Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #23, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Cubic realizations of Tamari interval lattices

Camille Combe

Institut de Recherche Mathématique Avancée UMR 7501, Université de Strasbourg et CNRS, France

Abstract. We introduce cubic coordinates, which are integer words encoding intervals in the Tamari lattices. Cubic coordinates are in bijection with interval-posets, themselves known to be in bijection with Tamari intervals. We show that in each degree the set of cubic coordinates forms a lattice, isomorphic to the lattice of Tamari intervals. Geometric realizations are naturally obtained by placing cubic coordinates in space, highlighting some of their properties. Finally, we consider the cellular structure of these realizations.

Keywords: Tamari lattices, Tamari intervals, interval-posets, posets, geometric realizations, cubical complexes.

1 Introduction

The Tamari lattices are partial orders having extremely rich combinatorial and algebraic properties. These partial orders are defined on the set of binary trees and rely on the rotation operation [12]. We are interested in the intervals of these lattices, meaning the pairs of comparable binary trees. Tamari intervals of size n also form a lattice. The number of these objects is given by a formula that was proved by Chapoton [4]:

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Strongly linked with associahedra, Tamari lattices have been recently generalized in many ways [1, 10]. In this process, the number of intervals of these generalized lattices have also been enumerated through beautiful formulas [3, 7]. Many bijections between Tamari intervals of size n and other combinatorial objects are known. For instance, a bijection with planar triangulations is presented by Bernardi and Bonichon in [2]. It has been proved by Châtel and Pons that Tamari intervals are in bijection with interval-posets of the same size [6].

We provide in this paper a new bijection with Tamari intervals, which is inspired by interval-posets. More precisely, we first build two words of size n from the Tamari diagrams [9] of a binary tree. Then, if they satisfy a certain property of compatibility, we build a Tamari interval diagram from these two words. We show that Tamari interval diagrams and interval-posets are in bijection. Then, we propose a new encoding of

Tamari intervals, by building $(n-1)$-tuples of numbers from Tamari interval diagrams. These tuples we refer to as cubic coordinates. This new coding has two obvious virtues: it is very compact and it gives a way of comparing in a simple manner two Tamari intervals, through a fast algorithm. On the other hand, some properties of Tamari intervals translate nicely in the setting of cubic coordinates. For instance, synchronized Tamari intervals [10] become cubic coordinates with no zero entry. Besides, cubic coordinates provide naturally a geometric realization of the lattice of Tamari intervals, by seeing them as space coordinates. Indeed, all cubic coordinates of size n can be placed in the space $\mathbb{R}^{n-1}$. By drawing their cover relations, we obtain an oriented graph. This gives us a realization of cubic coordinate lattices, which we call cubic realization. This realization leads us to many questions, in particular about the cells it contains.

This paper is organized as follows. In Section 2, we define Tamari interval diagrams from Tamari diagrams. Then we give a bijection between the set of Tamari interval diagrams and the set of interval-posets. Subsequently, we define in Section 3 cubic coordinates, and build a bijection between these and Tamari interval diagrams. By using these two bijections, and after endowing the set of cubic coordinates with a partial order structure, we show in Section 4 that there is an isomorphism of posets between cubic coordinate posets and Tamari interval lattices. Finally, the cubic realization and the cells it contains make the subject of Section 5. We also show here how to associate a synchronized cubic coordinate with each cell.

Some proofs of the presented results are omitted and some others are sketched in this extended abstract.

Notations. Throughout this article, for all words u, we denote by u_i the i-th letter of u. We use the notation $[n]$ to denote the set $\{1, \ldots, n\}$.

2 Tamari interval diagrams

In this section, we recall the definition of Tamari diagrams [9] and generalize this notion in order to define Tamari interval diagrams. Then, we establish a bijection between the set of Tamari interval diagrams and the set of interval-posets.

Definition 2.1. *A* Tamari diagram *is a word $u = u_1 u_2 \ldots u_n$ of integers such that*

(i) $0 \leqslant u_i \leqslant n - i$ *for all* $i \in [n]$*;*

(ii) $u_{i+j} \leqslant u_i - j$ *for all* $i \in [n]$ *and* $0 \leqslant j \leqslant u_i$*.*

The size of a Tamari diagram is its number of letters.

