

Resolving Stanley's conjecture on k -fold acyclic complexes

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Abstract. In 1993 Stanley showed that if a simplicial complex is acyclic over some field, then its face poset can be decomposed into disjoint rank 1 boolean intervals whose minimal faces together form a subcomplex. Stanley further conjectured that complexes with a higher notion of acyclicity could be decomposed in a similar way using boolean intervals of higher rank. We provide an explicit counterexample to this conjecture. We also prove both a weaker version and a special case of the original conjecture.

Keywords: Simplicial complexes, f -vectors, homology, face poset

1 Introduction

The interplay between combinatorial and topological properties of simplicial complexes has been a subject of great interest for researchers for many decades (see, e.g., [1, 2, 3, 6, 9, 11, 13, 15, 16, 18, 19]). One particularly beautiful result due to Stanley [18, Theorem 1.2] connects the homology of the geometric realization of a complex to a well-behaved decomposition of its face poset. In particular, if a simplicial complex Δ is acyclic over some field \mathbb{k} , then its face poset can be written as the disjoint union of rank 1 boolean intervals such that the minimal faces of these intervals together form a subcomplex of Δ . Stanley [18, Proposition 2.1] and Duval [6, Theorem 1.1] generalized this result. Stanley further conjectured [18, Conjecture 2.4] that complexes with a higher notion of acyclicity possess similar decompositions into boolean intervals of higher rank.

Definition 1. A simplicial complex is *k -fold acyclic* if $\text{link}_\Delta \sigma$ is acyclic (over a field \mathbb{k}) for all $\sigma \in \Delta$ such that $|\sigma| < k$.

Conjecture 2 ([18, Conjecture 2.4]). Let Δ be a k -fold acyclic simplicial complex. Then Δ can be decomposed into disjoint rank k boolean intervals, the minimal faces of which together form a subcomplex.

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Our main result is an explicit counterexample to [Conjecture 2](#). Our construction relies on reducing to relative complexes and follows ideas similar to those recently developed in [\[7\]](#) and [\[12\]](#).

Central to this problem is the f -polynomial $f(\Delta, t)$ of a d -dimensional simplicial complex, which we define as

$$f(\Delta, t) = \sum_{\sigma \in \Delta} t^{|\sigma|} = f_{-1} + f_0 t + f_1 t^2 + \cdots + f_d t^{d+1}$$

where $f_i = f_i(\Delta)$ is the number of i -dimensional faces of Δ . Stanley's result [\[18, Theorem 1.2\]](#) shows that if Δ is acyclic, then $f(\Delta, t) = (1+t)f(\Gamma, t)$ where Γ is a subcomplex. Earlier, Kalai [\[13\]](#) showed this equality holds for some complex Γ , not necessarily a subcomplex of Δ . Using results from Kalai's algebraic shifting [\[14\]](#), Stanley [\[18, Proposition 2.3\]](#) further showed that the f -polynomial of a k -fold acyclic complex can be written as

$$f(\Delta, t) = (1+t)^k f(\Gamma, t)$$

for some complex Γ (which is not necessarily a subcomplex of Δ). If it had been true, [Conjecture 2](#) would have provided a combinatorial witness for this Γ . We prove a weaker version of the original conjecture in [Theorem 9](#), which provides a witness Γ as a subcomplex of Δ .

In [Section 2](#), we review definitions and relevant background material. In [Section 3](#), we provide the construction of our counterexample. In [Section 4](#), we prove a weaker version of [Conjecture 2](#), replacing boolean intervals with boolean trees. In [Section 5](#), we prove a special case of the original conjecture. We end with a section on open questions.

