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A bijection between ordinary partitions and self-conjugate partitions with same disparity

Hyunsoo Cho[*1], JiSun Huh[†2], and Jaebum Sohn[‡1]

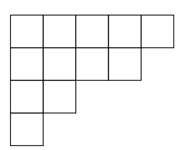
[1]*Department of Mathematics, Yonsei University, Seoul 03722, Republic of Korea*
[2]*Applied Algebra and Optimization Research Center, Sungkyunkwan University, Suwon 16420, Republic of Korea*

Abstract. We give a bijection between the set of ordinary partitions and that of self-conjugate partitions with some restrictions. Also, we show the relation between hook lengths of a self conjugate partition and its corresponding partition via the bijection. As a corollary, we give new combinatorial interpretations for the Catalan number and the Motzkin number in terms of self-conjugate simultaneous core partitions.

Keywords: partition, self-conjugate partition, hook length, simultaneous core partition

1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of n. The *Young diagram* of λ is a collection of n boxes in ℓ rows with λ_i boxes in row i. We label the columns of the diagram from left to right starting with column 1. The box in row i and column j is said to be in position (i, j). For example, the Young diagram for $\lambda = (5, 4, 2, 1)$ is below.



For the Young diagram of λ, the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$ is called the *conjugate* of λ, where λ'_j denotes the number of boxes in column j of λ. For each box in its Young diagram, we define its *hook length* by counting the number of boxes directly to its right or below, including the box itself. Equivalently, for the box in position (i, j), the hook length of λ is defined by

$$h_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1.$$

For example, the hook lengths in the first row above are 8, 6, 4, 3, and 1, respectively. We denote $h_\lambda(i, j)$ by $h(i, j)$ when λ is clear.

*coconut@yonsei.ac.kr
†hyunyjia@g.skku.edu JiSun Huh was supported by NRF grant #2015R1D1A1A01057476.
‡jsohn@yonsei.ac.kr Jaebum Sohn was supported by NRF grant #2017R1A2B4009501.

For a positive integer t, a partition λ is called a *t-core* if none of its hook lengths are multiples of t. The number of t-core partitions of n is denoted by $c_t(n)$. The study of core partitions arises from the representation theory of the symmetric group S_n. (See [11] for details.) Many researches on core partitions are being made through various ways, such as representation theory and analytic methods–see, for example, [5, 6, 7, 9, 10, 12, 13].

A partition whose conjugate is equal to itself is called *self-conjugate*. Let $sc_t(n)$ denote the number of t-core partitions of n which are self-conjugate. A number of properties of self-conjugate core partitions have been found and proved. (See [3, 4].)

Garvan, Kim, and Stanton [8] found the generating functions of $c_t(n)$ and $sc_t(n)$;

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{nt})^t}{1-q^n}. \tag{1.1}$$

$$\sum_{n=0}^{\infty} sc_{2t}(n)q^n = \prod_{n=1}^{\infty} (1-q^{4nt})^t(1+q^{2n-1}). \tag{1.2}$$

Now by combining (1.1), (1.2), and Gauss' well-known identity

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^{2n-1}} = \prod_{n=1}^{\infty} (1-q^{2n})(1+q^n),$$

one can obtain the following proposition which shows a relation between two numbers $c_t(n)$ and $sc_t(n)$.

Proposition 1.1.

$$\sum_{n=0}^{\infty} sc_{2t}(n)q^n = \left(\sum_{n=0}^{\infty} c_t(n)q^{4n}\right)\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right).$$

Also, if we let $p(n)$ be the number of partitions of n and let $sc(n)$ be the number of self-conjugate partitions of n, then $p(n)$ and $sc(n)$ have a similar relation.

Proposition 1.2.

$$\sum_{n=0}^{\infty} sc(n)q^n = \left(\sum_{n=0}^{\infty} p(n)q^{4n}\right)\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right).$$

In this paper, we construct a bijection between the set of ordinary partitions and the set of self-conjugate partitions with the same disparity. Our bijection can be obtained by combining Wright's bijection for proving Jacobi triple product identity and a bijection between self-conjugate partitions and diagonal sequence pairs. (See [14, 16].) The bijection leads to a new combinatorial proof for Proposition 1.2. Also, from the bijection, we find a relation between hook lengths of a self-conjugate partition and that of the corresponding partition. (See Theorem 4.4.) As a result of this relation, we can also reprove Proposition 1.1. Another result comes from Theorem 4.4 is Proposition 1.3 which is a generalization of Proposition 1.1. Here, we use the notation of a $(t_1, t_2, ..., t_p)$-core partition if it is simultaneously a t_1-core,..., and a t_p-core.