For instance, the fourteen Tamari diagrams of size 4 are

$$0000, 0010, 0100, 0200, 0210, 1000, 1010, 2000, 2100, 3000, 3010, 3100, 3200, 3210.$$

The set of Tamari diagrams of size n is in bijection with the set of binary trees of the same size [9]. In the sequel, we need to encode a pair of binary trees with n nodes by two words of size n. To this aim, we introduce here dual Tamari diagrams. The first binary tree of the pair is encoded by its Tamari diagram and the second is encoded by its dual Tamari diagram.

Definition 2.2. *A* dual Tamari diagram *is a word $v = v_1 v_2 \dots v_n$ of integers such that*

(i) $0 \leqslant v_i \leqslant i - 1$ *for all* $i \in [n]$;

(ii) $v_{i-j} \leqslant v_i - j$ *for all* $i \in [n]$ *and* $0 \leqslant j \leqslant v_i$.

The size of a dual Tamari diagram is its number of letters.

A word $v = v_1 v_2 \dots v_n$ is a dual Tamari diagram if and only if its reversal is a Tamari diagram. We can use for Tamari diagrams and dual Tamari diagrams the graphical representation proposed in [8]. The value of a letter of both diagrams gives the height of the corresponding column. Condition (ii) of Definition 2.1 (resp. (ii) of Definition 2.2) translates in the following way on the drawing: from each column, one draws a dotted line of slope -1 (resp. 1), and the column to its right (resp. its left) must not cross this line. An example is given in Figure 2.1.

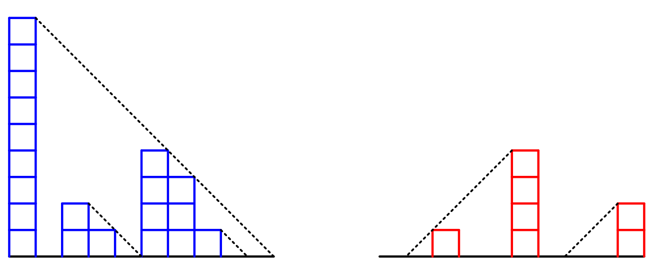
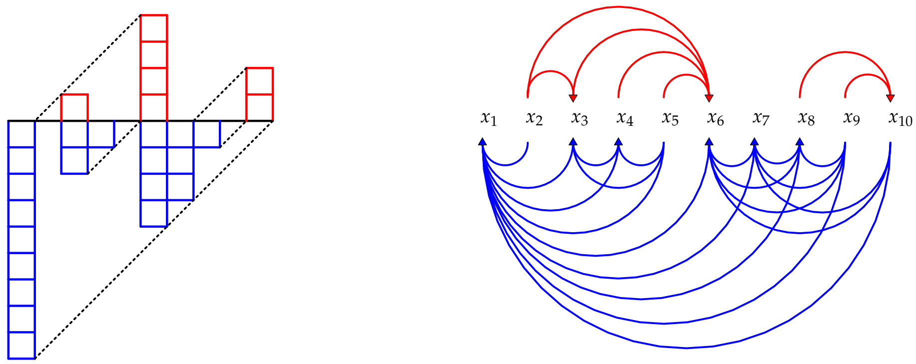


Figure 2.1: Representation of the Tamari diagram 9021043100 (left) and of the dual Tamari diagram 0010040002 (right), both of size 10.

Definition 2.3. *Let u be a Tamari diagram of size n and v be a dual Tamari diagram of size n. The diagrams u and v are* compatible *if for all $1 \leqslant i < j \leqslant n$ such that $j - i \leqslant u_i$, we have $v_j < j - i$.*

If u and v are compatible, then the pair (u, v) is a Tamari interval diagram*. The set of Tamari interval diagrams of size n is denoted by $\mathcal{TID}_n$.*

For example, the two diagrams of Figure 2.1 are compatible. Note that Definition 2.3 implies in particular that either $u_i = 0$ or $v_{i+1} = 0$.



(a) Tamari interval diagram. (b) Interval-poset.

Figure 2.2: Representation of the Tamari interval diagram $(9021043100, 0010040002)$ of size 10 and its corresponding interval-poset.

As previously, we draw Tamari interval diagrams through columns, as shown in Figure 2.2a which displays the Tamari interval diagram $(9021043100, 0010040002)$. The Tamari diagram u is drawn in blue (under) and its dual v is drawn in red (over).

