Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #37, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Thrall's problem: cyclic sieving, necklaces, and branching rules

Connor Ahlbach*[1] and Joshua P. Swanson†[2]

[1]*Department of Mathematics, University of Washington, Seattle, WA 98195, USA*
[2]*Department of Mathematics, University of California San Diego, La Jolla, CA 92093, USA*

Abstract. The cyclic sieving phenomenon (CSP) of Reiner–Stanton–White offers an explanation for why many polynomials evaluate to non-negative integers at roots of unity. We reverse the usual paradigm and show how certain cyclic sieving results can be used to give new proofs of Schur expansions due to Kraśkiewicz–Weyman, Stembridge, and Schocker. These results concern the so-called higher Lie modules and branching rules for inclusions $C_a \wr S_b \hookrightarrow S_{ab}$. Extending the approach gives monomial expansions for certain Frobenius series arising from a generalization of Thrall's problem.

Keywords: cyclic sieving; necklaces; branching rules; Thrall's problem; wreath products; major index

1 Introduction

Thrall's Problem was introduced by Thrall [9] in the 1940's and is known to be a difficult instance of a *plethysm* problem. It asks for the irreducible decomposition of a certain canonical $\mathrm{GL}(V)$-module decomposition $\oplus_\lambda \mathcal{L}_\lambda$ of the tensor algebra arising from the Poincaré–Birkhoff–Witt theorem. Using the Littlewood–Richardson rule, Thrall's problem can be reduced to the case when λ is a rectangle. Thrall's problem is currently open for rectangular λ with at least 2 rows and 3 columns.

Kraśkiewicz–Weyman [5] solved Thrall's problem in the important special case when $\lambda = (n)$. Their argument hinges upon the formula

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^r) = \chi^\lambda(\sigma_n^r) \tag{1.1}$$

where we write the major index generating function as

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)},$$

*ahlbach@uw.edu. Both authors were partially supported by the National Science Foundation grant DMS-1101017.

†jswanson@ucsd.edu

ω_n is a primitive nth complex root of unity, σ_n is the n-cycle $(1\ 2\ \cdots\ n)$ in the symmetric group S_n, and χ^λ is the character of the S_n-irreducible indexed by λ. Their analysis is somewhat indirect, involving results of Lusztig and Stanley on coinvariant algebras and an intricate though beautiful argument involving ℓ-decomposable partitions.

Equation (1.1) bears a striking resemblance to the *cyclic sieving phenomenon* (CSP) of Reiner–Stanton–White [6]. Let W be a finite set on which the cyclic group $C_n = \langle \sigma_n \rangle$ acts. Given a statistic stat: $W \to \mathbb{Z}_{\geq 0}$, we say the triple $(W, C_n, W^{\mathrm{stat}}(q))$ *exhibits the CSP* if

$$\begin{aligned} W^{\mathrm{stat}}(\omega_n^r) &= \# W^{\sigma_n^r} \\ &:= \#\{w \in W : \sigma_n^r \cdot w = w\} = \chi^W(\sigma_n^r), \end{aligned} \tag{1.2}$$

where χ^W is the character of W as a C_n-module. The analogy between (1.1) and (1.2) is apparent and suggests considering a triple of the form $(\mathrm{SYT}(\lambda), \langle \sigma_n \rangle, \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q))$. However, there is no obvious cyclic action on $\mathrm{SYT}(\lambda)$, and (1.1) generally involves negatives and hence cannot arise directly as a CSP triple. Nonetheless, given the similarity of (1.1) and (1.2), it is natural to ask precisely how the CSP relates to Thrall's problem.

In Section 2, we sketch a remarkably direct, new proof of Kraśkiewicz–Weyman's result using a fundamental cyclic sieving result on words also due to Reiner–Stanton–White [6]. See the published version [1] of this extended abstract for full details. Our argument involves two main steps: (a) building on work of Klyachko [4], we give a natural, explicit monomial expansion of the Schur character of $\mathcal{L}_{(n)}$ using necklaces; and (b) we use cyclic sieving on words and the Robinson–Schensted–Knuth correspondence to convert this to a Schur expansion. The generating function in (a) arises from the *flex* statistic on words, which the authors considered in [2] in the context of *refined cyclic sieving* as a "universal" cyclic sieving statistic on words. See Definition 2.4 below. In Section 2, flex arises from explicit necklace bases of the Schur–Weyl duals of $\exp(2\pi i r/n)\!\uparrow_{C_n}^{S_n}$. Our argument also provides a thus far rare example of an instance of cyclic sieving being used to prove other results rather than vice-versa.