2 Preliminaries

We let $[n]$ denote the set $\{1, \dots, n\}$. A **simplicial complex** Δ on $[n]$ is a subset of $2^{[n]}$ such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are **faces**, and maximal faces are **facets**. If F_1, \dots, F_j are the facets of Δ , we will often write $\Delta = \langle F_1, \dots, F_j \rangle$, since the facets uniquely determine Δ . The **dimension** of a face σ is $\dim \sigma = |\sigma| - 1$ and the dimension of Δ is $\dim \Delta = \max\{\dim \sigma \mid \sigma \in \Delta\}$. A complex is **pure** if all facets have the same dimension. For a pure complex, a **ridge** is a face of one dimension lower than the facets. Unless otherwise specified, we assume throughout that $\dim \Delta = d$.

A **subcomplex** of Δ is a simplicial complex Γ such that $\Gamma \subseteq \Delta$. If $W \subseteq [n]$, then the **induced subcomplex** on W is $\Delta|_W := \{\sigma \in \Delta \mid \sigma \subseteq W\}$. Given a face $\sigma \in \Delta$, the **link** and **star** of σ in Δ are

$$\begin{aligned} \text{link}_\Delta \sigma &= \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset\} \\ \text{star}_\Delta \sigma &= \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\} \end{aligned}$$

which we will often denote as $\text{link } \sigma$ and $\text{star } \sigma$ if there is no possibility of confusion. Given two complexes Δ_1 and Δ_2 on disjoint vertex sets, their **join** is $\Delta_1 \star \Delta_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$. If Δ_1 is a simplex on $[k]$, then this join is the k -**fold cone** of Δ_2 . We note that $\text{star } \sigma = \langle \sigma \rangle \star \text{link } \sigma$.

Throughout we fix a base field \mathbb{k} . The notation $\tilde{H}_i(X; \mathbb{k})$ denotes the i^{th} reduced homology group of the complex X with coefficients in \mathbb{k} . Since we have fixed \mathbb{k} , we drop it from the notation and instead write $\tilde{H}_i(X)$. The **(reduced) Betti numbers** of a complex Δ are $\tilde{\beta}_i = \dim_{\mathbb{k}} \tilde{H}_i(\Delta)$. A complex is **acyclic** (over \mathbb{k}) if all of its reduced homology groups are zero.

We note that 1-fold acyclicity is equivalent to acyclicity, so [18, Theorem 1.2] is the $k = 1$ case of **Conjecture 2**. When $k > 1$, k -fold acyclicity is not topological; for example, a triangle is 3-fold acyclic but its barycentric subdivision is 1-fold acyclic but not even 2-fold acyclic.

The construction of our counterexample relies on **relative simplicial complexes**; given a simplicial complex Δ and a subcomplex Γ , the relative complex $\Phi = (\Delta, \Gamma)$ is the set of all of the faces of Δ that are not faces of Γ .

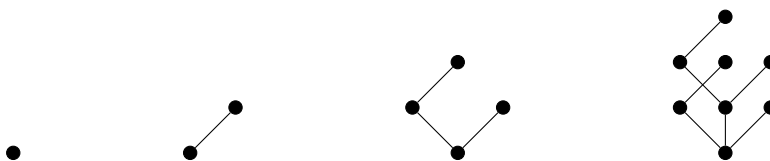
Given a poset P and two elements $x, y \in P$, the **interval** from x to y is $[x, y] = \{z \in P \mid x \leq z \leq y\}$. If $[x, y] = \{x, y\}$, then we say that y **covers** x , which is denoted $x < y$. An interval I is a **rank k boolean interval** if $I \cong 2^{[k]}$. A **boolean interval decomposition** of P is a collection \mathcal{B} of disjoint boolean intervals in P such that

$$P = \bigsqcup_{I \in \mathcal{B}} I.$$

Such a decomposition is a **rank k boolean interval decomposition** if all intervals in the decomposition are of rank k . We also refer to this as a rank k boolean decomposition.

Definition 3. A **boolean tree** of rank i is a subposet T_i of a poset P , that has a unique minimal element r , and is defined recursively as follows. Any subposet with exactly one element is a boolean tree of rank 0. Now assume T_1 and T_2 are two disjoint boolean trees of rank $(i - 1)$, each with minimal elements r_1 and r_2 respectively, such that r_2 covers r_1 in P . Then $T_1 \cup T_2$ is a boolean tree of rank i , with r_1 as its unique minimal element.