Proposition 1.3. *Let $c_{(t_1,t_2,\ldots,t_p)}(n)$ be the number of $(t_1, t_2, \ldots, t_p)$-core partitions of n and $sc_{(t_1,t_2,\ldots,t_p)}(n)$ be the number of self-conjugate $(t_1, t_2, \ldots, t_p)$-core partitions of n. Then we have*

$$\sum_{n=0}^{\infty} sc_{(2t_1,2t_2,\ldots,2t_p)}(n)q^n = \left(\sum_{n=0}^{\infty} c_{(t_1,t_2,\ldots,t_p)}(n)q^{4n}\right)\left(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\right).$$

At the end of this paper, new interpretations of the Catalan number and the Motzkin number in terms of self-conjugate simultaneous core partitions is given (see Corollary 4.11) as a corollary of Proposition 1.3.

This paper is organized as follows. In Section 2, we define the disparity and introduce new classification of the set of self-conjugate partitions. In Section 3, we give a bijection between the set of ordinary partitions and that of self-conjugate partitions with the same disparity. In Section 4, we explain the relation between even hook lengths in a self-conjugate partition and hook lengths in the corresponding partition via the bijection. Furthermore, we give some new results on counting self-conjugate simultaneous cores.

2 Self-conjugate partitions with same disparity

In this section, we give some basic notions and introduce a set partition of the set $\mathcal{SC}$ of self-conjugate partitions.

Let λ be a partition. We often use the notation δ_i for the hook length $h(i,i)$ of the ith box on the main diagonal. The set $D(\lambda) = \{\delta_i \ : \ i = 1, 2, \ldots\}$ is called the *set of main diagonal hook lengths of* λ. It is clear that if λ is self-conjugate, then $D(\lambda)$ determines λ, and elements of $D(\lambda)$ are all distinct and odd. Hence, for a self-conjugate partition λ, $D(\lambda)$ can be partitioned into the following two subsets;

$$\begin{aligned} D_1(\lambda) &= \{\delta_i \in D(\lambda) \ : \ \delta_i \equiv 1 \pmod 4\}, \\ D_3(\lambda) &= \{\delta_i \in D(\lambda) \ : \ \delta_i \equiv 3 \pmod 4\}. \end{aligned}$$

Example 2.1. *Let $\lambda = (4, 4, 4, 3)$ be a self-conjugate partition of 15. Figure 1 shows its Young diagram and the hook lengths. The set $D(\lambda) = \{7, 5, 3\}$ of main diagonal hook lengths is partitioned into $D_1(\lambda) = \{5\}$ and $D_3(\lambda) = \{7, 3\}$.*

The set of hook lengths of boxes in the first column of the Young diagram of λ is called the *beta-set* of λ and denoted by $\beta(\lambda)$.

7	6	5	3
6	**5**	4	2
5	4	**3**	1
3	2	1	

Figure 1: The Young diagram of a self-conjugate partition and its hook lengths

Let $\mathcal{SC}(n)$ be the set of self-conjugate partitions of n and $\lambda \in \mathcal{SC}(n)$. Using the value $|D_1(\lambda)| - |D_3(\lambda)|$, we split $\mathcal{SC}(n)$ as follows: For $m, n \geq 0$, we define a set $\mathcal{SC}^{(m)}(n)$ by

$$\mathcal{SC}^{(m)}(n) = \{\lambda \in \mathcal{SC}(n) \ : \ |D_1(\lambda)| - |D_3(\lambda)| = (-1)^{m+1} \left\lceil \frac{m}{2} \right\rceil \}.$$

We note that for a self-conjugate partition λ, if $|D_1(\lambda)| - |D_3(\lambda)| = k$ for $k \geq 1$, then $\lambda \in \mathcal{SC}^{(2k-1)}(n)$. Otherwise, if $|D_1(\lambda)| - |D_3(\lambda)| = -k$ for $k \geq 0$, then $\lambda \in \mathcal{SC}^{(2k)}(n)$. Therefore, $\mathcal{SC}(n) = \bigcup_{m=0}^{\infty} \mathcal{SC}^{(m)}(n)$.

