First, $f_{\mathsf{AP}}(\mathbf{a}^{(1)})$ is the poset consisting of a single element p_1. Supposing $P^{(k)} = f_{\mathsf{AP}}(\mathbf{a}^{(k)})$ for some $k \in [1, n)$, we have the following three cases for defining $P^{(k+1)} = f_{\mathsf{AP}}(\mathbf{a}^{(k+1)})$.

AP1 If $a_{k+1} \in [0, \ell^\star(P^{(k)})]$ then let $P^{(k+1)}$ be the result of adding to $P^{(k)}$ a new maximal element p_{k+1} that covers the same elements as do the elements in $L_{a_{k+1}}(P^{(k)})$.

AP2 If $a_{k+1} = 1 + \ell(P^{(k)})$ then let $P^{(k+1)}$ be the result of adding to $P^{(k)}$ a new element p_{k+1} covering all maximal elements of $P^{(k)}$.

AP3 If $a_{k+1} \in (\ell^\star(P^{(k)}), \ell(P^{(k)})]$ then let $P^{(k+1)}$ be the outcome of the following: to $P^{(k)}$, add a new element p_{k+1} covering the same elements as the elements in $L_{a_{k+1}}(P^{(k)})$. Let $\mathcal{M}$ be the set of maximal elements of $P^{(k)}$ lying at any level less than a_{k+1}. Add all relations $x \preceq y$ where $x \in \mathcal{M}$ and y is any element of $L_{a_{k+1}}(P^{(k)}) \cup \cdots \cup L_{\ell(P^{(k)})}(P^{(k)})$; here we do not consider the new element p_{k+1} to be an element of $L_{a_{k+1}}(P^{(k)})$.

Then $f_{\mathsf{AP}}(\mathbf{a}) = f_{\mathsf{AP}}(\mathbf{a}^{(n)}) = P^{(n)}$.

Example 2.2. Let $\mathbf{a} = (0, 0, 1, 2, 0, 1)$ as in Figure 3. Certainly, $P^{(1)}$ is the one-element poset. The recursive construction of $P = f_{\mathsf{AP}}(\mathbf{a})$ appears in Figure 2, where the dotted shapes depict the different levels. The element a_{k+1} appears above each arrow and the case name appears below each arrow. The labels p_i are just for expository purposes and are not part of $f_{\mathsf{AP}}(\mathbf{a})$. In the final step, $\mathcal{M} = \{p_5\}$. Note that the new element p_6 in the final step ends up on its own level, and this is true in general for applications of AP3.

3 Restricted sets

In this section we introduce and study a subset of ascent sequences that we term *restricted ascent sequences*.

Definition 3.1. Let RAsc_n be the subset of Asc consisting of those ascent sequences $\mathbf{a} = (a_1, \dots, a_n)$ such that

$$a_k \in [0, a_{k-1}] \cup \{1 + \mathsf{asc}(a_1, \dots, a_{k-1})\} \quad \text{for all } k > 1. \tag{3.1}$$

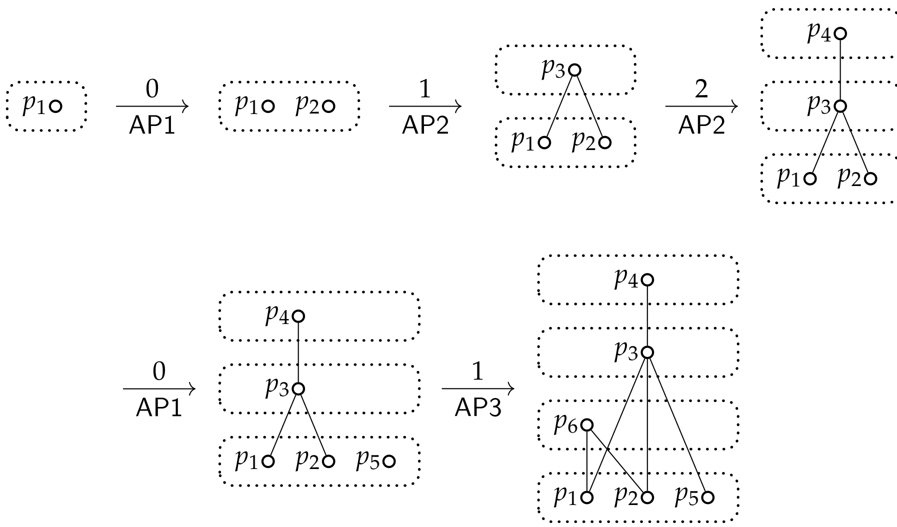

Figure 2: The recursive construction of Example 2.2.

