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Some identities involving second kind Stirling numbers of types B and D

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Abstract. Using Reiner's definition of Stirling numbers of the second kind in types B and D, we generalize two well known identities concerning the classical Stirling numbers of the second kind. The first relates them with Eulerian numbers and the second uses them as entries in a transition matrix between the elements of two standard bases of the polynomial ring in one variable.

Résumé. Les nombres de Stirling de la deuxième espèce en type B et D ont été introduits par Reiner. En utilisant ces nombres, nous généralisons deux identités bien connues sur les nombres classiques de Stirling de la deuxième espèce aux groupes de Coxeter de type B et D. La première identité relie ces nombres avec les nombres Euleriens, et la deuxième interprète les nombres de Stirling de la deuxième espèce comme les entrées de la matrice de passage entre deux bases classiques de l'anneau des polynômes en une variable.

Keywords: Coxeter groups, Stirling numbers of the second kind, Eulerian numbers, falling factorial, hyperplane arrangements, descent number, permutation statistics

1 Introduction

The *Stirling number of the second kind*, denoted $S(n,k)$, is defined as the number of partitions of the set $[n] := \{1,\ldots,n\}$ into k blocks (see [19, page 81]). Stirling numbers of the second kind arise in a variety of problems in enumerative combinatorics; they have many combinatorial interpretations, and have been generalized in various contexts and in different ways.

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In the geometric theory of Coxeter groups they appear as follows. For any finite Coxeter group W, there is a corresponding hyperplane arrangement $\mathcal{W}$, whose elements are the reflecting hyperplanes of W. Associated with $\mathcal{W}$, there is the set of all the intersections of these hyperplanes, ordered by reverse inclusion, called the *intersection lattice*, and denoted $L(\mathcal{W})$ (see e.g. [7, 18]). It is well known that in type A, $L(\mathcal{A}_n)$ is isomorphic to the lattice of the set partitions of $[n]$. By this identification, the subspaces of dimension $n-k$ are counted by $S(n,k)$. In this geometric context, Stirling numbers of the second kind are usually called Whitney numbers (see [20, 21] for more details).

In types B and D, Zaslavsky [21] gave a description of $L(\mathcal{B}_n)$ and $L(\mathcal{D}_n)$ by using the general theory of signed graphs. Then, Reiner [15] gave a different combinatorial representation of $L(\mathcal{B}_n)$ and $L(\mathcal{D}_n)$ in terms of new types of set partitions, called B_n- and D_n-partitions. We call the number of B_n- (resp. D_n-) partitions with k pairs of nonzero blocks the *Stirling numbers of the second kind of type B* (resp. *type D*) and denote them by $S_B(n,k)$ (resp. $S_D(n,k)$).

The posets of B_n- and D_n-partitions, as well as their corresponding intersection lattices, have been studied in several papers [5, 6, 7, 10, 11, 20], from algebraic, topological and combinatorial points of view. However, to our knowledge, two famous identities concerning the classical Stirling numbers of the second kind (see e.g. Bona [8, Theorems 1.8 and 1.17]) have not been generalized to types B and D in a combinatorial way: the first identity involves the Eulerian numbers, and the second one formulates a change of bases in $\mathbb{R}[x]$, both are described below.

The original definition of the *Eulerian numbers* was first given by Euler in an analytic context [12, §13]. Later, they began to appear in combinatorial problems, as the Eulerian number $A(n,k)$ counts the number of permutations in the symmetric group S_n having $k-1$ descents, where a *descent* of $\sigma \in S_n$ is an element of the *descent set* of σ, defined by:

$$\mathrm{Des}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}. \tag{1.1}$$

We denote by $\mathrm{des}(\sigma) := |\mathrm{Des}(\sigma)|$ the *descent number*.