The following are examples of boolean trees of ranks zero through three. Boolean trees have the same size and ranks as a boolean interval, but relations between certain elements may be missing.



A **(rank k) boolean tree decomposition** of a poset is defined the same as a (rank k) boolean interval decomposition, except that boolean intervals are replaced with boolean trees.

Definition 4. A simplicial complex Δ is a **stacked simplicial complex** if Δ is pure of dimension d with a facet order F_1, \dots, F_j such that for each $i \in [j - 1]$, $\langle F_1, \dots, F_i \rangle \cap \langle F_{i+1} \rangle$ is a $(d - 1)$ -simplex. Such an order is known as a **stacked shelling**.

3 Construction

To construct our counterexample, we require complexes to glue together, and a way to glue them. First, we need a gluing lemma to maintain k -fold acyclicity. We use a variant of [7, Theorem 3.1], which allows us to construct a counterexample to **Conjecture 2** by reducing the problem to finding a relative complex (Δ, Γ) with appropriate properties. Second, we must actually find such a pair (Δ, Γ) . We begin with the gluing lemma.

Lemma 5. Let Δ_1 and Δ_2 be simplicial complexes such that Δ_1 is j -fold acyclic, Δ_2 is k -fold acyclic, and $\Delta_1 \cap \Delta_2$ is ℓ -fold acyclic. Then $\Delta_1 \cup \Delta_2$ is m -fold acyclic, where $m = \min\{j, k, \ell\}$.

This lemma follows from a Mayer-Vietoris sequence. It is used to preserve k -fold acyclicity in the following theorem, which is a k -fold acyclic version of [7, Theorem 3.1].

Theorem 6. Let $\Phi = (\Delta, \Gamma)$ be a relative complex such that

1. Δ and Γ are k -fold acyclic;
2. Γ is an induced subcomplex of Δ ; and
3. Φ cannot be written as a disjoint union of rank k boolean intervals.

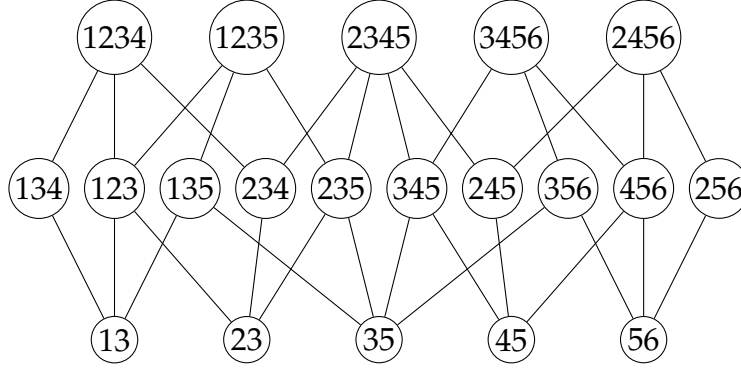
Let ℓ be the total number of faces of Γ and let $N > \ell/2^k$. If $\Omega = \Omega_N$ is the complex formed by gluing N copies of Δ together along Γ , then Ω is a k -fold acyclic complex that cannot be written as a disjoint union of rank k boolean intervals.

The proof follows from **Lemma 5** and the pigeonhole principle.

We now start the construction of our counterexample, beginning with the following relative complex Ψ , which is inspired by the complex in [7, Remark 3.6]. We have shortened the notation so instead of writing $\{1, 2, 3, 4\}$ we write 1234, for example.

$$\begin{aligned}\Sigma &= \langle 1234, 1235, 2345, 2456, 3456 \rangle \\ Y &= \langle 125, 124, 246, 346 \rangle \\ \Psi &= (\Sigma, Y)\end{aligned}$$

Both Σ and Y are 2-fold acyclic and the face poset of Ψ cannot be decomposed into disjoint rank 2 boolean intervals. Since Y is not an induced subcomplex of Σ , we cannot immediately apply [Theorem 6](#) to produce a counterexample to [Conjecture 2](#). However, this complex is the foundation of our counterexample and will be referred to repeatedly in our construction. The face poset of Ψ is given below, for the reader to verify that it cannot be decomposed into disjoint rank 2 boolean intervals.