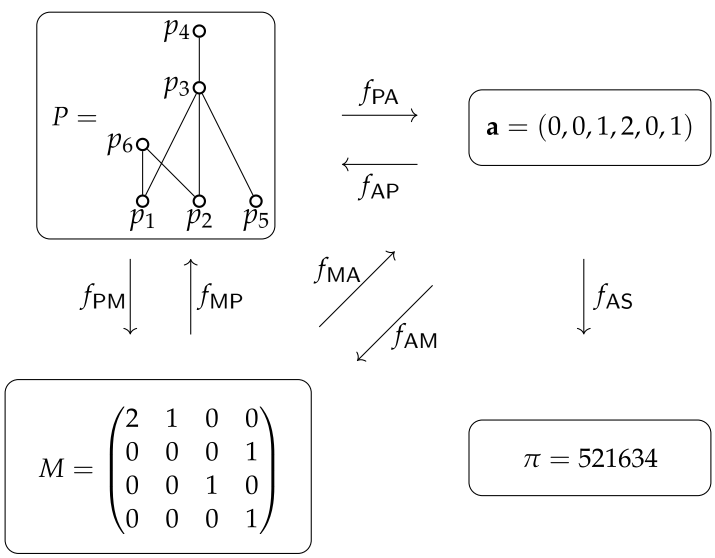

Figure 3: An example of the bijections.

In other words, if a_k is larger than a_{k-1}, then a_k must be the largest it can be under the conditions on an ascent sequence. For example, $(0,1,0,2)$ is in RAsc whereas $(0,1,0,1)$ is not.

Corollary 3.7 is the main result of this section. It states that, under the classical bijections of [3, 8, 10], RAsc_n maps to the sets we now define.

Definition 3.2.

- Let $\mathsf{RMatrices}_n$ be the set of matrices in $\mathsf{Matrices}_n$ all of whose diagonal entries are positive.
- Let $\mathsf{RPosets}_n$ be the set of those posets in Posets_n that have a chain of length $\ell(P)$.
- Let $\mathsf{RPerms}_n = \mathcal{S}_n(3\bar{1}52\bar{4})$, the set of length n permutations π such that every occurrence of the pattern 231 in π plays the role of 352 in an occurrence of the pattern 31524 in π.

The following observation is crucial for several results stated in this abstract. It not only gives the image of an element of RMatrices under f_{MA} but, combined with Corollary 3.7(a), shows that all elements of RAsc take a particular form. We use i^j to denote a sequence of j copies of i.

Lemma 3.3. *Suppose that $M \in \mathsf{RMatrices}$ with $\dim(M) = d$. Then*

$$f_{\mathsf{MA}}(M) = (0^{m_{11}}, 1^{m_{22}}, 0^{m_{12}}, 2^{m_{33}}, 1^{m_{23}}, 0^{m_{13}}, \ldots, (d-1)^{m_{dd}}, (d-2)^{m_{d-1\,d}}, \ldots, 0^{m_{1d}}). \quad (3.2)$$

Example 3.4. $M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is in bijection with the ascent sequence $\mathbf{a} = (0,0,1,2,0)$.

The essential condition for the desired bijection between RAsc_n and $\mathsf{RMatrices}_n$ is as follows:

Proposition 3.5. *Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathsf{Asc}_n$ and $M = f_{\mathsf{AM}}(\mathbf{a})$ with $\dim(M) = d$. There will be a zero on the diagonal of M if and only if there exists $i \in [1, n-1]$ such that $a_i < a_{i+1} \leq \mathsf{asc}(a_1, \ldots, a_i)$.*

We next provide a similarly essential ingredient for the desired bijection between RPosets and RMatrices.