The first mentioned famous identity relating Stirling numbers of the second kind and Eulerian numbers is the following one, see e.g. [8, Theorem 1.17]:

Theorem 1. *For all positive integers n, r where $n \geq r$, we have*

$$S(n,r) = \frac{1}{r!} \sum_{k=0}^{r} A(n,k) \binom{n-k}{r-k}. \tag{1.2}$$

The second identity arises when one expresses the standard basis of the polynomial ring $\mathbb{R}[x]$ as a linear combination of the basis consisting of the falling factorials (see e.g. [1, Prop. 3.24(i)] and Boyadzhiev [9]):

Theorem 2. *Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have*

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k, \tag{1.3}$$

where $[x]_k := x(x-1)\cdots(x-k+1)$ is the falling factorial of degree k and $[x]_0 := 1$.

There are some known proofs for the last identity. A combinatorial one, realizing x^n as the number of functions from the set $\{1,\ldots,n\}$ to the set $\{1,\ldots,x\}$ (for an integer x), is presented in [19, Eqn. (1.94d)]. A first geometric proof is due to Knop [13]. A similar geometric way to interpret this equality was suggested to us by Reiner [16], and will be used later on.

In this paper, we use Stirling numbers of the second kind of types B and D, in order to generalize the identities stated above in Equations (1.2) and (1.3). Theorems 3 and 4 below are generalizations of the first identity for types B and D. The way to prove them is by providing explicit procedures to construct ordered set partitions starting from the elements of the corresponding Coxeter groups.

Theorems 5 and 6 generalize the second identity. We present here a geometric approach, suggested to us by Reiner [16], which is based on some geometric characterizations of the intersection lattice of types B and D.

The rest of the paper is organized as follows. Sections 2 and 3 present the known generalizations of Eulerian numbers and set partitions, respectively, to the Coxeter groups of types B and D. In Sections 4 and 5, we state our generalizations and prove some of them.

2 Eulerian numbers of types B and D

We start with some notation. For $n \in \mathbb{N}$, we set

$$[n] := \{1,\ldots,n\} \text{ and } [\pm n] := \{\pm 1,\ldots,\pm n\}.$$

For a subset $B \subseteq [\pm n]$, we denote by $-B$ the set obtained by negating all the elements of B, and by $\pm B$ we denote the unordered pair of sets $B, -B$.

Let (W,S) be a Coxeter system. As usual, denote by $\ell(w)$ the *length* of $w \in W$, which is the minimum k satisfying $w = s_1 \cdots s_k$ with $s_i \in S$. The *(right) descent set* of $w \in W$ is defined to be $\mathrm{Des}(w) := \{s \in S \mid \ell(ws) < \ell(w)\}$. A combinatorial characterization of $\mathrm{Des}(w)$ in type A, is given by Equation (1.1) above. Now we recall analogous descriptions for types B and D.

We denote by B_n the group of all bijections β of the set $[\pm n]$ onto itself such that $\beta(-i) = -\beta(i)$ for all $i \in [\pm n]$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $[n]$, or as the *hyperoctahedral group* of rank n. If $\beta \in B_n$, we write $\beta = [\beta(1), \ldots, \beta(n)]$ and we call this the *window* notation of β.

As a set of generators for B_n we take $S_B := \{s_0^B, s_1^B, \ldots, s_{n-1}^B\}$ where for $i \in [n-1]$

$$s_i^B := [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ and } s_0^B := [-1, 2, \ldots, n].$$

It is well known that (B_n, S_B) is a Coxeter system of type B (see e.g. [4, §8.1]). The following characterization of the (right) descent set of $\beta \in B_n$ is well known [4]:

Proposition 1. *Let $\beta \in B_n$. Then*

$$\mathrm{Des}_B(\beta) = \{i \in [0, n-1] \mid \beta(i) > \beta(i+1)\},$$

where $\beta(0) := 0$ (we use the usual order on the integers). In particular, $0 \in \mathrm{Des}_B(\beta)$ is a descent if and only if $\beta(1) < 0$. We set $\mathrm{des}_B(\beta) := |\mathrm{Des}_B(\beta)|$.

For all positive integers $n \geq k$, we set

$$A_B(n, k) := |\{\beta \in B_n \mid \mathrm{des}_B(\beta) = k\}|,$$

and we call them the *Eulerian numbers of type B*.