We use the notation $sc^{(m)}(n)$ for $|\mathcal{SC}^{(m)}(n)|$ and $\mathcal{SC}^{(m)}$ for $\bigcup_{n \geq 0} \mathcal{SC}^{(m)}(n)$.

For a partition λ, we define the *disparity* of λ by

$$\mathrm{dp}(\lambda) = |\{(i,j) \in \lambda \ : \ h(i,j) \text{ is odd}\}| - |\{(i,j) \in \lambda \ : \ h(i,j) \text{ is even}\}|.$$

For example, for $\lambda = (4,4,4,3)$ given in Example 2.1, $|D_1(\lambda)| - |D_3(\lambda)| = -1$, and λ is an element of $\mathcal{SC}^{(2)}(15)$. Moreover, the disparity of λ is $\mathrm{dp}(\lambda) = 9 - 6 = 3$.

It is not hard to show that each element of $\mathcal{SC}^{(m)}(n)$ has the same disparity.

Proposition 2.2. *For $m \geq 0$, if λ is in the set $\mathcal{SC}^{(m)}$, then its disparity* $\mathrm{dp}(\lambda)$ *is* $\frac{m(m+1)}{2}$.

By Proposition 2.2, one may notice that the disparity of a self-conjugate partition is a triangular number $\frac{m(m+1)}{2}$, and the set of self-conjugate partitions with the disparity $\frac{m(m+1)}{2}$ is $\mathcal{SC}^{(m)}$. In fact, the disparity of any ordinary partition is a triangular number.

3 Bijections between $\mathcal{SC}^{(m)}$ and $\mathcal{P}$

The set of partitions of n is denoted by $\mathcal{P}(n)$, and the set of partitions is denoted by $\mathcal{P}$. In this section we construct bijections between two sets $\mathcal{SC}^{(m)}(4n + m(m+1)/2)$ and $\mathcal{P}(n)$ which play a key role throughout the paper.

Before constructing bijections, we give a notation. For a self-conjugate partition λ, if

$$\begin{aligned} D_1(\lambda) &= \{4a_1+1, 4a_2+1, \ldots, 4a_r+1\}, \\ D_3(\lambda) &= \{4b_1-1, 4b_2-1, \ldots, 4b_s-1\}, \end{aligned}$$

we say that λ has the *diagonal sequence pair* $((a_1, a_2, \ldots, a_r), (b_1, b_2, \ldots, b_s))$, where $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_s \geq 1$. For convenience, we allow an empty sequence if r or s is equal to 0.

For $\lambda = (4,4,4,3) \in \mathcal{SC}^{(2)}(15)$, its diagonal sequence pair is $((1),(2,1))$.

We note that if $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$ is the diagonal sequence pair of a self-conjugate partition $\lambda \in \mathcal{SC}^{(m)}(4n + m(m+1)/2)$, then

$$r - s + (-1)^m \left\lceil \frac{m}{2} \right\rceil = 0$$

and

$$4\left(\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j\right) + r - s = 4n + \frac{m(m+1)}{2}.$$

Now, we are ready to construct our mapping.

Mapping $\phi_n^{(m)}$: $\mathcal{SC}^{(m)}(4n + m(m+1)/2) \;\to\; \mathcal{P}(n)$

Let $\lambda \in \mathcal{SC}^{(m)}(4n + m(m+1)/2)$ be a self-conjugate partition with the diagonal sequence pair $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$. We define $\phi_n^{(m)}(\lambda)$ by the partition $\mu = (\mu_1, \ldots, \mu_\ell)$ such that

$$\mu_i = a_i + i + s - r \qquad \text{for} \quad i \leq r,$$

and $(\mu_{r+1}, \ldots, \mu_\ell)$ is the conjugate of the partition $\gamma = (b_1 - s, b_2 - s + 1, \ldots, b_s - 1)$. (We allow that γ has some zero parts.)