In Section 3 and Section 4, we summarize how the approach in Section 2 extends in two different directions to yield new, combinatorial proofs of results due to Stembridge [8] and Schocker [7] using cyclic sieving. Kraśkiewicz–Weyman's result can be thought of as determining the branching rules for $C_n \hookrightarrow S_n$ as

$$\sum_{r=1}^{n} q^r \operatorname{ch}\left(\exp(2\pi i r/n)\!\uparrow_{C_n}^{S_n}\right) = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}_n(T)} s_\lambda(\mathbf{x}) \tag{1.3}$$

where $1 \leq \mathrm{maj}_n(T) \leq n$ is the major index modulo n. Extracting the coefficient of q solves Thrall's problem for $\mathcal{L}_{(n)}$. Stembridge's result can be phrased as determining the branching rules for $\langle \sigma \rangle \hookrightarrow S_n$ for an arbitrary element σ of cycle type ν and order ℓ as

$$\sum_{r=1}^{\ell} q^r \operatorname{ch}\left(\exp(2\pi i r/\ell)\!\uparrow_{\langle \sigma \rangle}^{S_n}\right) = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}_\nu(T)} s_\lambda(\mathbf{x}). \tag{1.4}$$

See Definition 3.6 for maj_ν and a related statistic $\mathbf{maj}_\nu$, both of which arise naturally from our combinatorial manipulations with necklace generating functions and a desire to apply cyclic sieving.

Schocker gave a formula for the Schur expansion of $\mathcal{L}_{(a^b)}$ generalizing Kraśkiewicz–Weyman's result, though it typically involves many subtractions and divisions. In Section 4, we sketch a new proof of this formula as well, which makes use of the $\mathbf{maj}_\nu$ statistic from Section 3. Moreover, the preceding arguments and results strongly suggest the need to consider Thrall's problem in the larger context of general branching rules. The plethystic identity

$$\mathrm{ch}\,\mathcal{L}_{(a^b)} = h_b[\mathcal{L}_{(a)}]$$

is easy to see from representation theory. It follows that the Schur–Weyl dual of $\mathcal{L}_{(a^b)}$ is a certain representation of S_{ab} induced from a one-dimensional representation of the wreath product $C_a \wr S_b$. Consequently, we give the following generalization of Schocker's formula arising from inducing all one-dimensional representations of $C_a \wr S_b$; see Section 4 for missing definitions. Schocker's result is the $\mathcal{L}^{1,1}_{(a^b)}$ case.

Theorem 1.1 (See [7, Thm. 3.1])**.** *For all integers $a, b \geq 1$ and $r = 1, \ldots, a$, we have*

$$\mathrm{ch}\,\mathcal{L}^{r,1}_{(a^b)} = \sum_{\lambda \vdash ab} \left(\sum_{\nu \vdash b} \frac{1}{z_\nu} \sum_{\tau \mid r*\nu} \mu_\tau(\nu, r*\nu) \mathbf{a}^{a*\nu}_{\lambda,\tau} \right) s_\lambda(\mathbf{x}) \qquad \textit{and}$$

$$\mathrm{ch}\,\mathcal{L}^{r,\epsilon}_{(a^b)} = \sum_{\lambda \vdash ab} \left(\sum_{\nu \vdash b} \frac{(-1)^{b-\ell(\nu)}}{z_\nu} \sum_{\tau \mid r*\nu} \mu_\tau(\nu, r*\nu) \mathbf{a}^{a*\nu}_{\lambda,\tau} \right) s_\lambda(\mathbf{x}),$$

where

$$\mathbf{a}^{a*\nu}_{\lambda,\tau} := \#\{Q \in \mathrm{SYT}(\lambda) : \mathbf{maj}_{a*\nu}(Q) = \tau\}.$$

In our approach, the subtractions and divisions arise from the underlying combinatorics using Möbius inversion and Burnside's lemma, respectively.

In Section 5, we discuss applying aspects of our approach to Thrall's problem in general. Combinatorially, our generalization of Schocker's formula involves considering only a single one-dimensional representation of $C_a \wr S_b$ at a time, which may explain its failure to be cancellation-free. We thus consider all irreducible representations $S^{\underline{\lambda}}$ of $C_a \wr S_b$, which are indexed by the set of a-tuples $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(a)})$ of partitions with $\sum_{r=1}^a |\lambda^{(r)}| = b$. We first give the following necklace generating function expression for the corresponding characteristics.

Theorem 1.2. *For all integers $a, b \geq 1$, we have*

$$\mathrm{ch}\, S^{\underline{\lambda}}\uparrow^{S_{ab}}_{C_a \wr S_b} = \prod_{r=1}^{a} s_{\lambda^{(r)}}[\mathrm{NFD}^{\mathrm{cont}}_{a,r}(\mathbf{x})].$$

We then identify the analogues of maj_n in this context, which sends words to such a-tuples of partitions. We consequently give the following monomial expansion of the corresponding graded Frobenius series.