Let χ be the map sending a Tamari interval diagram (u, v) of size n to the binary relation

$$(\{x_1, \ldots, x_n\}, \lhd), \tag{2.1}$$

where for all $i \in [n]$ and $0 \leqslant l \leqslant u_i$, $x_{i+l} \lhd x_i$, and for all $i \in [n]$ and $0 \leqslant k \leqslant v_i$, $x_{i-k} \lhd x_i$.

We recall that an *interval-poset* P of size n is a partial order $\lhd$ on the set $\{x_1, \ldots, x_n\}$ such that, for $i < k$, if $x_k \lhd x_i$ then for all x_j such that $i < j < k$, one has $x_j \lhd x_i$, and if $x_i \lhd x_k$ then for all x_j such that $i < j < k$, one has $x_j \lhd x_k$ [6] (see Figure 2.2b for instance). We denote $\mathcal{IP}_n$ the set of interval-posets of size n.

Theorem 2.4. *The application χ is a bijection from $\mathcal{TID}_n$ to $\mathcal{IP}_n$.*

Proof. We set $P = \chi(u, v)$. First, we check that P satisfies all axioms of the definition of interval-posets. Then we show that χ is surjective: for every interval-poset P, the pair of words $(u_1 u_2 \ldots u_n, v_1 v_2 \ldots v_n) \in \mathbb{N}^n \times \mathbb{N}^n$, where for all $i, j \in [n]$,

$$u_i = \#\{x_j \in P : x_j \lhd x_i \text{ and } i < j\}, \tag{2.2}$$

$$v_j = \#\{x_i \in P : x_i \lhd x_j \text{ and } i < j\}, \tag{2.3}$$

has image P and is a Tamari interval diagram. The proof that χ is injective is direct. $\square$

3 Cubic coordinates

We now build the set of cubic coordinates and provide a bijection between this set and the set of Tamari interval diagrams. We conclude this section by reviewing some properties of cubic coordinates.

Let $(u, v) \in \mathcal{TID}_n$. We build a $(n-1)$-tuple $(u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n)$ from letters of (u, v). This $(n-1)$-tuple can be defined by using the definition of Tamari interval diagrams.

Definition 3.1. *Let c be a $(n-1)$-tuple with entries in $\mathbb{Z}$. We say that c is a* cubic coordinate *if the pair (u, v), where u is the word defined by $u_n = 0$ and for all $i \in [n-1]$ by*

$$u_i = \max(c_i,\ 0),$$

and v is the word defined by $v_1 = 0$ and for all $2 \leqslant i \leqslant n$ by

$$v_i = |\min(c_{i-1},\ 0)|,$$

is a Tamari interval diagram. The size of a cubic coordinate is its number of entries plus one. The set of cubic coordinates of size n is denoted by $\mathcal{CC}_n$.

For example, in Figure 2.2, the Tamari interval diagram has for cubic coordinate $(9, -1, 2, 1, -4, 4, 3, 1, -2)$.

Let ϕ be the application which maps a cubic coordinate c to a Tamari interval diagram as stated in Definition 3.1.

Theorem 3.2. *The application $\phi : \mathcal{CC}_n \to \mathcal{TID}_n$ is bijective.*

Proof. Let $c, c' \in \mathcal{CC}_n$ such that $c \neq c'$. Then, there is an entry $c_i \neq c'_i$, with $i \in [n-1]$. By the definition of the application ϕ, there is a letter $u_i \neq u'_i$, or a letter $v_{i+1} \neq v'_{i+1}$, meaning that $(u, v) \neq (u', v')$. This shows the injectivity of ϕ.