In Figure 2, the diagram after deleting the shaded area is the Young diagram of μ.

Theorem 3.1. *For nonnegative integers m and n, the mapping $\phi_n^{(m)}$ is bijective.*

We define the bijection $\phi^{(m)} : \mathcal{SC}^{(m)} \to \mathcal{P}$ by $\phi_n^{(m)}(\lambda)$, for a partition $\lambda \in \mathcal{SC}^{(m)}$ of $4n + \frac{m(m+1)}{2}$. We say that μ is the *corresponding partition* of λ when $\phi^{(m)}(\lambda) = \mu$.

We give two examples of the bijection $\phi^{(m)}$.

Example 3.2. *We consider two self-conjugate partitions λ and $\tilde{\lambda}$ with the set of main diagonal hook lengths $D(\lambda) = \{21, 15, 13, 9, 3, 1\}$ and $D(\tilde{\lambda}) = \{31, 19, 11, 5\}$, respectively.*

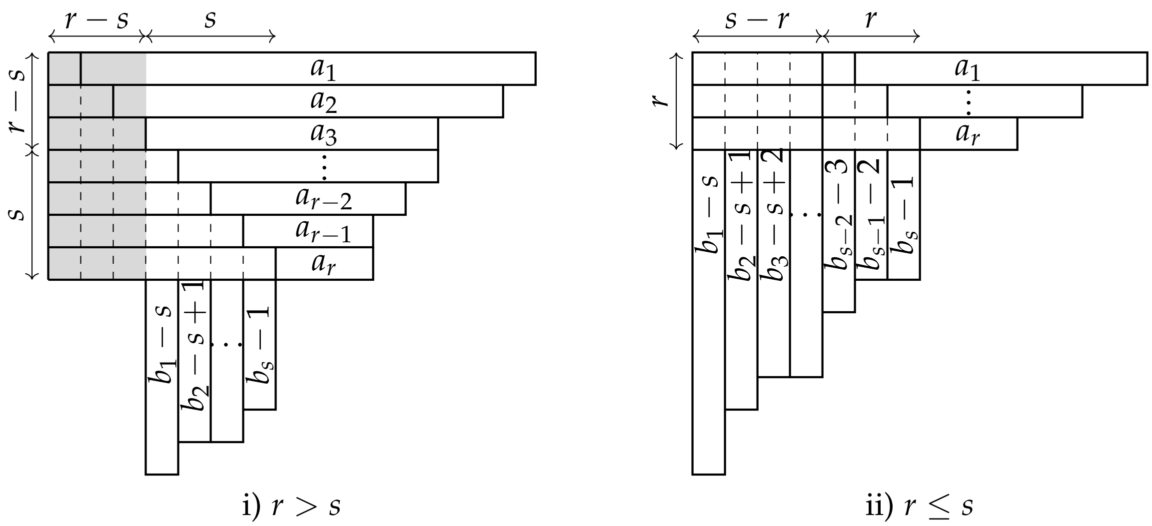


Figure 2: Graphical interpretations of mapping $\phi_n^{(m)}$

- *Since* $D_1(\lambda) = \{21, 13, 9, 1\}$ *and* $D_3(\lambda) = \{15, 3\}$, $\lambda \in \mathcal{SC}^{(3)}$ *and* $((5, 3, 2, 0), (4, 1))$ *is the diagonal sequence pair of* λ. *If we let* μ *be the partition* $\phi_{14}^{(3)}(\lambda)$, *then*

  $$\mu_1 = 5 + 1 - 2 = 4, \quad \mu_2 = 3 + 2 - 2 = 3, \quad \mu_3 = 2 + 3 - 2 = 3, \quad \mu_4 = 0 + 4 - 2 = 2$$

  and $(\mu_5, \mu_6, \dots)$ *is the conjugate of the partition* $(4 - 2, 1 - 2 + 1)$.
  Therefore, $\mu = (4, 3, 3, 2, 1, 1)$.