Theorem 1.3. *For all integers $a, b \geq 1$, we have*

$$\sum_{\underline{\lambda}} \dim S^{\underline{\lambda}} \cdot \mathrm{ch}\left(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}}\right) q^{\underline{\lambda}} = \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{maj}_a^b}(\mathbf{x}; q)$$

where the $q^{\underline{\lambda}}$ are independent indeterminates.

Having accomplished step (a) of our earlier outline, we are at present unable to apply step (b). A key missing ingredient is the lack of a useful generalization of cyclic sieving to the non-abelian groups $C_a \wr S_b$ together with a corresponding fundamental sieving result on words. In Section 5, we conclude by describing the properties of a generalized major index statistic which would allow one to convert the monomial expansions in Theorem 1.3 to Schur expansions, which would solve Thrall's problem and more generally fully determine the branching rules for the inclusions $C_a \wr S_b \hookrightarrow S_{ab}$.

2 Cyclic Sieving and Kraśkiewicz–Weyman's Result

Klyachko first observed that both $\mathcal{L}_{(n)}$ and the Schur–Weyl dual of $\exp(2\pi i/n)\uparrow_{C_n}^{S_n}$ have explicit weight space bases indexed by primitive necklaces [4, Prop. 1]. Here a *necklace* is a circularly ordered word in the letters $1, 2, \ldots$, which is *primitive* if all of its cyclic rotations are distinct. We begin by generalizing this observation.

Definition 2.1. The *frequency* of a word is the order of its stabilizer under cyclic rotations, and its *period* is the size of its orbit. These notions are constant on necklaces. Let

$$\mathrm{NFD}_{n,r} := \{\underline{\text{n}}\text{ecklaces of length } n \text{ words with } \underline{\text{f}}\text{requency } \underline{\text{d}}\text{ividing } r\}.$$

The *content* of a word or necklace is $(m_1, m_2, \ldots)$ where m_i is the number of times the letter i appears.

Theorem 2.2. *There is an explicit weight space basis for $(\mathbb{C}^m)^{\otimes n} \otimes_{\mathbb{C}S_n} \left(\exp(2\pi i r/n)\uparrow_{C_n}^{S_n}\right)$ indexed by necklaces of length n words with letters from $[m]$ and with frequency dividing r. Moreover,*

$$\mathrm{ch}\exp(2\pi i r/n)\uparrow_{C_n}^{S_n} = \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}). \tag{2.1}$$

Proof Sketch. The tensors $e_{w_1} \otimes \cdots \otimes e_{w_n} \otimes 1$ indexed by words $w_1 \cdots w_n \in [m]^n$ form a spanning set for the isomorphic module $(\mathbb{C}^m)^{\otimes n} \otimes_{\mathbb{C}C_n} \exp(2\pi i r/n)$. The relation obtained by commuting the cyclic action ensures these are well-defined on necklaces up to a scalar multiple. The result follows by carefully tracking this scalar multiple. □

Remark 2.3. Hall showed that $\mathcal{L}_{(n)}$, the degree n component of the free Lie algebra on m generators, has a weight space basis indexed by primitive necklaces of length n words with letters from $[m]$. Setting $r = 1$ in Theorem 2.2 thus gives Klyachko's observation,

$$\operatorname{ch} \mathcal{L}_{(n)} = \operatorname{ch} \exp(2\pi i r/n)\uparrow_{C_n}^{S_n} \tag{2.2}$$

in the "stable limit" as $m \to \infty$. This observation is the fundamental connection between Thrall's problem and symmetric group branching rules.

The indexing sets $\mathrm{NFD}_{n,r}$ in Theorem 2.2 are *not* disjoint. We use the following statistic to choose particular cyclic rotations of the necklaces in $\mathrm{NFD}_{n,r}$ for all $r = 1, \ldots, n$ simultaneously in such a way that the corresponding words *are* disjoint.

Definition 2.4 ([2, §8])**.** Given $w \in \mathrm{W}_n$, let $\operatorname{lex}(w)$ denote the position at which w appears in the lexicographic order of its rotations, starting at 1. The *flex* statistic is given by

$$\operatorname{flex}(w) = \operatorname{freq}(w) \cdot \operatorname{lex}(w).$$

Example 2.5. If $w = 21132113$, its necklace is the orbit

$$\{11321132, 13211321, 21132113, 32113211\}$$

listed in lexicographic order. Since w is in the third position, $\operatorname{lex}(w) = 3$. Here $\operatorname{freq}(w) = 2$, so $\operatorname{flex}(w) = 6$.

Lemma 2.6. *Each necklace in* $\mathrm{NFD}_{n,r}$ *has a unique cyclic rotation with* flex r*, for* $r = 1, \ldots, n$.

It follows from Theorem 2.2 and Lemma 2.6 that we have the following generating function tracking branching rules for $C_n \hookrightarrow S_n$.