Let $(u, v) \in \mathcal{TID}_n$ and let $c = (u_1 - v_2, u_2 - v_3, \dots, u_{n-1} - v_n)$ be the $(n-1)$-tuple whose entries are given by the difference $u_i - v_{i+1}$, for all $i \in [n-1]$. If $u_i \neq 0$ then by Definition 2.3, $v_{i+1} = 0$, so we have $\phi(c) = (u, v)$. As (u, v) is a Tamari interval diagram by hypothesis and by Definition 3.1, the application ϕ is surjective. □

Lemma 3.3. *Let $c \in \mathcal{CC}_n$ with $c_i \neq 0$ for some $i \in [n-1]$. We denote by c' the $(n-1)$-tuple such that $c'_i = 0$ and all entries having indices different from i are the ones of c. Then c' is a cubic coordinate.*

Proof. We set $(u, v) = \phi(c')$ and $c'_i = 0$. Then we can check all axioms of Definitions 2.1 to 2.3 for (u, v) with the pair of letters $(u_i, v_{i+1}) = (0, 0)$. □

Definition 3.4. *A cubic coordinate c of size n is* synchronized *if for all $i \in [n-1]$, $c_i \neq 0$. The set of synchronized cubic coordinates of size n is denoted by $\mathcal{CC}_n^{sync}$.*

This definition can be translated in terms of Tamari interval diagrams: a Tamari interval diagram (u, v) of size n is *synchronized* if for all $i \in [n-1]$, $u_i \neq 0$ or $v_{i+1} \neq 0$.

We recall that a Tamari interval $[S, T]$ is synchronized if and only if the binary trees S and T have the same canopy [10]. Let $\mathcal{TI}_n$ be the set of Tamari intervals of size n and ρ be the bijection from $\mathcal{IP}_n$ to $\mathcal{TI}_n$ [6].

Proposition 3.5. *Let $(u, v) \in \mathcal{TID}_n$. The Tamari interval diagram (u, v) is synchronized if and only if $\rho(\chi(u, v)) = [S, T]$ is a synchronized Tamari interval.*

Proof. Arguing by contradiction, we translate the fact that $u_i = 0$ and $v_{i+1} = 0$ to the trees S and T, and deduce that their canopies are distinct. Reciprocally, if the canopies of S and of T are distinct, we can find an index $i \in [n]$ such that $u_i = 0$ and $v_{i+1} = 0$. □

Definition 3.6. *A Tamari interval diagram (u, v) of size n is* new *if the following conditions are satisfied:*

(i) $0 \leqslant u_i \leqslant n - i - 1$ *for all* $i \in [n-1]$;

(ii) $0 \leqslant v_j \leqslant j - 2$ *for all* $j \in \{2, \dots, n\}$;

(iii) $u_k < l - k - 1$ *or* $v_l < l - k - 1$ *for all* $k, l \in [n]$ *such that* $k + 1 < l$.

The definition of new interval-posets is given in [11]. The three conditions of this original definition imply the three conditions of Definition 3.6 and reciprocally. Then, we have the following proposition.

Proposition 3.7. *Let $(u, v) \in \mathcal{TID}_n$. The Tamari interval diagram (u, v) is new if and only if $\chi(u, v) = P$ is a new interval-poset.*

In [11], Rognerud shows that an interval-poset P is new if and only if $\rho(P)$ is a new Tamari interval (see [5] for more about this notion). Then, one has the following result.

Corollary 3.8. *Let $(u, v) \in \mathcal{TID}_n$. The Tamari interval diagram (u, v) is new if and only if $\rho(\chi(u, v))$ is a new Tamari interval.*

Proposition 3.9. *Let $(u, v) \in \mathcal{TID}_n$. If (u, v) is synchronized then (u, v) is not new.*

Proof. Let us suppose that (u, v) is synchronized and new. Then, by using (iii) from Definition 3.6, we come to a contradiction. □

4 Isomorphism of posets

In this section, we define the poset of cubic coordinates and we show that there is an isomorphism between this poset and the poset of Tamari intervals. To this aim, both bijections seen in Section 2 and in Section 3 are used.

Let $c, c' \in \mathcal{CC}_n$. We set $c \leqslant_{\mathrm{cc}} c'$ if and only if $c_i \leqslant c'_i$ for all $i \in [n-1]$. The set of cubic coordinates of size n endowed with the binary relation $\leqslant_{\mathrm{cc}}$ is a poset, the *cubic coordinate poset*.

Let us denote by $\lessdot$ the covering relation of $(\mathcal{CC}_n, \leqslant_{\mathrm{cc}})$. We admit here the following result.