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an *even* number of negative entries in their window notation. This subgroup is usually called the *even-signed permutation group*. As a set of generators for D_n we take $S_D := \{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$ where for $i \in [n-1]$

$$s_i^D := s_i^B \text{ and } s_0^D := [-2, -1, 3, \ldots, n].$$

It is well known that (D_n, S_D) is a Coxeter system of type D, and there is a direct combinatorial way to compute the (right) descent set of $\delta \in D_n$ (see e.g. [4, §8.2]):

Proposition 2. *Let $\delta \in D_n$. Then*

$$\mathrm{Des}_D(\delta) = \{i \in [0, n-1] \mid \delta(i) > \delta(i+1)\},$$

where $\delta(0) := -\delta(2)$. In particular, $0 \in \mathrm{Des}_D(\delta)$ if and only if $\delta(1) + \delta(2) < 0$. We set $\mathrm{des}_D(\delta) := |\mathrm{Des}_D(\delta)|$.

For all positive integers $n \geq k$, we set:

$$A_D(n, k) := |\{\delta \in D_n \mid \mathrm{des}_D(\delta) = k\}|,$$

and we call them the *Eulerian numbers of type D*.

For example, if $\delta = [1, -3, 4, -5, -2, -6]$, then:

$$\mathrm{Des}_D(\delta) = \{0, 1, 3, 5\}, \text{ but } \mathrm{Des}_B(\delta) = \{1, 3, 5\}.$$

3 Set partitions of types B and D

In this section, we introduce the set partitions of types B and D as defined by Reiner [15].

As mentioned above, we denote by $L(\mathcal{W})$ the intersection lattice corresponding to the Coxeter hyperplane arrangement $\mathcal{W}$ of a finite Coxeter group W. We will focus only on the hyperplane arrangements of types A, B and D. In terms of the coordinate functions $x_1, \ldots, x_n$ in $\mathbb{R}^n$, they can be defined as follows:

$$\begin{array}{rcl} \mathcal{A}_n & := & \{\{x_i = x_j\} \mid 1 \le i < j \le n\}, \\ \mathcal{B}_n & := & \{\{x_i = \pm x_j\} \mid 1 \le i < j \le n\} \cup \{\{x_i = 0\} \mid 1 \le i \le n\}, \\ \mathcal{D}_n & := & \{\{x_i = \pm x_j\} \mid 1 \le i < j \le n\}. \end{array}$$

It is well known that in type A, the intersection lattice $L(\mathcal{A}_n)$ is isomorphic to the lattice of set partitions of $[n]$. In type B, let us consider the following element of $L(\mathcal{B}_9)$:

$$\{x_1 = -x_3 = x_6 = x_8 = x_9, x_2 = x_4 = 0, x_5 = -x_7\}.$$

It can be easily presented as the following set partition of $[\pm 9]$:

$$\{\{1,-3,6,8,-9\},\{-1,3,-6,-8,9\},\{2,-2,4,-4\},\{5,-7\},\{-5,7\}\}.$$

This probably was Reiner's motivation to define the set partitions of type B as follows.

Definition 1. *A B_n-*partition *is a set partition of $[\pm n]$ into blocks such that the following conditions are satisfied.*

- *There exists at most one block satisfying $-C = C$. This block is called the* zero-block *(if it exists). It is a subset of $[\pm n]$ of the form $\{\pm i \mid i \in S\}$ for some $S \subseteq [n]$.*
- *Now let C be an arbitrary block. If C appears as a block in the partition, then $-C$ also appears in that partition.*

A similar definition holds for set partitions of type D.

Definition 2. *A D_n-*partition *is a B_n-partition such that the zero-block, if exists, contains at least two positive elements.*

We denote by $S_B(n,r)$ (*resp.* $S_D(n,r)$) the number of B_n- (*resp.* D_n-) partitions having exactly r pairs of nonzero blocks. These numbers are called *Stirling numbers (of the second kind) of type B* (*resp.* D).

We now define the concept of an *ordered* set partition.

Definition 3. *A B_n-partition (or D_n-partition) is* ordered *if the set of blocks is totally ordered and the following conditions are satisfied.*

- *If the zero-block exists, then it appears as the first block.*
- *For each block C which is not the zero-block, the blocks C and $-C$ occupy adjacent places.*

We indicate ordered partitions by square brackets.