Corollary 2.7. *We have*

$$\sum_{r=1}^{n} q^r \operatorname{ch} \exp(2\pi i r/n)\uparrow_{C_n}^{S_n} = \mathrm{W}_n^{\mathrm{cont;flex}}(\mathbf{x}; q). \tag{2.3}$$

An easy observation is that the flex statistic is "universal" for cyclic rotations on words in the following sense.

Lemma 2.8. *[2, Lemma 8.3] Let W be a finite set of length n words closed under the C_n-action of cyclic rotation. Then, the following triple exhibits the CSP:*

$$\left(W, C_n, W^{\mathrm{flex}}(q)\right).$$

Proof sketch. We may suppose W is a single orbit and directly evaluate the generating function at roots of unity. Alternatively, we may use the "stabilizer–order" as in [6]. □

A much more difficult observation is that maj is "universal" for S_n-actions on words in the following sense, which is a corollary of [6, Theorem 8.3, Proposition 4.4]. A very similar observation appeared in [3, Prop. 3.1].

Theorem 2.9 ([6, Theorem 8.3, Proposition 4.4])**.** *Let W be a finite set of length n words closed under the S_n-action. Then, the triple*

$$\left(W, C_n, W^{\mathrm{maj}}(q)\right)$$

exhibits the CSP.

Corollary 2.10. *We have*

$$\mathrm{W}_n^{\mathrm{cont;flex}}(\mathbf{x};q) = \mathrm{W}_n^{\mathrm{cont;maj}_n}(\mathbf{x};q). \tag{2.4}$$

Proof. The polynomial in a CSP triple is determined uniquely modulo $q^n - 1$. By Lemma 2.8 and Theorem 2.9, flex and maj_n are equidistributed for each fixed content. □

The RSK algorithm gives a bijection from words w of length n to pairs of tableaux (P, Q) of the same shape $\lambda \vdash n$. The content of w agrees with the content of the semistandard tableau P and the descent set of w agrees with the descent set of the standard tableau Q. Since the Schur $s_\lambda(\mathbf{x})$ is the content generating function over semistandard tableaux of shape λ, and since maj_n can be computed in terms of the descent set, we have

$$\mathrm{W}_n^{\mathrm{cont;maj}_n}(\mathbf{x};q) = \sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}_n(T)} s_\lambda(\mathbf{x}). \tag{2.5}$$

Kraśkiewicz–Weyman's result, (1.3), now follows immediately by combining (2.3), (2.4), and (2.5). Their solution to Thrall's problem for $\mathcal{L}_{(n)}$ follows by using (2.2) and comparing q^1 coefficients in (1.3).

Remark 2.11. In [2], the authors proved a refinement of the cyclic sieving result underlying Theorem 2.9. Since earlier approaches involving representation theory could not readily be adapted to this refinement, the argument instead uses completely different and highly combinatorial techniques. Thus, the arguments above together with [2] give a self-contained proof of Kraśkiewicz–Weyman's result. It would be very interesting to find a refinement of Thrall's problem that corresponds to the refined cyclic sieving result in [2].

3 Induced Representations of Cyclic Subgroups of S_n

We now briefly sketch how to modify the argument in Section 2 to give Stembridge's generalization (1.4) of Kraśkiewicz–Weyman's result. Throughout this section, let $\sigma \in S_n$, let C be the cyclic group generated by σ, and let $\ell := \#C$ be the order of σ.

The relevant analogue of the sets of necklaces $\mathrm{NFD}_{n,r}$ is as follows. The argument in Theorem 2.2 holds verbatim in this larger context.

Definition 3.1. Suppose $\mathcal{O}$ is an orbit of W_n under the C-action. The *frequency* of $\mathcal{O}$ is the stabilizer-order of any element of $\mathcal{O}$, and the *period* of $\mathcal{O}$ is $\#\mathcal{O}$. Let

$$\mathrm{OFD}_{C,r} := \{C\text{-orbits of } W_n \text{ with frequency dividing } r\}.$$

Theorem 3.2. *There is an explicit weight space basis for* $(\mathbb{C}^m)^{\otimes n} \otimes_{\mathbb{C}S_n} \left(\exp(2\pi i r/\ell)\uparrow_C^{S_n}\right)$ *indexed by C-orbits of length n words with letters from $[m]$ and with frequency dividing r. Moreover,*

$$\mathrm{ch}\left(\exp(2\pi i r/\ell)\uparrow_C^{S_n}\right) = \mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x}). \tag{3.1}$$

Our approach is broadly to replace $\mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x})$ with a necklace generating function, apply the cyclic sieving results from Section 2 to get a major index generating function on words, and then apply RSK to get a Schur expansion.