Lemma 4.1. *Let $c, c' \in \mathcal{CC}_n$ such that $c \leqslant_{\mathrm{cc}} c'$. Then, $c \lessdot c'$ if and only if there is exactly one $i \in [n-1]$ such that $c_i < c'_i$, and if there is a $c'' \in \mathcal{CC}_n$ such that $c \leqslant_{\mathrm{cc}} c'' \leqslant_{\mathrm{cc}} c'$, then either $c = c''$ or $c' = c''$.*

We denote by $\leqslant_{\mathrm{t}}$ the Tamari order on the set of binary trees [12]: for S and T, two binary trees of size n, $S \leqslant_{\mathrm{t}} T$ if and only if T can be obtained by performing an arbitrary number of rotations from S. If T is obtained by only one rotation in S, then T covers S. Let $[S, T], [S', T'] \in \mathcal{TI}_n$. We set $[S, T] \leqslant_{\mathrm{ti}} [S', T']$ if and only if $S \leqslant_{\mathrm{t}} S'$ and $T \leqslant_{\mathrm{t}} T'$. Then, S' covers S and $T = T'$ or T' covers T and $S = S'$ if and only if $[S', T']$ covers $[S, T]$.

Let $\psi = \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$ be the application from the set of Tamari intervals to the set of cubic coordinates.

Theorem 4.2. *The application ψ is an isomorphism of posets.*

Proof. Let $[S, T], [S', T'] \in \mathcal{TI}_n$, and $\psi([S, T]) = c$, $\psi([S', T']) = c'$. Then, let $\phi(c) = (u, v)$, $\phi(c') = (u', v')$ and $\chi(u, v) = P$, $\chi(u', v') = P'$. Let us show that $[S', T']$ covers $[S, T]$ in $(\mathcal{TI}_n, \leqslant_{\mathrm{ti}})$ if and only if c' covers c in $(\mathcal{CC}_n, \leqslant_{\mathrm{cc}})$.

Let $(\star)$ (resp. $(\diamond)$) be the following condition: P' is obtained from P by only adding (resp. removing) some decreasing (resp. increasing) relations ending at a vertex x_i, such that if we remove (resp. add) one of these decreasing (resp. increasing) relations, then either we obtain P or the resulting object is not an interval-poset.

We admit here that P and P' satisfy $(\star)$ (resp. $(\diamond)$) for the vertex x_i if and only if S' (resp. T') is obtained by a unique rotation of the node of index i in S (resp. T) and $T' = T$ (resp. $S' = S$). In other words, P and P' satisfy either $(\star)$ or $(\diamond)$ if and only if $[S', T']$ covers $[S, T]$. It only remains to show that c' covers c with $c_i < c'_i$ if and only if P and P' satisfy either $(\star)$ or $(\diamond)$ for the vertex x_i.

We assume that $c \lessdot c'$ with $c_i < c'_i$. Then, there are two cases:

(1) If c'_i is positive, then c_i is not negative due to Lemma 3.3. Hence $c'_i = u'_i$ and $c_i = u_i$. The image by ϕ of c and of c' differs only for the letter u_i. Besides, the fact that $c \lessdot c'$ implies in particular that if there is a word u'' of size n such that $u''_i = u'_i - 1$ and $u''_j = u'_j$ for any $j \neq i$, then either $(u'', v') = (u, v)$ or (u'', v') is not a Tamari interval diagram. Indeed, let us suppose that there is a Tamari interval diagram (u'', v') such as previously described, different from (u, v). Then, let c'' be the cubic coordinate associated with (u'', v') by ϕ^{-1}. Since $u''_i = u'_i - 1$ and $u''_j = u'_j$ for any $j \neq i$, one has $c'' \leqslant_{\mathrm{cc}} c'$. Moreover, since $u'' \neq u$, one has $c \leqslant_{\mathrm{cc}} c''$. Knowing these

two facts, we have a contradiction with our hypothesis $c \lessdot c'$. The difference of a unique letter u_i between (u, v) and (u', v') is directly translated by χ: The interval-poset P' has more decreasing relations ending at x_i than the vertex x_i has in P. Moreover, the fact that there is no other Tamari interval diagram between (u, v) and (u', v') implies that the number of decreasing relations added in P' compared to P is minimal. This means that if one decreasing relation ending at x_i is removed from P', then either we obtain P or the resulting object is not an interval-poset. Hence P and P' satisfy $(\star)$.