- *Since* $D_1(\tilde\lambda) = \{5\}$ *and* $D_3(\tilde\lambda) = \{31, 19, 11\}$, $\tilde\lambda \in \mathcal{SC}^{(4)}$ *and* $((1), (8, 5, 3))$ *is the diagonal sequence pair of* $\tilde\lambda$. *If we let* $\tilde\mu$ *be the partition* $\phi_{14}^{(4)}(\tilde\lambda)$, *then* $\mu_1 = 1 + 1 + 2 = 4$ *and* $(\mu_2, \mu_3, \dots)$ *is the conjugate of the partition* $(8 - 3, 5 - 3 + 1, 3 - 3 + 2)$.
  Therefore, $\tilde\mu = (4, 3, 3, 2, 1, 1)$.

For given $\mu \in \mathcal{P}$ *and* $m \geq 0$, *we consider the following diagram to find* λ *such that* $\phi^{(m)}(\lambda) = \mu$. *For convenience, even if* $i \leq 0$, *we set the* i*th column is the column on the left side of the* $(i+1)$*st column and the* i*th row is on the above of the* $(i+1)$*st row.*

- *For* $m = 2k - 1$, *we consider the diagram* ν *obtained from the Young diagram of* μ *by attaching* $\frac{k(k-1)}{2}$ *boxes on the left side such that* ν *has* $\mu_i + k - i$ *boxes in row* i *for* $i < k$ *and* μ_i *boxes in row* i *for* $i \geq k$. *Then, the number of (white) boxes* (i, j) *in row* i *such that* $i - j < k$ *is equal to* a_i *and the number of (gray) boxes* (i, j) *in column* j *such that* $i - j \geq k$ *is equal to* b_j. *See the first diagram in Figure 3 for* $\mu = (4, 3, 3, 2, 1, 1)$ *and* $m = 3$.

- *For* $m = 2k$, *we consider the diagram* ν *obtained from the Young diagram of* μ *by attaching* $\frac{k(k+1)}{2}$ *boxes on the above such that* ν *has* $k - i$ *boxes in row* $-i$ *for* $i = 0, 1, \dots, k-1$

and μ_i boxes in row i for $i > 0$. Then, the number of (white) boxes (i, j) in row i such that $i - j < -k$ is equal to a_i and the number of (gray) boxes (i, j) in column j such that $i - j \geq -k$ is equal to b_j. See the second diagram in Figure 3 for $\mu = (4, 3, 3, 2, 1, 1)$ and $m = 4$.

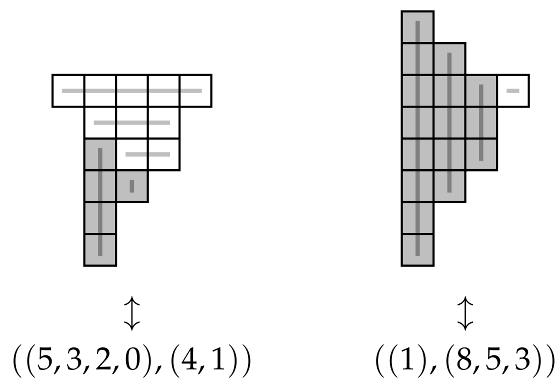


Figure 3: Graphical interpretations for odd m and even m of the bijection $\phi^{(m)}$

Proposition 3.3. *For $m \geq 0$, the number of self-conjugate partitions of n with the disparity $\frac{m(m+1)}{2}$ is*

$$sc^{(m)}(n) = \begin{cases} p(k) & \text{if } n = 4k + \frac{m(m+1)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3.1 and Proposition 3.3, we have the following corollary and as a consequence of Corollary 3.4, we have Proposition 1.2.

Corollary 3.4. *For a nonnegative integer m, we have*

$$\sum_{\lambda \in \mathcal{SC}^{(m)}} q^{|\lambda|} = q^{\frac{m(m+1)}{2}} \sum_{\mu \in \mathcal{P}} q^{4|\mu|}.$$

4 Properties of hook lengths of $\mathcal{SC}^{(m)}$

In this section we provide some properties of hook lengths of $\lambda \in \mathcal{SC}^{(m)}$.