Example 1. *The following partitions*

$$P_1 = \{\{\pm 3\}, \pm\{-2,1\}, \pm\{-4,5\}\}, \qquad P_2 = \{\pm\{1\}, \pm\{2\}, \pm\{-4,3\}\},$$
$$P_3 = [\{\pm 1, \pm 3\}, \{-2\}, \{2\}, \{-4,5\}, \{-5,4\}],$$

are respectively, a B_5-partition which is not a D_5-partition, a D_4-partition with no zero-block, and an ordered D_5-partition having a zero-block.

4 Connections between Stirling and Eulerian numbers of types B and D

In this section, we present two generalizations of Theorem 1 for Coxeter groups of types B and D and we provide a combinatorial proof for type B in the next subsection.

Theorem 3. *For all positive integers n, r with $n \geq r$, we have*

$$S_B(n,r) = \frac{1}{2^r r!} \sum_{k=0}^{r} A_B(n,k) \binom{n-k}{r-k}.$$

Theorem 4. *For all positive integers n, r with $n \geq r$, we have*

$$S_D(n,r) = \frac{1}{2^r r!} \left(n \cdot 2^{n-1}(r-1)! \cdot S(n-1,r-1) + \sum_{k=0}^{r} A_D(n,k) \binom{n-k}{r-k} \right),$$

where $S(n-1,r-1)$ is the usual Stirling number of the second kind.

Now, by inverting these formulas, similarly to [8, Corollary 1.18], we get the following expression for the Eulerian numbers of type B (*resp.* type D) in terms of the Stirling numbers of type B (*resp.* type D):

Corollary 1. *For all positive integers n and k with $n \geq k$, we have*

$$A_B(n,k) = \sum_{r=1}^{k} (-1)^{k-r} \cdot 2^r r! \cdot S_B(n,r) \binom{n-r}{k-r}.$$

Corollary 2. *For all positive integers n and k with $n \geq k$, we have*

$$A_D(n,k) = \left[\sum_{r=0}^{k} (-1)^{k-r} \cdot 2^r r! \cdot S_D(n,r) \cdot \binom{n-r}{k-r} \right] - n \cdot 2^{n-1} \cdot A(n-1,k-1).$$

4.1 Proof for type B

The proof uses arguments similar to Bona's proof of Theorem 1.17 in [8] for the corresponding identity for type A.

Proof of Theorem 3. We have to prove the following equality:

$$2^r r! S_B(n,r) = \sum_{k=0}^{r} A_B(n,k) \binom{n-k}{r-k}.$$

The number $2^r r! S_B(n,r)$ in the left-hand side is the number of ordered B_n-partitions having r pairs of nonzero blocks. Now, let us show that the right-hand side counts the same set of partitions in a different way.

Let $\beta \in B_n$ be a signed permutation with $\mathrm{des}_B(\beta) = k$, written in its window notation. We start by adding a separator after each descent of β and after $\beta(n)$. If $0 \in \mathrm{Des}_B(\beta)$, this means that a separator is added before $\beta(1)$. If $r > k$, we add extra $r-k$ *artificial separators* in some of the $n-k$ empty (i.e. without a descent) spots, where a *spot* is a gap between two consecutive entries of β or the gap before the first entry $\beta(1)$. This splits β into a set of r blocks, where the block C_i is defined as the set of entries between the ith and the $(i+1)$-st separators for $1 \le i \le r$. Now, this set of blocks is transformed into the ordered B_n-partition with r pairs of nonzero blocks and an optional *zero-block* C_0:

$$[C_0, C_1, -C_1, \ldots, C_r, -C_r],$$

where the zero-block C_0 is a set equal to $\{\pm\beta(1), \ldots, \pm\beta(j)\}$ if the first separator is after $\beta(j)$, for $j \ge 1$, and C_0 does not exist if the first separator is before $\beta(1)$.