(2) Symmetrically, if c'_i is nonpositive, then by using the same arguments, we obtain that the interval-poset P' has less increasing relations ending at x_{i+1} than the vertex x_{i+1} from P, in a minimal way. This leads us to the fact that P and P' satisfy $(\diamond)$.

Reciprocally, suppose that P and P' satisfy either $(\star)$ or $(\diamond)$ for a vertex x_i.

(1) Suppose that P and P' satisfy $(\star)$. Since only decreasing relations ending at x_i are added in P', then only the letter u'_i from u' is increased compared to u, and $v' = v$. Besides, since the number of decreasing relations added in P is minimal, there is no Tamari interval diagram between (u, v) and (u', v'), and so there is no cubic coordinate between c and c'. Hence $c \lessdot c'$.

(2) Suppose that P and P' satisfy $(\diamond)$. Since only increasing relations ending at x_i are removed in P', then only the letter v'_i from v' is decreased compared to v, and $u' = u$. Then, just like in case (1), $c \lessdot c'$.

One can now conclude that ψ is an isomorphism of posets. □

To sum up the applications seen in Sections 2 to 4, let us recall that $\psi = \phi^{-1} \circ \chi^{-1} \circ \rho^{-1}$. Then, we have the following diagram of poset isomorphisms:

$$\begin{tikzcd} \mathcal{TID}_n \arrow[r, "\chi"] & \mathcal{IP}_n \arrow[d, "\rho"] \\ \mathcal{CC}_n \arrow[u, "\phi"] & \mathcal{TI}_n \arrow[l, "\psi"'] \end{tikzcd}$$

One consequence of the isomorphism ψ is that the order dimension [13] of the poset of Tamari intervals is at most equal to $n - 1$.

5 Cubic realization and cells

Now, we regard the poset of cubic coordinates defined in Section 4 as a natural geometric object. We study this geometric realization by giving a theoretical definition of the cells it contains.

All cubic coordinates of size n can be placed in the space $\mathbb{R}^{n-1}$, as space coordinates. For all cubic coordinates c and c' such that $c \leqslant_{cc} c'$, we connect c to c' with an arrow if and only if there is no other cubic coordinate c'' such that $c \leqslant_{cc} c'' \leqslant_{cc} c'$, meaning that just one entry increases between c and c''. This oriented graph is the *cubic realization* of the cubic coordinate lattice.

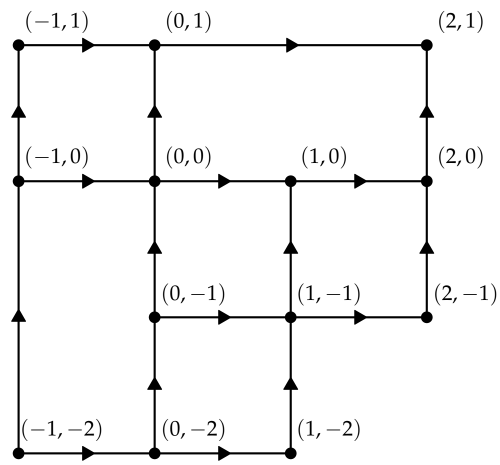


Figure 5.1: Cubic realization of $\mathcal{CC}_3$.

Figure 5.1 is the cubic realization of $\mathcal{CC}_3$, where the elements of $\mathcal{CC}_3$ are vertices and the cover relations are arrows orientated to the covering cubic coordinates.

Definition 5.1. *Let $c \in \mathcal{CC}_n$. Suppose that there is $c' \in \mathcal{CC}_n$ such that $c'_i > c_i$ and $c'_j = c_j$ for all $j \neq i$, with $i, j \in [n-1]$. We define then the application of* minimal increase $\uparrow_i$ *as follows*

$$\uparrow_i (c) = (c_1, \dots, c_{i-1}, \widehat{c}_i, c_{i+1}, \dots, c_{n-1}), \tag{5.1}$$

such that $c \lessdot \uparrow_i (c)$ and $c_i < \widehat{c}_i \leqslant c'_i$.

Definition 5.2. *Let $c \in \mathcal{CC}_n$. We say that c is* minimal-cellular *if for all $i \in [n-1]$, $\uparrow_i (c)$ is well-defined.*

We notice that the cubic coordinates which are minimal-cellular are the elements that are covered by exactly $n-1$ elements in $(\mathcal{CC}_n, \leqslant_{cc})$.