Notation 3.3. For the rest of the section, suppose that σ has disjoint cycle decomposition $\sigma = \sigma_1 \cdots \sigma_k$ with $\nu_i := |\sigma_i|$. Consequently, $\ell = |\langle\sigma\rangle| = \mathrm{lcm}(\nu_1, \ldots, \nu_k)$. Further, write

$$C_\nu := \{\sigma_1^{r_1} \cdots \sigma_k^{r_k} \in S_n : r_1, \ldots, r_k \in \mathbb{Z}\} \cong C_{\nu_1} \times \cdots \times C_{\nu_k}$$

where $C_{\nu_i} := \langle\sigma_i\rangle \subset S_n$. Thus, we have $C \subset C_\nu \subset S_n$.

In Section 2, we considered the C_n-orbits of W_n as necklaces N. We may group together C_n-orbits of W_n according to their stabilizers by letting

$$\mathrm{NF}_{n,r} := \{\text{necklaces of length } n \text{ words with frequency } r\}. \tag{3.2}$$

Similarly, $\mathrm{NFD}_{n,r}$ consists of C_n-orbits of W_n whose stabilizer is contained in the common stabilizer of $\mathrm{NF}_{n,r}$.

Analogously, the C_ν-orbits of W_n can be identified with products of necklaces $N_1 \times \cdots \times N_k$ or equivalently with tuples $(N_1, \ldots, N_k)$ where N_j is a necklace of length ν_j. Since

$$\mathrm{Stab}_{C_\nu}(N_1 \times \cdots \times N_k) = \prod_{j=1}^{k} \mathrm{Stab}_{C_{\nu_j}}(N_j),$$

we may group together C_ν-orbits of W_n according to their stabilizers as follows.

Definition 3.4. For any $\nu = (\nu_1, \ldots, \nu_k)$ and $\rho = (\rho_1, \ldots, \rho_k)$, let

$$\begin{aligned} \mathrm{NF}_{\nu,\rho} &:= \mathrm{NF}_{\nu_1,\rho_1} \times \cdots \times \mathrm{NF}_{\nu_k,\rho_k}, \\ \mathrm{NFD}_{\nu,\rho} &:= \mathrm{NFD}_{\nu_1,\rho_1} \times \cdots \times \mathrm{NFD}_{\nu_k,\rho_k}. \end{aligned}$$

The elements of $\mathrm{NF}_{\nu,\rho}$ all have the same stabilizer, and the elements of $\mathrm{NFD}_{\nu,\rho}$ are precisely those whose stabilizer is contained in the common stabilizer of elements of $\mathrm{NF}_{\nu,\rho}$. We write $\rho \mid \nu$ to mean that $\rho_i \mid \nu_i$ for all $i = 1, \ldots, r$. Note that $\mathrm{NF}_{\nu,\rho} \neq \varnothing$ if and only if $\rho \mid \nu$.

Given a group G acting on a set W and a subgroup H of G, each G-orbit of W is partitioned into H-orbits. Consequently, C_ν-orbits of W_n are unions of C-orbits. Using this idea, one may decompose $\mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x})$ as a sum of necklace generating functions $\mathrm{NF}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x})$. In order to apply cyclic sieving, we must instead use the related generating functions $\mathrm{NFD}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x})$. Doing so, one eventually arrives at the following decomposition.

Lemma 3.5. *For $r = 1, \ldots, \ell$,*

$$\mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x}) = \sum \mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}),$$

where the sum is over all k-tuples of integers $\tau \in [\nu_1] \times \cdots \times [\nu_k]$ such that $\sum_{j=1}^k \frac{\ell}{\nu_j}\tau_j \equiv_\ell r$.

Let $\mathrm{M}_{n,r} := \{w \in \mathrm{W}_n : \mathrm{maj}_n(w) = r\}$. It follows from (2.4) that

$$\mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}) = \prod_{j=1}^{k} \mathrm{NFD}^{\mathrm{cont}}_{\nu_j,\tau_j}(\mathbf{x}) = \prod_{j=1}^{k} \mathrm{M}^{\mathrm{cont}}_{\nu_j,\tau_j}(\mathbf{x}). \tag{3.3}$$

Interpreting the right-hand side of (3.3) in terms of words and comparing with the indexing set in Lemma 3.5 motivates the following variations on the major index.