For example, if $\beta = [-2,3,5,1,-4] \in B_5$, then after adding the separators induced by the descents, we get $\beta = [\ | -2,3,5 \mid 1 \mid -4 \mid\ ]$. So, we get a partition with three non-zero blocks. On the other hand, if $\beta' = [2,3,5,1,-4] \in B_5$, then after adding the separators induced by the descents, we have $\beta' = [\ 2,3,5 \mid 1 \mid -4 \mid\ ]$. So, we get a partition with only two non-zero blocks, and a zero-block $\{\pm 2, \pm 3, \pm 5\}$.

There are exactly $\binom{n-k}{r-k}$ ordered B_n-partitions obtained from β in this way. From now on, we refer to this process of creating this set of B_n-partitions starting from a single signed permutation $\beta \in B_n$, as the *B-procedure*.

It is easy to see that the B-procedure applied to different signed permutations produces disjoint sets of ordered B_n-partitions; therefore, $\sum\limits_{k=0}^{r} A_B(n,k)\binom{n-k}{r-k}$ distinct ordered B_n-partitions with r pairs of nonzero blocks can be created by this way.

Let us show that any ordered B_n-partition

$$\lambda = [C_0, C_1, -C_1, \ldots, C_r, -C_r]$$

can be obtained by the B-procedure. If λ contains a zero-block C_0, then put the positive elements of C_0 in increasing order at the beginning of a sequence $\mathbf{S}$, and add a separator after them. Then, order in increasing way the elements in each of the blocks $C_1, \ldots, C_r$, and write them sequentially in **S** (after the first separator if exists), by adding a separator after the last entry coming from each block. Reading the formed sequence **S** left-to-right, one obtains a signed permutation $\beta \in B_n$. Note that the number of descents in β is smaller than or equal to r, since the elements in each block are ordered increasingly. Now, it is clear that λ can be obtained by applying the B-procedure to β, where the artificial separators are easily recovered. $\square$

Example 2. *The signed permutation $\beta = [\,1,4 \mid -5,-3,2 \mid\,] \in B_5$ has one descent in position 2. It produces the following ordered B_5-partition with one pair of nonzero blocks*

$$[\{\pm 1, \pm 4\}, \{-5,-3,2\}, \{5,3,-2\}],$$

and exactly $\binom{4}{1}$ ordered B_5-partitions with two pairs of non-zero blocks, namely:

$$[\{1,4\},\{-1,-4\},\{-5,-3,2\},\{5,3,-2\}], \;\; [\{\pm 1\},\{4\},\{-4\},\{-5,-3,2\},\{5,3,-2\}],$$
$$[\{\pm 1,\pm 4\},\{-5\},\{5\},\{-3,2\},\{3,-2\}], \;\; [\{\pm 1,\pm 4\},\{-5,-3\},\{5,3\},\{2\},\{-2\}],$$

obtained by placing one artificial separator before positions 1,2,4 and 5, respectively.

Conversely, let $\lambda = [\{\pm 1, \pm 4\}, \{5\}, \{-5\}, \{-3,2\}, \{3,-2\}]$ be an ordered B_5-partition. The corresponding signed permutation with the added separators is $\beta = [\,1,4 \vdots 5 \mid -3,2 \mid\,] \in B_5$. Note that although $C_1 = \{5\}$ is a separate block, there is no descent between 4 and 5, meaning that λ is obtained by adding an artificial separator in the spot between these two entries.

The proof for type D (Theorem 4) is more tricky. The basic idea is the same as before: obtaining the set of ordered D_n-partitions with r pairs of nonzero blocks from elements in D_n with at most r descents. In addition to the B-procedure, there is a need to an extra step which uses the set $B_n - D_n$ in order to take care of the special structure of the D_n-partitions. A certain subset of D_n-partitions cannot be obtained in this way and hence should be added manually. More details can be found in [2, Section 4.2].

5 Falling factorials for Coxeter groups of types B and D

In this section, we present generalizations of Theorem 2 for Coxeter groups of types B and D and provide a combinatorial proof for both types.