4.1 Hook lengths of the first row or column

For the partitions $\lambda \in \mathcal{SC}^{(m)}$ and $\mu = \phi^{(m)}(\lambda)$, we give a relation between their hook lengths in the first row or the first column.

For a self-conjugate partition λ, we define the *half-even beta set* of λ by

$$\beta_{e/2}(\lambda) = \{h(i, 1)/2 \, : \, h(i, 1) \text{ is even}, \ 1 \leq i \leq \lambda_1\}.$$

Proposition 4.1. *Let $\lambda \in \mathcal{SC}^{(m)}$ with $D(\lambda) = \{\delta_1, \dots, \delta_d\}$ and $\mu = \phi^{(m)}(\lambda) = (\mu_1, \dots, \mu_\ell)$. Then the half-even beta set of λ is*

$$\beta_{e/2}(\lambda) = \begin{cases} \beta(\mu') & \text{if} \quad \delta_1 \in D_1(\lambda), \\ \beta(\mu) & \text{if} \quad \delta_1 \in D_3(\lambda). \end{cases}$$

Example 4.2. *Let λ, $\tilde{\lambda}$ be self-conjugate partitions we considered in Example 3.2. We remind that $\phi^{(3)}(\lambda) = \phi^{(4)}(\tilde{\lambda}) = \mu = (4,3,3,2,1,1)$. We note that $h_\lambda(1,1) = 21 \in D_1(\lambda)$ and $h_{\tilde{\lambda}}(1,1) = 31 \in D_3(\tilde{\lambda})$. As in Proposition 4.1, $\beta_{e/2}(\lambda) = \beta(\mu') = \{9,6,4,1\}$ and $\beta_{e/2}(\tilde{\lambda}) = \beta(\mu) = \{9,7,6,4,2,1\}$. See Figure 4 for the Young diagrams of μ, λ, $\tilde{\lambda}$, and their hook lengths.*

4.2 The relation between hook lengths of $\mathcal{SC}^{(m)}$ and $\mathcal{P}$

We start this subsection by stating a proposition.

Proposition 4.3. *For $\lambda \in \mathcal{SC}^{(m)}$ with $D(\lambda) = \{\delta_1, \delta_2, \dots, \delta_d\}$, let $\bar{\lambda}$ be the self-conjugate partition with $D(\bar{\lambda}) = \{\delta_i \in D(\lambda) \ : \ 2 \le i \le d\}$, and let $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ and $\bar{\mu}$ be the corresponding partitions of λ and $\bar{\lambda}$, respectively. If $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$, then*

$$\bar{\mu} = \begin{cases} (\mu_2, \mu_3, \dots, \mu_\ell) & \text{if} \quad \delta_1 \in D_1(\lambda) \\ (\mu_1 - 1, \mu_2 - 1, \dots, \mu_\ell - 1) & \text{if} \quad \delta_1 \in D_3(\lambda) \end{cases}$$

One may notice that there are more relations between hook lengths of corresponding partitions from Figure 4. By using Propositions 4.1 and 4.3, we have the following theorem.

Theorem 4.4. *Let $\lambda \in \mathcal{SC}^{(m)}$ be a self-conjugate partition with the disparity $m(m+1)/2$. If $\phi(\lambda) = \mu$, then for each positive integer k, the number of boxes (i,j) with $h_\lambda(i,j) = 2k$ is equal to twice the number of boxes $(\tilde{i}, \tilde{j})$ with $h_\mu(\tilde{i}, \tilde{j}) = k$.*

The following corollary is obtained directly from Theorem 4.4.

Corollary 4.5. *For a self-conjugate partition λ with the disparity $m(m+1)/2$, let $\phi(\lambda) = \mu$. Then λ is a $(2t_1, 2t_2, \dots, 2t_p)$-core partition if and only if μ is a $(t_1, t_2, \dots, t_p)$-core partition.*

We denote the set of self-conjugate $(t_1, t_2, \dots, t_p)$-core partitions $\lambda \in \mathcal{SC}^{(m)}$ of n by $\mathcal{SC}^{(m)}_{(t_1,\dots,t_p)}(n)$, and use notation $sc^{(m)}_{(t_1,\dots,t_p)}(n)$ for $|\mathcal{SC}^{(m)}_{(t_1,\dots,t_p)}(n)|$.