Lemma 5.3. *Let c be minimal-cellular cubic coordinate of size n and $i \in [n-1]$. If*

$$c' = \uparrow_{i+1} (\uparrow_{i+2} (\dots (\uparrow_{n-1} (c)) \dots)), \tag{5.2}$$

is well-defined, then $\uparrow_i (c')$ is well-defined.

Definition 5.4. *Let c be a minimal-cellular cubic coordinate of size n and let c' be a cubic coordinate of size n. We set that c' is the* maximal-cellular correspondent *of c if*

$$c' = \uparrow_1 (\uparrow_2 (\ldots (\uparrow_{n-1} (c)) \ldots)). \tag{5.3}$$

For instance, $c = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is minimal-cellular, and its maximal-cellular correspondent is $c' = (1, 0, 2, 0, -4, 3, 2, 0, -2)$. Such an element always exists, by Lemma 5.3. Note that by performing minimal increases in another order does not always lead to the maximal-cellular correspondent (see Figure 5.1 for example).

Definition 5.5. *Let c^m be minimal-cellular cubic coordinate of size n and let c^M be its maximal-cellular correspondent. The pair (c^m, c^M) is a* cell*, denoted by $\langle c^m, c^M \rangle$. The size of the cell is the size of c^m.*

Lemma 5.6. *Let $\langle c^m, c^M \rangle$ be a cell of size n and $i \in [n-1]$,*

(i) if $c_i^m < 0$ then $c_i^M \leqslant 0$;

(ii) if $c_i^m \geqslant 0$ then $c_i^M > 0$.

Theorem 5.7. *Let $\langle c^m, c^M \rangle$ be a cell of size n. Let c be any $(n-1)$-tuple whose entries c_i are equal to c_i^m, or to c_i^M, for all $i \in [n-1]$. Then c is a cubic coordinate.*

One of the consequences of Theorem 5.7 is that for every cell, we have at least 2^{n-1} cubic coordinates between c^m and c^M. There can be strictly more. Now, since we have a definition of cells, we show that from a cell, we can build a synchronized cubic coordinate.

Consider $\langle c^m, c^M \rangle$ a cell of size n. Let γ be the map defined for all $i \in [n-1]$ by

$$\gamma(c_i^m, c_i^M) = \begin{cases} c_i^m & \text{if } c_i^m < 0, \\ c_i^M & \text{if } c_i^m \geqslant 0. \end{cases} \tag{5.4}$$

Let Γ be the map from the set of cells of size n to the set of $(n-1)$-tuples defined by

$$\Gamma(\langle c^m, c^M \rangle) = (\gamma(c_1^m, c_1^M), \gamma(c_2^m, c_2^M), \ldots, \gamma(c_{n-1}^m, c_{n-1}^M)). \tag{5.5}$$

For example, the cell $\langle (0, -1, 1, -1, -5, 0, 1, -1, -3), (1, 0, 2, 0, -4, 3, 2, 0, -2) \rangle$ is sent to $(1, -1, 2, -1, -5, 3, 2, -1, -3)$.

Theorem 5.8. *The application Γ is a bijection from the set of cells of size n to CC_n^{sync}.*

Proof. The entries of $\Gamma(\langle c^m, c^M \rangle)$ belong either to c^m or to c^M. In both cases, the entries are not zero. Hence, by Theorem 5.7, $\Gamma(\langle c^m, c^M \rangle)$ is a cubic coordinate of size n. Additionally, by Definition 3.4 and Lemma 5.6, this cubic coordinate is synchronized.

Let us first show that Γ is injective. Let $\langle c^m, c^M \rangle$ and $\langle e^m, e^M \rangle$ be two cells of size n, such that $\Gamma(\langle c^m, c^M \rangle) = \Gamma(\langle e^m, e^M \rangle)$. Then let (u_i^j, v_{i+1}^j) and (x_i^j, y_{i+1}^j) be the two pairs of letters corresponding respectively to c_i^j and to e_i^j by ϕ with $j \in \{m, M\}$ and $i \in [n-1]$.