Our goal is to create a new pair (Δ, Γ) that does meet the conditions of [Theorem 6](#). We now consider the following complex, Γ . It is straightforward to check that Γ is 2-fold acyclic. In particular, Γ is a simplicial 3-ball with no interior vertices.

$$\Gamma = \langle ABCE, BCEF, BCDF, ABCG, BCGH, BCDH, ABEG, BEFG, BFHG \rangle$$

Within Γ there are the following six pairs of triangles:

$$\begin{aligned} & \{ABC, BCD\}, \quad \{ABE, BEF\}, \quad \{ABG, BGH\}, \\ & \{CDF, CEF\}, \quad \{CDH, CGH\}, \quad \{EFG, FGH\}. \end{aligned} \quad (3.1)$$

To each of the edges AB, CD, EF, GH in Γ we add a vertex, forming four triangles which are not in Γ :

$$ABI, CDJ, EFK, GHL. \quad (3.2)$$

For any two triangles from (3.2) there is a unique pair of triangles in (3.1) so that the four triangles together form a complex isomorphic to Y . For example, the two triangles $\{ABI, CDJ\}$ from (3.2) together with $\{ABC, BCD\}$ form a complex isomorphic to Y . Given these four triangles, we glue a copy of Σ to Γ along this Y in the natural way.

We obtain Δ as the result of gluing six copies of Σ to Γ in this way, one for each choice of two triangles from (3.2). For clarity, we list all the facets of Δ that are not in Γ .

$$\begin{aligned} & ABCJ, \quad ABIJ, \quad BCIJ, \quad BCDI, \quad CDIJ, \\ & ABEK, \quad ABIK, \quad BEIK, \quad BEFI, \quad EFIK, \\ & ABGL, \quad ABIL, \quad BGIL, \quad BGHI, \quad GHIL, \\ & CDFK, \quad CDJK, \quad CFJK, \quad CEFJ, \quad EFJK, \\ & CDHL, \quad CDJL, \quad CHJL, \quad CGHJ, \quad GHJL, \\ & EFGL, \quad EFKL, \quad FGKL, \quad FGHK, \quad GHKL. \end{aligned} \quad (3.3)$$

It is straightforward to verify that Δ is 2-fold acyclic and that Γ is an induced subcomplex of Δ . It only remains to be shown that (Δ, Γ) is not decomposable into rank 2 boolean intervals; this is done using a similar argument as in the proof of [Theorem 6](#).

Theorem 7. $\Phi = (\Delta, \Gamma)$ is not rank 2 boolean decomposable.

Since $f(\Gamma) = (1, 8, 22, 24, 9)$, [Theorem 6](#) immediately implies that Ω_{17} is a counterexample to [Conjecture 2](#), since $17 > 64/4$.

Remark 8. A linear program [5] verifies that $\Omega = \Omega_3$ is a counterexample to [Conjecture 2](#). The f -polynomial of this counterexample is $f(\Omega_3, t) = 1 + 20t + 136t^2 + 216t^3 + 99t^4 = (1 + t)^2(1 + 18t + 99t^2)$. This is the smallest known counterexample to [Conjecture 2](#).

4 Boolean Trees

While [Conjecture 2](#) is false, we will use this section to prove a weakened version of it by replacing boolean intervals with boolean trees. We will rely on algebraic shifting, developed by Kalai in [14], and iterated homology, developed by Duval and Rose in [9] and Duval and Zhang in [10]. We refer the reader to these sources for more details.

Theorem 9. Let Δ be k -fold acyclic. Then Δ can be written as a disjoint union of boolean trees of rank k . Furthermore, the minimal faces of these boolean trees together form a subcomplex Δ' .