Definition 3.6. Suppose $\nu \vDash n$, $\tau \in [\nu_1] \times \cdots \times [\nu_k]$, and $\ell = \mathrm{lcm}(\nu_1, \ldots, \nu_k)$. Let $\mathbf{maj}_\nu \colon \mathrm{W}_n \to [\nu_1] \times \cdots \times [\nu_k]$ be defined as follows. For $w \in \mathrm{W}_n$, write $w = w^1 \cdots w^k$ where each w^j is a word in W_{ν_j}. Set

$$\mathbf{maj}_\nu(w) := (\mathrm{maj}_{\nu_1}(w^1), \ldots, \mathrm{maj}_{\nu_k}(w^k)),$$

where maj_n is the usual major index modulo n normalized to lie in the range $1, \ldots, n$. Furthermore, let $\mathrm{maj}_\nu \colon \mathrm{W}_n \to [\ell]$ be defined by

$$\mathrm{maj}_\nu(w) := \sum_{j=1}^{k} \frac{\ell}{\nu_j}\, \mathbf{maj}_\nu(w)_j \qquad (\mathrm{mod}\ \ell),$$

Note that both $\mathbf{maj}_\nu$ and maj_ν are functions of $\mathrm{Des}(w)$. We may thus define both $\mathbf{maj}_\nu$ and maj_ν on $Q \in \mathrm{SYT}(n)$ using only $\mathrm{Des}(Q)$ in the same way. Equivalently, we may set $\mathbf{maj}_\nu(Q) := \mathbf{maj}_\nu(w)$ and $\mathrm{maj}_\nu(Q) := \mathrm{maj}_\nu(w)$ for any w such that $Q = Q(w)$.

Example 3.7. Let $\nu = (5, 3, 3)$ and $w = 44121361631$, so that $\ell = 15$, $w_1 = 44121$, $w_2 = 361$, and $w_3 = 631$. We have

$$\mathbf{maj}_\nu(w) = (\mathrm{maj}_5(w_1), \mathrm{maj}_3(w_2), \mathrm{maj}_3(w_3)) = (1, 2, 3)$$

and, hence, $\mathrm{maj}_\nu(w) = \frac{15}{5} \cdot 1 + \frac{15}{3} \cdot 2 + \frac{15}{3} \cdot 3 = 13 \ (\mathrm{mod}\ 15)$.

Using (3.1), (3.3), and RSK exactly as in Section 2 now gives Stembridge's result, (1.4).

4 Inducing linear characters from $C_a \wr S_b$ to S_{ab}

We next sketch how to apply the approach of Section 2 and Section 3 to prove the generalization of Schocker's formula for the Schur expansion of $\mathcal{L}_{(a^b)}$ in Theorem 1.1.

Let $\chi^{r,1}$ denote the one-dimensional representation of the group $C_a \wr S_b$ of a-colored permutations of $[b]$ obtained by taking the wreath product $\exp(2\pi i/a) \wr 1_b$ where 1_b is the trivial character of S_b. Analogously, let $\chi^{r,\epsilon} = \exp(2\pi i/a) \wr \operatorname{sgn}_b$. For $b > 1$, these are precisely the inequivalent one-dimensional representations of $C_a \wr S_b$. There is a natural inclusion $C_a \wr S_b \hookrightarrow S_{ab}$. Using well-known properties of wreath products, one may show that $\operatorname{ch} \mathcal{L}_{(a^b)} = \chi^{1,1}\uparrow_{C_a \wr S_b}^{S_{ab}}$. Consequently, we write

$$\mathcal{L}^{r,1}_{(a^b)} := \chi^{r,1}\uparrow_{C_a \wr S_b}^{S_{ab}} \qquad \text{and} \qquad \mathcal{L}^{r,\epsilon}_{(a^b)} := \chi^{r,\epsilon}\uparrow_{C_a \wr S_b}^{S_{ab}}.$$

In this notation, Thrall's problem reduces to finding the Schur expansion of $\mathcal{L}^{1,1}_{(a^b)}$.

It is possible to extend the argument in Theorem 2.2 to the Schur–Weyl duals of $\mathcal{L}^{1,1}_{(a^b)}$ and $\mathcal{L}^{1,\epsilon}_{(a^b)}$ using explicit weight space bases indexed in terms of necklaces. Alternatively, one may apply well-known properties of plethysms and wreath products to Theorem 2.2 directly. In either case, we arrive at the following identities.

Lemma 4.1. *We have*

$$\operatorname{ch} \mathcal{L}^{r,1}_{(a^b)} = \left(\!\!\binom{\mathrm{NFD}_{a,r}}{b}\!\!\right)^{\mathrm{cont}}(\mathbf{x}) \qquad \text{and} \qquad \operatorname{ch} \mathcal{L}^{r,\epsilon}_{(a^b)} = \binom{\mathrm{NFD}_{a,r}}{b}^{\mathrm{cont}}(\mathbf{x}). \tag{4.1}$$

Here $\left(\!\binom{A}{k}\!\right)$ and $\binom{A}{k}$ are the sets of all k-element multisubsets or subsets of A, respectively.

In words, $\operatorname{ch} \mathcal{L}^{r,1}_{(a^b)}$ is the content generating function of multisets of b necklaces each of length a and with frequency dividing r. Such multisets may be thought of as S_b-orbits of length a lists of necklaces in $\mathrm{NFD}_{a,r}$. We may thus use Burnside's lemma to count the number of these multisets by averaging over the number of such lists fixed by permutations in S_b. Doing so, we arrive at the following.