5.1 Type B

The following theorem is natural generalization of Theorem 2 for the Stirling numbers of type B, and it is a particular case of a formula appearing in Bala [3], where the numbers $S_B(n,k)$ correspond to the sequence denoted there by $S_{(2,0,1)}$. Bala uses generating function techniques for proving this identity.

Theorem 5 (Bala). *Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Then we have*

$$x^n = \sum_{k=0}^{n} S_B(n,k)[x]_k^B, \tag{5.1}$$

where $[x]_k^B := (x-1)(x-3)\cdots(x-2k+1)$ *and* $[x]_0^B := 1$.

A combinatorial interpretation of $S_B(n,k)$ using the model of k-attacking rooks was given by Remmel and Wachs [17] (specifically, this is $S_{n,k}^{0,2}(1,1)$ in their notation). More information on the rook interpretation of this and other factorization theorems can be found in the paper of Miceli and Remmel [14].

Here we provide a kind of a geometric proof, suggested to us by Reiner, and it is related to a method used by Blass and Sagan [7] to compute the characteristic polynomial of the poset $L(\mathcal{B}_n)$.

Proof. Being a polynomial identity, it is sufficient to prove it only for odd integers $x = 2m+1$.

The left-hand side of Equation (5.1) counts the number of lattice points in the n-dimensional cube $\{-m, -m+1, \ldots, -1, 0, 1, \ldots, m\}^n$. We show that the right-hand side of Equation (5.1) counts the same set of points using the maximal intersection subsets of hyperplanes the points lie on. More precisely, let $P = \{C_0, \pm C_1, \ldots, \pm C_k\}$ be a B_n-partition with k pairs of nonzero blocks.

For $k = 0$, P consists of only the zero block $\{\pm 1, \ldots, \pm n\}$. This set partition corresponds to the single lattice point $(0, \ldots, 0)$.

For $0 < k < n$, let $P = \{C_0, \pm C_1, \ldots, \pm C_k\}$ be a B_n-partition with k pairs of nonzero blocks and one (possibly nonexisting) zero-block C_0. We associate to this partition the set of lattice points of the form $(x_1, \ldots, x_n)$ with $x_{j_1} = x_{j_2}$ (*resp.* $x_{j_1} = -x_{j_2}$) whenever j_1, j_2 (*resp.* $j_1, -j_2$) belong to the same block C_i or $-C_i$.

For the first pair of nonzero blocks $C_1, -C_1$ of the set partition P, if $j_1 \in C_1 \cup -C_1$ then there are $x-1$ possibilities (excluding the value 0) to choose the value of x_{j_1}.

Next, for the second pair of blocks $C_2, -C_2$ of the partition P, we have $x-3$ possibilities (excluding the value 0 and the value x_{j_1} and its negative chosen for $\pm C_1$) and so on, until we get $x-(2k-3)$ possibilities for the last pair of blocks $\pm C_k$.

If the zero-block C_0 is not empty, then the coordinates corresponding to the indices in C_0 will be assigned the value 0.

Now, for $k = n$, the only B_n-partition having n nonzero pairs of blocks is

$$\{\{1\}, \{-1\}, \ldots, \{n\}, \{-n\}\}$$

which corresponds to the points $(x_1, \ldots, x_n)$ such that $x_i \neq \pm x_j \neq 0$ for all $i \neq j$. For this we have $(x-1)(x-3)\cdots(x-(2n-1))$ possibilities. Note that these are the lattice points which do not lie on any hyperplane. $\square$

Example 3. *Let $n = 2$ and $m = 3$, so we have that $x = 2m + 1 = 7$. The lattice $([-3,3] \times [-3,3]) \cap \mathbb{Z}^2$ is presented in Figure 1.*

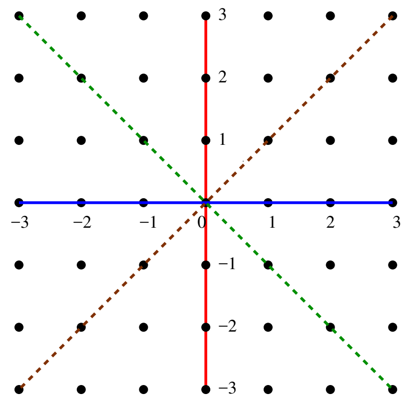

Figure 1: Lattice points for type B.