The proof is similar to the proof of [10, Corollary 3.5]. It follows from [9, Theorem 4.1], [10, Theorem 3.2], and [14, Theorem 4.2].

The subcomplex Δ' in [Theorem 9](#) is a combinatorial witness to the subcomplex in [18, Proposition 2.3]. This shows that the correct generalization of Stanley's acyclic matching is to boolean trees rather than boolean intervals.

We note the similarity between this case and the so-called Partitionability Conjecture (see, e.g., [8], [17]). A complex Δ is **partitionable** if its face poset can be written as the disjoint union of boolean intervals whose maximal faces are the *facets* of Δ . Though there exist Cohen-Macaulay complexes which are not partitionable [7], all Cohen-Macaulay complexes do have a similar decomposition if "boolean interval" is replaced in the definition of partitionable with "boolean tree" [10, Theorem 5.4].

5 d -fold Acyclic Complexes

In this section, we will show that [Conjecture 2](#) holds for d -fold acyclic complexes where $d = \dim \Delta$. We first show that [Conjecture 2](#) holds for stacked complexes. We then show that d -dimensional d -fold acyclic complexes must be stacked. Thus [Conjecture 2](#) holds when $k = \dim \Delta$.

Our interest in this case was sparked by the following result.

Theorem 10 (Duval–Klivans–Martin, unpublished). *If Δ is 2-dimensional and 2-fold acyclic, then Δ is stacked.*

Theorem 10 together with the following proposition shows that **Conjecture 2** holds if $\dim \Delta \leq 2$.

Proposition 11. *Let Δ be a d -dimensional stacked simplicial complex. Then Δ is d -fold acyclic and Δ can be written as the disjoint union of rank d boolean intervals, the minimal elements of which form a subcomplex $\Delta' \subseteq \Delta$. In other words, **Conjecture 2** holds for stacked complexes.*

The proof of this proposition follows from a straightforward induction on dimension. The following are two technical results needed in the proof of **Theorem 14**.

Lemma 12. *Let Δ be d -dimensional and d -fold acyclic. Then the f -polynomial of Δ is $f(\Delta, t) = (1 + t)^d(1 + nt)$ where n is the number of facets of Δ .*

Lemma 13. *Let Δ be d -dimensional and d -fold acyclic. Then Δ is pure and its facet-ridge graph is connected.*

Lemma 12 follows immediately from [18, Proposition 2.3], and **Lemma 13** follows from straightforward elementary arguments.

Theorem 14. *If Δ is d -dimensional and d -fold acyclic, then Δ is stacked.*

Using the above lemmas, **Theorem 14** is proved by first producing a partitioning of a given d -fold acyclic complex Δ and then showing that this partitioning gives rise to a stacked shelling.

Combining **Theorem 14** and **Proposition 11**, we see that a d -dimensional complex Δ is stacked if and only if it is d -fold acyclic. This leads immediately to our main result of this section.

Corollary 15. ***Conjecture 2** holds when $k = \dim \Delta$.*

6 Open Questions

While our construction gives a counterexample to **Conjecture 2**, our result in **Theorem 9** provides an explicit witness to the structure of the f -polynomials of k -fold acyclic complexes. Perhaps the most interesting questions in light of **Remark 8** are in determining any additional conditions that would make the conjecture hold. We know that Ω_3 is the lowest dimensional counterexample possible, but we have no reason to suspect that is in other senses the smallest.

Question 1. *What is a minimal counterexample to **Conjecture 2** with respect to the total number of faces, vertices, or facets, respectively?*

Though our counterexample is three-dimensional, it cannot be embedded into \mathbb{R}^3 . It is unknown if this non-embedding is necessary to be a counterexample.

Question 2. *Is it possible to find a counterexample to [Conjecture 2](#) that embeds into \mathbb{R}^3 ? In general, is it possible to find a d -dimensional counterexample that embeds into \mathbb{R}^d ?*

It is also unknown whether complexes with additional topological or combinatorial structure could be counterexamples.