Proposition 3.6. *Let $P \in \mathsf{Posets}_n$ and let $M = f_{\mathsf{PM}}(P)$ with $dim(M) = \ell(P) + 1$. All the entries on the diagonal of M will be non-zero if and only if P has a chain of length $\ell(P)$.*

Combining the previous two propositions gives the main result of this section, which shows that RAsc, RMatrices, RPosets and RPerms are all in bijection, and the bijections we need are exactly the restricted versions of the classical ones. Part (c) of the theorem was already proved as [3, Prop. 10]

Corollary 3.7. *The classical bijections among* Asc_n*,* $\mathsf{Matrices}_n$*,* Posets_n *and* Perms_n *restrict to bijections among* RAsc_n*,* $\mathsf{RMatrices}_n$*,* $\mathsf{RPosets}_n$ *and* RPerms_n*, Specifically,*

(a) f_{AM} *maps* RAsc_n *bijectively to* $\mathsf{RMatrices}_n$*;*

(b) f_{PM} *maps* $\mathsf{RPosets}_n$ *bijectively to* $\mathsf{RMatrices}_n$*;*

(c) *[3]* f_{AS} *maps* RAsc_n *bijectively to* RPerms_n*.*

4 Poset duality under the bijections

If a poset P is (2+2)-free, then it is clear that the dual poset P^* obtained by reversing all its inequalities is also (2+2)-free. An open question in [10] asks how $\mathbf{a}$ is related to $\mathbf{a}^*$, where $\mathbf{a}$ and $\mathbf{a}^*$ are the ascent sequences corresponding to P and P^* respectively. While this question appears intractable for general (2+2)-free posets, in this section we answer it for RPosets. In addition, we extend the answer to give the corresponding notion of duality for RPerms. Combined with the duality result for RMatrices given by [8, Theorem 10], we get a complete understanding of how poset duality acts on our four R-families according to our bijections. In fact, one major motivation for our restriction to the R-families is their amenability to adopting an analogue of poset duality.

In view of Corollary 3.7, we can abuse notation by using the same f notation for our bijections even though our domains will now be R-families as opposed to the domains of Asc_n, $\mathsf{Matrices}_n$, Posets_n and Perms_n that we had before.

Definition 4.1. Let $f : \mathsf{RPosets}_n \to \mathsf{Struct}_n$ be a bijection where Struct_n is a collection of objects. Given $P \in \mathsf{RPosets}$ with $f(P) = s$, we write s^* for the unique object $f(P^*)$, and we call s^* the *dual* of s according to f.

Observation 4.2. As a first example, we consider the dual of an element M of RMatrices according to the bijection f_{PM}. Define $\mathsf{flip}(M)$ to be the reflection of M through its antidiagonal, i.e., if $\dim(M) = d$, then $\mathsf{flip}(M)_{ij} = m_{d+1-j,d+1-i}$. Observe that $M \in \mathsf{RMatrices}_n$ if and only if $\mathsf{flip}(M) \in \mathsf{RMatrices}_n$. Theorem 10 from [8] states that $M^* = \mathsf{flip}(M)$.

4.1 Duality for ascent sequences

We will use Observation 4.2 as a basis for determining the dual of an element of RAsc according to f_{PA}.

Theorem 4.3. *Let* $\mathbf{a} \in \mathsf{RAsc}_n$*. By Lemma 3.3, we have*

$$\mathbf{a} = (0^{m_{11}}, 1^{m_{22}}, 0^{m_{12}}, 2^{m_{33}}, 1^{m_{23}}, 0^{m_{13}}, \ldots, (d-1)^{m_{dd}}, (d-2)^{m_{d-1\,d}}, \ldots, 0^{m_{1d}}) \tag{4.1}$$

where $M = f_{\mathsf{AM}}(\mathbf{a})$. The dual ascent sequence $\mathbf{a}^*$ according to f_{PA} is given by

$$\mathbf{a}^* = (0^{m_{dd}}, 1^{m_{d-1\,d-1}}, 0^{m_{d-1\,d}}, 2^{m_{d-2\,d-2}}, 1^{m_{d-2\,d-1}}, 0^{m_{d-2\,d}}, \ldots, (d-1)^{m_{11}}, (d-2)^{m_{12}}, \ldots, 0^{m_{1d}}). \quad (4.2)$$

Example 4.4. If $\mathbf{a} = (0,0,1,2,0)$, then $M = \begin{pmatrix} 2&0&1\\0&1&0\\0&0&1 \end{pmatrix}$. Thus $\mathsf{flip}(M) = \begin{pmatrix} 1&0&1\\0&1&0\\0&0&2 \end{pmatrix}$, and so $\mathbf{a}^* = (0,1,2,2,0)$.