Lemma 4.2. *We have*

$$\left(\!\!\binom{\mathrm{NFD}_{a,r}}{b}\!\!\right)^{\mathrm{cont}}(x_1, x_2, \ldots) = \sum_{\nu \vdash b} \frac{1}{z_\nu} \prod_{j=1}^{\ell(\nu)} \mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^{\nu_j}, x_2^{\nu_j}, \ldots).$$

Applying a similar argument together with a sign-reversing involution gives an analogous expression for $\mathcal{L}^{r,\epsilon}_{(a^b)}$ which differs only by the factor $(-1)^{b-\ell(\nu)}$.

We may interpret $\mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^k, x_2^k, \ldots)$ as the generating function for k-tuples of length a necklaces with frequency dividing r of the form $(N, \ldots, N)$, repeating the same necklace

k times. By concatenation, we may equivalently view such tuples as length ak necklaces whose frequency f satisfies $k \mid f \mid rk$. Hence,

$$\mathrm{NFD}_{a,r}^{\mathrm{cont}}(x_1^k, x_2^k, \ldots) = \sum_{k|f|rk} \mathrm{NF}_{ak,f}^{\mathrm{cont}}(x_1, x_2, \ldots). \tag{4.2}$$

Definition 4.3. Suppose $d \mid e$ and $f \mid e$. Set

$$\mu_f(d, e) := \sum_{\substack{g \\ \text{s.t. } \operatorname{lcm}(f,d)|g|e}} \mu\left(\frac{g}{f}\right),$$

and extend $\mu_f(d, e)$ to sequences multiplicatively.

Applying Lemma 4.2, (4.2), and Möbius inversion, we arrive at the following expression. Note the appearance of the expressions $\mathrm{NFD}_{\nu,\tau}^{\mathrm{cont}}(\mathbf{x})$ from Section 3.

Lemma 4.4. *We have*

$$\operatorname{ch} \mathcal{L}_{(a^b)}^{r,1} = \sum_{\nu \vdash b} \frac{1}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r*\nu)\, \mathrm{NFD}_{a*\nu,\tau}^{\mathrm{cont}}(\mathbf{x}),$$

$$\operatorname{ch} \mathcal{L}_{(a^b)}^{r,\epsilon} = \sum_{\nu \vdash b} \frac{(-1)^{b-\ell(\nu)}}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r*\nu)\, \mathrm{NFD}_{a*\nu,\tau}^{\mathrm{cont}}(\mathbf{x}).$$

Here $r * \nu := (r\nu_1, r\nu_2, \ldots)$.

Finally, one may deduce Theorem 1.1 from Lemma 4.4 by applying (3.3), interpreting the result in terms of $\mathbf{maj}_\nu$, and applying RSK as before.

5 Higher Lie Modules and Branching Rules

The argument in Section 2 solves Thrall's problem for $\lambda = (n)$ by considering all branching rules for $C_n \hookrightarrow S_n$ simultaneously and using cyclic sieving and RSK to convert from the monomial to the Schur basis. We now turn to analogous considerations for the higher Lie modules and more generally branching rules for $C_a \wr S_b \hookrightarrow S_{ab}$. We give an analogue of the flex statistic and the monomial basis expansion for such branching rules from Section 2. We then show how to convert from the monomial to the Schur basis assuming the existence of a certain statistic on words we call mash which interpolates between maj_n and the shape under RSK.

Theorem 1.2 follows from Theorem 2.2, standard properties of plethysm and wreath products, and a straightforward lemma concerning the interaction of induction and wreath products. The monomial expansions in Theorem 1.3 use the following statistics. The proof of Theorem 1.3 proceeds by applying RSK to expressions arising from Theorem 1.2.

Definition 5.1. Fix integers $a, b \geq 1$. Construct a statistic

$$\mathrm{maj}_a^b \colon \mathrm{W}_{ab} \to \{a\text{-tuples of partitions with total size } b\}$$

as follows. Given $w \in \mathrm{W}_{ab}$, write $w = w^1 \cdots w^b$ where $w^j \in \mathrm{W}_a$. In this way, consider w as a word of size b whose letters are in W_a. For each $r \in [a]$, let $w^{(r)}$ denote the subword of w whose letters are those w^j such that $\mathrm{maj}_a(w^j) = r$. Totally order W_a lexicographically, so that RSK is well-defined for words with letters from W_a. Set