Question 3. *Do all k -fold acyclic simplicial balls have a rank k boolean interval decomposition? If they do, must there be a decomposition so that the bottoms of these intervals forms a subcomplex?*

Although a bit further afield from the techniques developed in this paper, one can ask about random simplicial complexes.

Question 4. *For a fixed triple of k, d, v , there are finite k -fold acyclic complexes of dimension d with v vertices. Sampling from this set with the uniform distribution, what is the probability the chosen complex has a rank k boolean decomposition? What is the limiting probability as v goes towards ∞ ?*

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References

- [1] A. Björner. “Topological methods”. *Handbook of Combinatorics, Vol. 1, 2*. Elsevier Sci. B. V., Amsterdam, 1995, pp. 1819–1872.
- [2] A. Björner and G. Kalai. “An extended Euler-Poincaré theorem”. *Acta Math.* **161**.3-4 (1988), pp. 279–303. [Link](#).
- [3] K. Borsuk. “On the imbedding of systems of compacta in simplicial complexes”. *Fundam. Math.* **35** (1948), pp. 217–234. [Link](#).
- [4] T. S. Developers. “SageMath, the Sage Mathematics Software System (Version 8.4)”. 2018.
- [5] J. Doolittle and B. Goeckner. “A counterexample to Stanley’s k -fold acyclic boolean interval decomposition conjecture.” Available online. 2018. [Link](#).

- [6] A. Duval. "A combinatorial decomposition of simplicial complexes". *Isr. J. Math.* **87**.1-3 (1994), pp. 77–87. [Link](#).
- [7] A. Duval, B. Goekner, C. Klivans, and J. Martin. "A non-partitionable Cohen–Macaulay complex". *Adv. in Math.* **299** (2016), pp. 381–395. [Link](#).
- [8] A. Duval, C. Klivans, and J. Martin. "The partitionability conjecture". *Not. Amer. Math. Soc.* **64**.2 (2017), pp. 117–122. [Link](#).
- [9] A. Duval and L. Rose. "Iterated homology of simplicial complexes". *J. Algebr. Comb.* **12**.3 (2000), pp. 277–292. [Link](#).
- [10] A. Duval and P. Zhang. "Iterated homology and decompositions of simplicial complexes". *Isr. J. Math.* **121** (2001), pp. 313–331. [Link](#).
- [11] R. Forman. "A user's guide to discrete Morse theory". *Sém. Lothar. Combin.* **48** (2002), Art. B48c, 35 pp. [Link](#).
- [12] M. Juhnke-Kubitzke and L. Venturullo. "A balanced non-partitionable Cohen–Macaulay complex". 2017. [arXiv:1711.05529](#).
- [13] G. Kalai. " f -vectors of acyclic complexes". *Discrete Math.* **55** (1985), pp. 97–99. [Link](#).
- [14] G. Kalai. "Algebraic shifting". *Computational Commutative Algebra and Combinatorics (Osaka, 1999)*. Adv. Stud. Pure Math. 33. Math. Soc. Japan, Tokyo, 2002, pp. 121–163.
- [15] V. Klee. "A combinatorial analogue of Poincaré's duality theorem". *Canad. J. Math.* **16** (1964), pp. 517–531. [Link](#).
- [16] I. Novik. "Upper bound theorems for homology manifolds". *Isr. J. Math.* **108** (1998), pp. 45–82. [Link](#).
- [17] R. Stanley. "Balanced Cohen–Macaulay complexes". *Trans. Amer. Math. Soc.* **249** (1979), pp. 139–157. [Link](#).
- [18] R. Stanley. "A combinatorial decomposition of acyclic simplicial complexes". *Discrete Math.* **120**.1-3 (1993), pp. 175–182. [Link](#).
- [19] D. W. Walkup. "The lower bound conjecture for 3- and 4-manifolds". *Acta Math.* **125** (1970), pp. 75–107. [Link](#).