4.2 Duality for permutations

We use the definition of the dual of an ascent sequence to determine the dual π^* of an element π of $\mathsf{RPerms} = \mathcal{S}_n(3\bar{1}52\bar{4})$ according to $f_{\mathsf{PS}} := f_{\mathsf{AS}} \circ f_{\mathsf{PA}}$.

Our definition of π^* requires the use of what are perhaps the three best known involutions on a permutation $\pi = (\pi_1, \ldots, \pi_n)$: its inverse π^{-1}, its *reverse* $\mathrm{rev}(\pi) := (\pi_n, \ldots, \pi_1)$, and its *complement* $\mathrm{comp}(\pi) := (n+1-\pi_1, \ldots, n+1-\pi_n)$. Together, these involutions allow us to state the result with the most technical proof of this abstract.

Theorem 4.5. *Let $\pi \in \mathcal{S}_n(3\bar{1}52\bar{4}) = \mathsf{RPerms}$. The dual permutation according to f_{PS} is given by $\pi^* = (\mathrm{comp}(\mathrm{rev}(\pi)))^{-1}$.*

5 A Catalan restriction and series-parallel posets

In this section, we consider subsets of the R-families whose cardinalities are given by the Catalan numbers; we thus use the C prefix for naming these subsets. The study of these subsets is also motivated by the results of applying our bijections to these subsets, which allow us to draw connections between some natural families: pattern-avoiding ascent sequences, pattern-avoiding permutations, and series-parallel (2+2)-free posets. Pattern avoidance in ascent sequences is studied from an enumerative perspective in [11].

Definition 5.1. Let CAsc_n denote the set of 101-avoiding ascent sequences of length n.

As shown [11], the cardinality of CAsc_n is the Catalan number C_n. As an example, of the 15 ascent sequences of length 4, the only one containing 101 is 0101. Thus $14 = C_4 = |\mathsf{CAsc}_4|$. It is certainly not the case that $\mathsf{CAsc}_n = \mathsf{RAsc}_n$ since $01021 \in \mathsf{RAsc}_n \setminus \mathsf{CAsc}_n$. However, we do have the following relationship.

Proposition 5.2. $\mathsf{CAsc}_n \subseteq \mathsf{RAsc}_n$.

We next turn to determining the image of CAsc_n under our bijections. We begin with posets since this case is perhaps the most interesting. Recall from Figure 1 that $\mathsf{CPosets}_n$ denotes the series-parallel posets in Posets_n. In other words, $\mathsf{CPosets}_n$ consists of those posets with n elements that are both $(2+2)$-free and N-free.

Proposition 5.3. *$f_{\mathsf{AP}} : \mathsf{CAsc}_n \to \mathsf{CPosets}_n$, and f_{AP} is a bijection.*

We follow Jelínek [14] to define the family $\mathsf{CMatrices}_n$.

Definition 5.4. An SE-pair of a matrix $M \in \mathsf{Matrices}$ is a pair of non-zero entries m_{ij} and $m_{i'j'}$ such that $i < i'$, $j < j'$ and $i' < j$. We say that M is *SE-free* if it contains no SE-pair.

In [14, Lem. 1.2], Jelínek shows that $P \in \mathsf{Posets}$ is series-parallel if and only if $f_{\mathsf{PM}}(P)$ is SE-free.

Definition 5.5. Define $\mathsf{CMatrices}_n$ to be the subset of $\mathsf{Matrices}_n$ consisting of those elements that are SE-free.

Finally, we determine the C-family for permutations. The answer is quite appealing, namely $\mathsf{CPerms} = \mathcal{S}_n(231)$.

Proposition 5.6. *f_{AS} maps CAsc_n bijectively to CPerms.*

The above results now give the following “Catalan” refinement of Corollary 3.7

Corollary 5.7. *The classical bijections among Asc_n, $\mathsf{Matrices}_n$, Posets_n and Perms restrict to bijections among CAsc_n, $\mathsf{CMatrices}_n$, $\mathsf{CPosets}_n$ and CPerms. Specifically,*

(a) *f_{AM} maps CAsc_n bijectively to $\mathsf{CMatrices}_n$;*

(b) *[14] f_{PM} maps $\mathsf{CPosets}_n$ bijectively to $\mathsf{CMatrices}_n$;*

(c) *f_{AS} maps CAsc_n bijectively to CPerms.*

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