By using Theorems 3.1 and 4.4, we obtain the cardinality of $\mathcal{SC}^{(m)}_{(2t_1,\dots,2t_p)}(n)$.

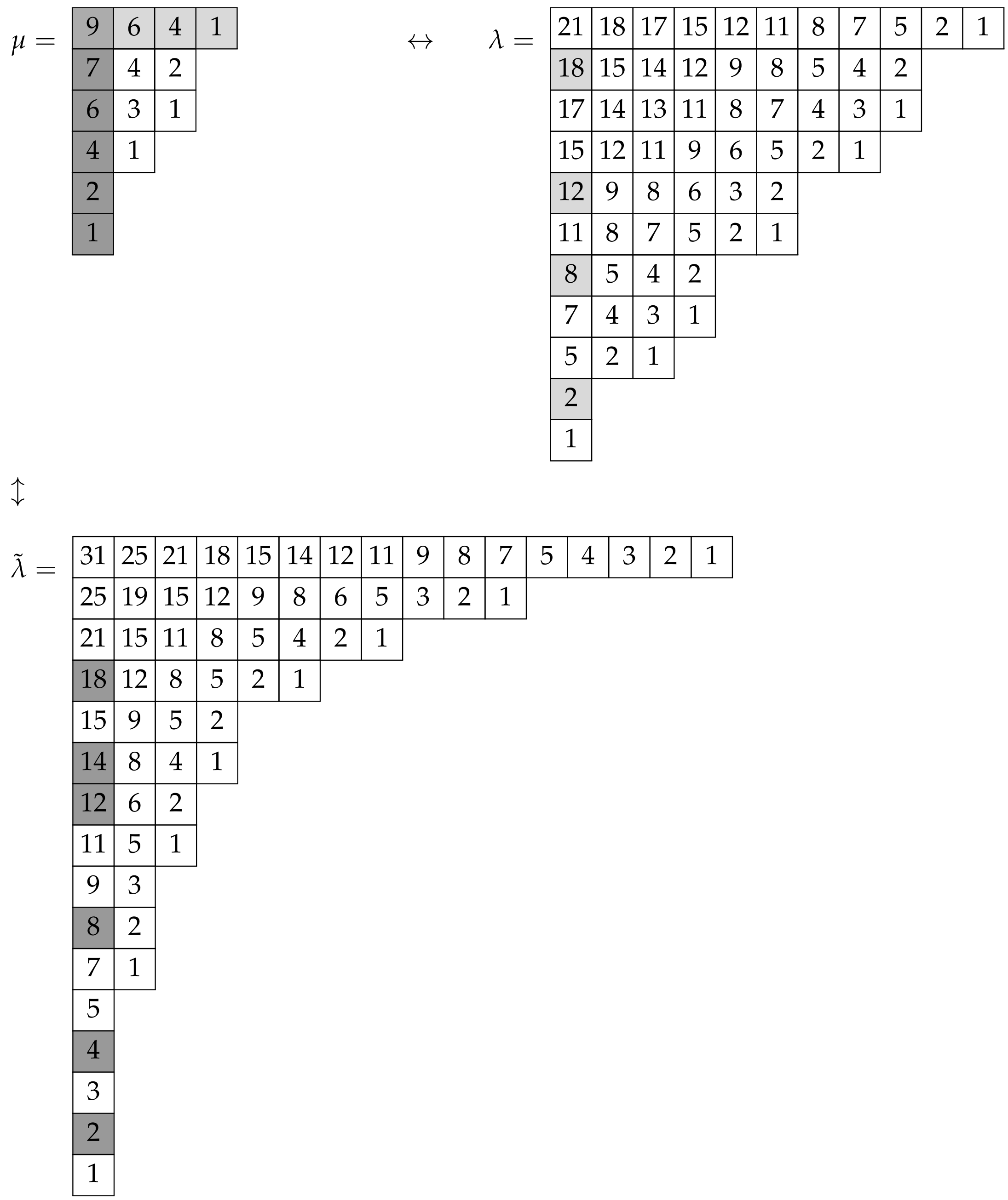


Figure 4: Hook length relations between corresponding partitions

Proposition 4.6. *For a nonnegative integer m, the number of self-conjugate $(2t_1, 2t_2, \dots, 2t_p)$-core partitions of n with the disparity $\frac{m(m+1)}{2}$ is*