By hypothesis, $\gamma(c_i^m, c_i^M) = \gamma(e_i^m, e_i^M)$ for all $i \in [n-1]$. We need to prove that $c_i^j = e_i^j$ for all $i \in [n-1]$ with $j \in \{m, M\}$. Thus, two cases are possible:

(1) either $\gamma(c_i^m, c_i^M) = u_i^M$. In this case, $\gamma(e_i^m, e_i^M) = x_i^M$ and $u_i^M = x_i^M$;

(2) or $\gamma(c_i^m, c_i^M) = -v_{i+1}^m$. Then, $\gamma(e_i^m, e_i^M) = -y_{i+1}^m$ and $v_{i+1}^m = y_{i+1}^m$.

- Let us assume that (1) holds. Since $u_i^M \neq 0$, by Definition 2.3, $v_{i+1}^M = 0$. Likewise, $x_i^M \neq 0$ implies that $y_{i+1}^M = 0$. Hence $c_i^M = e_i^M$.

  By Lemma 5.6, if $u_i^M \neq 0$ (resp. $x_i^M \neq 0$) then $0 \leqslant u_i^m < u_i^M$ and $v_{i+1}^m = 0$ (resp. $0 \leqslant x_i^m < x_i^M$ and $y_{i+1}^m = 0$). It remains only to show that $u_i^m = x_i^m$. Suppose that $u_i^m < x_i^m$, and let $c = \uparrow_{i+1} (\dots (\uparrow_{n-1} (c^m)) \dots)$ and $e = \uparrow_{i+1} (\dots (\uparrow_{n-1} (e^m)) \dots)$. By Lemma 5.3, c and e are cubic coordinates. Therefore, by Definition 5.4, $c_k = c_k^M$ and $e_k = e_k^M$ for all $i+1 \leqslant k \leqslant n-1$. By hypothesis, if c_k is positive, then $c_k = u_k^M$ and since in this case $u_k^M = x_k^M$, one has $e_k = x_k^M$. Let c' be a $(n-1)$-tuple such that $c'_i = x_i^m$ and $c'_j = c_j$ for any $j \neq i$. By hypothesis, we also know that $\widehat{c}_i = u_i^M$, $\widehat{e}_i = x_i^M$, and $u_i^M = x_i^M$. Since $x_i^m < x_i^M$, then $x_i^m < u_i^M$. Hence $u_i^m < x_i^m < u_i^M$. Let (u, v) and (u', v') be the two pairs of words corresponding respectively with c and c'. The $(n-1)$-tuple c' is a cubic coordinate. Indeed, since $v = v'$ and c is a cubic coordinate, v' is a dual Tamari diagram. Therefore, since e is a cubic coordinate, $c'_k = e_k$ if c'_k is positive for all $i \leqslant k \leqslant n-1$, and one has that $\uparrow_i (c)$ is also a cubic coordinate, then u' is a Tamari diagram. Finally, since $\uparrow_i (c) \in \mathcal{CC}_n$, we can conclude that Definition 2.3 is satisfied and c' is a cubic coordinate. It leads us to the fact that there is a cubic coordinate c' distinct from c and $\uparrow_i (c)$ such that $c \leqslant_{\mathrm{cc}} c' \leqslant_{\mathrm{cc}} \uparrow_i (c)$, which is impossible by Definition 5.1. Whence $c_i^m = e_i^m$.

- Let us suppose that (2) holds. We show that $c_i^m = e_i^m$ and $c_i^M = e_i^M$ in the same way as (1), by reformulating the arguments for the dual case.

By definition of γ, we cannot have any other case. Therefore Γ is injective.

Now, let us show that the cardinality of the set of cells of size n is equal to the cardinality of $\mathcal{CC}_n^{sync}$. Recall that the set of cells of size n is exactly the set of minimal-cellular cubic coordinates of size n. Moreover, this is also the set of cubic coordinates which are covered by exactly $n-1$ elements in $(\mathcal{CC}_n, \leqslant_{\mathrm{cc}})$. Besides, by the isomorphism of posets ψ, we know that these elements correspond to the ones with the same property in the poset of Tamari intervals. In [5], Chapoton shows that the set of these elements have the same cardinality as the set of synchronized Tamari intervals (see Theorem 2.1

and Theorem 2.3 there). By Proposition 3.5, it may be inferred that the cardinality of $\mathcal{CC}_n^{sync}$ and the cardinality of the set of cells of size n are equal.

One can conclude that Γ is bijective. $\square$

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