$$\mathrm{maj}_a^b(w) := (\mathrm{sh}(w^{(1)}), \ldots, \mathrm{sh}(w^{(a)})).$$

Example 5.2. Let $w = 212023101241$ and suppose $a = 3$, $b = 4$. Grouping into 3-letter sequences, write $w = (212)(023)(101)(241)$. The parenthesized terms have maj_3 statistics $1, 3, 1, 2$. When computing $\mathrm{maj}_3^4(w)$, we have $w^{(1)} = (212)(101), w^{(2)} = (241), w^{(3)} = (023)$. Since $(101) <_{\mathrm{lex}} (212)$, $\mathrm{sh}(w^{(1)}) = \mathrm{sh}(21) = (1,1)$. Hence,

$$\mathrm{maj}_3^4(212023101241) = ((1,1),(1),(1)).$$

While Theorem 1.3 determines the monomial expansion of the graded Frobenius series tracking branching rules for $C_a \wr S_b \hookrightarrow S_{ab}$, we are really interested in the Schur expansion. The key properties used in the proof of Kraśkiewicz–Weyman's result to convert from the monomial basis to the Schur basis were (i) that maj_n is equidistributed with flex on each W_α and (ii) $\mathrm{maj}_n(w)$ depends only on $Q(w)$. In order to apply a similar argument for $\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}})$, we need a statistic with the following properties.

Problem 5.3. *Fix $a, b \geq 1$. Find a statistic*

$$\mathrm{mash}_a^b \colon \mathrm{W}_{ab} \to \{a\text{-tuples of partitions with total size } b\}$$

(i) where for all $\alpha \vDash ab$, maj_a^b and mash_a^b are equidistributed on W_α; and

(ii) if $v, w \in \mathrm{W}_{ab}$ satisfy $Q(v) = Q(w)$, then $\mathrm{mash}_a^b(v) = \mathrm{mash}_a^b(w)$.

Corollary 5.4. *Suppose mash_a^b satisfies Properties (i) and (ii) in Problem 5.3. Then*

$$\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}}) = \sum_{\nu \vdash ab} \frac{\#\{Q \in \mathrm{SYT}(\nu) : \mathrm{mash}_a^b(Q) = \underline{\lambda}\}}{\dim(S^{\underline{\lambda}})} s_\nu(\mathbf{x}),$$

where $\mathrm{mash}_a^b(Q) := \mathrm{mash}_a^b(w)$ for any $w \in \mathrm{W}_{ab}$ with $Q(w) = Q$.

When $a = 1$ and $b = n$, we may replace $\underline{\lambda}$ with $\lambda \vdash n$, Under this identification, $\mathrm{maj}_1^n(w) = \mathrm{sh}(w)$. When $a = n$ and $b = 1$, we may replace $\underline{\lambda}$ with an element $r \in [n]$. Under this identification, we may set $\mathrm{mash}_n^1(w) = \mathrm{maj}_n(w)$. In each case, Properties (i) and (ii) are satisfied. In this sense mash_a^b interpolates between the major index maj_n and the shape under RSK, hence the name. While maj_a^b trivially satisfies Property (i), it is easy to check that it fails Property (ii) already when $a = b = 2$, regardless of the total ordering.

Acknowledgements

Many thanks go to our advisor, Sara Billey, for extensive comments related to this project. We would also like to thank Vic Reiner, Christophe Reutenauer, Brendon Rhoades, John Stembridge, and Sheila Sundaram for helpful conversations on this material.

References

[1] C. Ahlbach and J. P. Swanson. “Cyclic sieving, necklaces, and branching rules related to Thrall’s problem”. *Electron. J. Combin.* **25**.4 (2018), Paper 4.42, 38 pp. Link.

[2] C. Ahlbach and J. P. Swanson. “Refined cyclic sieving on words for the major index statistic”. *European J. Combin.* **73** (2018), pp. 37–60. Link.

[3] A. Berget, S.-P. Eu, and V. Reiner. “Constructions for cyclic sieving phenomena”. *SIAM J. Discrete Math.* **25**.3 (2011), pp. 1297–1314. Link.

[4] A. A. Klyachko. “Lie elements in the tensor algebra”. *Siberian Mathematical Journal* **15**.6 (1974), pp. 914–920.

[5] W. Kraśkiewicz and J. Weyman. “Algebra of coinvariants and the action of a Coxeter element”. *Bayreuth. Math. Schr.* 63 (2001), pp. 265–284.

[6] V. Reiner, D. Stanton, and D. White. “The cyclic sieving phenomenon”. *J. Combin. Theory Ser. A* **108**.1 (2004), pp. 17–50. Link.

[7] M. Schocker. “Multiplicities of higher Lie characters”. *J. Aust. Math. Soc.* **75**.1 (2003), pp. 9–21. Link.

[8] J. R. Stembridge. “On the eigenvalues of representations of reflection groups and wreath products”. *Pacific J. Math.* **140**.2 (1989), pp. 353–396. Link.

[9] R. M. Thrall. “On symmetrized Kronecker powers and the structure of the free Lie ring”. *Amer. J. Math.* **64** (1942), pp. 371–388. Link.