$$sc^{(m)}_{(2t_1,\dots,2t_p)}(n) = \begin{cases} c_{(t_1,\dots,t_p)}(k) & \text{if} \quad n = 4k + \frac{m(m+1)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{SC}_{(t_1,\ldots,t_p)} = \bigcup_{m\geq 0} \mathcal{SC}^{(m)}_{(t_1,\ldots,t_p)}$, where $\mathcal{SC}^{(m)}_{(t_1,\ldots,t_p)}$ denote the set of self-conjugate $(t_1, t_2, \ldots, t_p)$-core partitions λ with the disparity $\frac{m(m+1)}{2}$. From the previous proposition, we have the following corollary and Proposition 1.3.

Corollary 4.7.

$$\sum_{\lambda\in\mathcal{SC}^{(m)}_{(2t_1,\ldots,2t_p)}} q^{|\lambda|} = q^{\frac{m(m+1)}{2}} \sum_{\mu\in\mathcal{P}_{(t_1,\ldots,t_p)}} q^{4|\mu|}.$$

4.3 Counting self-conjugate $(2t_1, \ldots, 2t_p)$-cores with same disparity

In this subsection, we give some sets of self-conjugate partitions each of them is counted by known special numbers.

It is well-known that there are finitely many $(t_1, ..., t_p)$-core partitions when $t_1, ..., t_p$ are relatively prime positive integers. From Proposition 4.6, we have the following result.

Corollary 4.8. *For relatively prime positive integers $t_1, ..., t_p$, the number of self-conjugate $(2t_1, ..., 2t_p)$-core partitions with the disparity $\frac{m(m+1)}{2}$ is equal to the number of $(t_1, ..., t_p)$-core partitions.*

Anderson [2] gives an interpretation for the Catalan number in terms of simultaneous core partitions, and Amderberhan and Leven [1], Yang, Zhong, and Zhou [17], Wang [15], respectively, gives an identity for the Motzkin number.

Theorem 4.9 ([2])**.** *For relatively prime integers $t_1, t_2 \geq 1$, the number of (t_1, t_2)-core partitions is*

$$c_{(t_1,t_2)} = \frac{1}{t_1+t_2}\binom{t_1+t_2}{t_1}.$$

In particular, $c_{(n,n+1)} = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

Theorem 4.10 ([15])**.** *For relatively prime integers $n, d \geq 1$, the number of $(n, n+d, n+2d)$-core partitions is*

$$c_{(n,n+d,n+2d)} = \frac{1}{n+d}\sum_{i=0}^{\lfloor n/2 \rfloor}\binom{n+d}{i, i+d, n-2i}.$$

In particular, $c_{(n,n+1,n+2)}$ is the nth Motzkin number $M_n = \sum_{i\geq 0}\frac{1}{i+1}\binom{n}{2i}\binom{2i}{i}$.

By using Corollary 4.8 and the above known results, we have the following corollary.

Corollary 4.11. *Let $m \geq 0$ be an integer.*

(a) *For relatively prime integers* $t_1, t_2 \geq 1$*, the number of self-conjugate* $(2t_1, 2t_2)$*-core partitions with the disparity* $\frac{m(m+1)}{2}$ *is*

$$sc^{(m)}_{(2t_1,2t_2)} = \frac{1}{t_1+t_2}\binom{t_1+t_2}{t_1}.$$

In particular, $sc^{(m)}_{(2n,2n+2)} = C_n$*, where* C_n *is the nth Catalan number.*

(b) *For relatively prime integers* $n, d \geq 1$*, the number of self-conjugate* $(2n, 2n+2d, 2n+4d)$*-core partitions with the disparity* $\frac{m(m+1)}{2}$ *is*

$$sc^{(m)}_{(2n,2n+2d,2n+4d)} = \frac{1}{n+d}\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n+d}{i, i+d, n-2i}.$$

In particular, $sc^{(m)}_{(2n,2n+2,2n+4)}$ *is the nth Motzkin number.*

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