For $k = 0$, we have exactly one B_2-partition λ_0 consisting only of the zero-block: $\lambda_0 = \{\{\pm 1, \pm 2\}\}$. The corresponding subspace is $\{x_1 = x_2 = 0\}$, which counts only the lattice point $(0,0)$.

For $k = 1$, we have 4 B_2-partitions, two of them contain a zero-block:

$$\lambda_1 = \{\{\pm 1\}, \{2\}, \{-2\}\}; \qquad \lambda_2 = \{\{\pm 2\}, \{1\}, \{-1\}\},$$

and two of them do not:

$$\lambda_3 = \{\{1,2\}, \{-1,-2\}\}; \qquad \lambda_4 = \{\{1,-2\}, \{-1,2\}\}.$$

The partitions λ_1 and λ_2 correspond to the axes $x_1 = 0$ and $x_2 = 0$, respectively. The second pair λ_3 and λ_4 corresponds to the diagonals $x_1 = x_2$ and $x_1 = -x_2$ respectively. Each of these four hyperplanes contains 6 points (since the origin is already counted and hence excluded).

For $k = 2$, the single B_2-partition: $\lambda_5 = \{\pm\{1\}, \pm\{2\}\}$ corresponds to the set of lattice points (x_1, x_2) with $x_1 \neq \pm x_2 \neq 0$, which are those not lying on any hyperplane.

Remark 1. *Note that [7, Theorem 2.1] shows that, when x is an odd number, the cardinality of the set of lattice points not lying on any hyperplane is counted by the characteristic polynomial of the lattice $L(\mathcal{B}_n)$, denoted by $\chi(\mathcal{B}_n, x)$ which is exactly $[x]_n^B$.*

5.2 Type D

The falling factorial in type D is as follows:

$$[x]_k^D := \begin{cases} 1, & k = 0 \\ (x-1)(x-3)\cdots(x-(2k-1)), & 1 \leq k < n \\ (x-1)(x-3)\cdots(x-(2n-3))(x-(n-1)), & k = n \end{cases}$$

We have found no generalization of Equation (1.3) for type D in the literature, so we supply one here.

Theorem 6. *For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$:*

$$x^n = n\left((x-1)^{n-1} - [x]^D_{n-1}\right) + \sum_{k=0}^{n} S_D(n,k)[x]^D_k. \tag{5.2}$$

Proof. For D_n-partitions having $0 \leq k < n$ pairs of nonzero blocks the proof goes verbatim as in type B, so let $k = n$.

In this case, we have only one possible D_n-partition having n pairs of nonzero blocks: $\{\{1\},\{-1\},\ldots,\{n\},\{-n\}\}$. We associate this D_n-partition with the points of the form $(x_1,\ldots,x_n)$ such that $x_i \neq \pm x_j$ for $i \neq j$, having at most one appearance of the value 0. Note that the points with exactly one appearance of 0 cannot be obtained by any D_n-partition having $k < n$ blocks, since the zero-block cannot consist of exactly one pair. If 0 does appear, then we have to place it in exactly one of the n coordinates and then we are left with $(x-1)(x-3)\cdots(x-(2n-3))$ possibilities for the rest, while if 0 does not exist, then we have $(x-1)(x-3)\cdots(x-(2n-1))$ possibilities. These two values sum up to a total of $[x]^D_n = (x-1)(x-3)\cdots(x-(2n-3))(x-(n-1))$. As in type B, this number is equal to the evaluation of the characteristic polynomial $\chi(\mathcal{D}_n, x)$ of $L(\mathcal{D}_n)$, where x is odd.

Note that during the above process of collecting lattice points of the n-dimensional cube, the points containing exactly one appearance of 0 and at least two non-zero coordinates with the same absolute value are not counted. The reason is that the zero-block (if exists) must contain at least two elements. This phenomenon happens when $n > 2$, and the number of such points is $n((x-1)^{n-1} - [x]^D_{n-1})$. This concludes the proof. □

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