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Factorization problems in complex reflection groups

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Abstract. We enumerate factorizations of a Coxeter element into arbitrary factors in the complex reflection groups $G(d,1,n)$ (the wreath product of the symmetric group with a cyclic group) and its subgroup $G(d,d,n)$, applying combinatorial and algebraic methods, respectively. After a change of basis, the coefficients that appear are the same as those that appear in the corresponding enumeration in the symmetric group.

Résumé. Nous comptons les factorisations d'un élément de Coxeter en facteurs arbitraires dans les groupes de réflexion complexes $G(d,1,n)$ (le produit en couronne du groupe symétrique avec un groupe cyclique) et de son sous-groupe $G(d,d,n)$, en appliquant des méthodes combinatoires et algébriques, respectivement. Après un changement de base, les coefficients qui apparaissant sont le même que ceux figurant dans l'énumération correspondante du groupe symétrique.

Keywords: factorizations, complex reflection groups, wreath product, permutations

1 Introduction

The motivation for this abstract is the following uniform formula of Chapuy–Stump for the generating function for the number of factorizations of a fixed Coxeter element by reflections in any complex reflection group.

Theorem 1.1 (Chapuy–Stump [5]). *Let W be an irreducible well-generated complex reflection group of rank n. Let c be a Coxeter element in W, let $\mathcal{R}$ and $\mathcal{R}^*$ be the set of all reflections and all reflecting hyperplanes in W, and for $\ell \geq 1$ let $N_\ell(W) := \#\{(\tau_1, \ldots, \tau_\ell) \in \mathcal{R}^\ell \mid \tau_1 \cdots \tau_\ell = c\}$ be the number of factorizations of c as a product of ℓ reflections in $\mathcal{R}$. Then*

$$\sum_{\ell \geq 0} N_\ell(W) \frac{t^\ell}{\ell!} = \frac{1}{|W|} \left(e^{t|\mathcal{R}|/n} - e^{-t|\mathcal{R}^*|/n} \right)^n.$$

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Near $t = 0$, the generating function gives the well known result that the number of shortest factorizations of a Coxeter element is $n!h^n/|W|$, where h is the Coxeter number of W. In addition, when W is the symmetric group $\mathfrak{S}_n$, the result above reduces to a result of Jackson [7] on the generating function for factorizations of the n-cycle $(12\cdots n)$ into transpositions. Chapuy and Stump prove their result by an algebraic approach with irreducible characters that dates back to Frobenius. (Recently, Michel [9] gave a uniform proof of this result when W is a Weyl group using Deligne–Lusztig representations.)

A natural question is whether there are extensions to complex reflection groups of other factorization results in the symmetric group. In the same paper, Jackson [7] gave formulas for the generating polynomial of factorizations of a fixed n-cycle as a product of a fixed number of factors, keeping track of the number of cycles of each factor. We state the result for two factors, as reformulated by Schaeffer and Vassilieva.

Theorem 1.2 (Jackson, Schaeffer–Vassilieva [7, 12])**.** *Let c be a fixed n-cycle in $\mathfrak{S}_n$, and for integers r, s let $a_{r,s}$ be the number of pairs (u, v) of elements in $\mathfrak{S}_n$ such that u has r cycles, v has s cycles, and $u \cdot v = c$. Then*

$$\frac{1}{n!} \cdot \sum_{r,s\geq 1} a_{r,s} x^r y^s = \sum_{p,q\geq 1} \binom{n-1}{p-1; q-1; n-p-q+1} \frac{(x)_p}{p!} \frac{(y)_q}{q!},$$

where $(x)_p$ denotes the falling factorial $(x)_p := x(x-1)\cdots(x-p+1)$. In particular, the leading coefficient $a_{r,n-1-r}$ is the Narayana number $\mathrm{Nar}_A(n,r) := \frac{1}{n}\binom{n}{r}\binom{n}{r-1}$.

In this abstract we give analogues of this result, and its generalizations to k factors, for two infinite families of complex reflection groups: the group $G(d, 1, n)$ of $n \times n$ matrices having one nonzero entry in every row and column, each of which is a dth root of unity (*i.e.*, the wreath product $(\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$; at $d = 2$, the Coxeter group of type B_n) and its subgroup $G(d, d, n)$ (at $d = 2$, the Coxeter group of type D_n). The analogue of an n-cycle in a complex reflection group is a *Coxeter element*. In $G(d, 1, n)$, the Coxeter elements are the matrices whose underlying permutation is an n-cycle and in which the product of the nonzero elements is a primitive nth root of unity. (The Coxeter elements in $G(d, d, n)$ are described in Section 4.) The analogue of the number of cycles of a group element is the *fixed space dimension*. Our results for $G(d, 1, n)$ are in terms of the polynomials

$$(x)_k^{(d)} := x(x-d)(x-2d)\cdots(x-(k-1)d) = \prod_{i=1}^{k}(x - e_i^*).$$

Here the roots e_i^* are the *coexponents* of this group, one of the fundamental sets of invariants associated to every complex reflection group.

Theorem 1.3. *For $d > 1$, let $G = G(d, 1, n)$ and let $a_{r,s}^{(d)}$ be the number of factorizations of a fixed Coxeter element c in G as a product of two elements of G with fixed space dimensions r and* *s, respectively. Then*

$$\frac{1}{|G|}\sum_{r,s\geq 0} a^{(d)}_{r,s}x^r y^s = \sum_{p,q\geq 0}\binom{n}{p;q;n-p-q}\frac{(x-1)^{(d)}_p}{d^p p!}\frac{(y-1)^{(d)}_q}{d^q q!}. \tag{1.1}$$

In particular, the leading coefficient $a^{(d)}_{r,n-r}$ is the type B Narayana number $\mathrm{Nar}_B(n,r) := \binom{n}{r}^2$.

In Section 3, we give a combinatorial proof of this result and its generalization to k factors. As a special case, we recover the Chapuy–Stump result for $G(d,1,n)$ (see Section 6). Our proof works directly with the group elements and permutations. However, the proof could also be written in terms of *maps* [8, 11], as in [1, 2, 4, 12] (for example).

Our results for the subgroup $G(d,d,n)$ involve a notion of *transitive* factorization, defined in Section 4 below, and are in terms of the polynomials

$$P^{(d)}_x(k) := (x-(k-1)(d-1))\cdot(x-1)^{(d)}_{k-1} = \prod_{i=1}^{k}(x-e^*_i),$$

where again the e^*_i are the coexponents of the group.

Theorem 1.4. *For $d>1$, let $G=G(d,d,n)$ and let $b^{(d)}_{r,s}$ be the number of* transitive *factorizations of a Coxeter element c in G as a product of two elements of G with fixed space dimensions r and s, respectively. Then*

$$\frac{1}{d^{n-1}(n-1)(n-1)!}\sum_{r,s\geq 0} b^{(d)}_{r,s}x^r y^s = \sum_{p,q\geq 1}\binom{n-2}{p-1;q-1;n-p-q}\frac{P^{(d)}_p(x)}{d^{p-1}p!}\frac{P^{(d)}_q(x)}{d^{q-1}q!}. \tag{1.2}$$

In particular, the leading term $b^{(d)}_{r,n-r}$ equals $dn\cdot\mathrm{Nar}_A(n-1,r)$.

In Section 4, we sketch a proof of this result and its generalization to k factors using character theory. As a special case, we recover the Chapuy–Stump result for $G(d,d,n)$.

In Section 5, we also include some suggestive evidence from factorization data of rank 2 complex reflection groups that indicates that the results above could be particular cases of a uniform statement like the Chapuy–Stump result.

2 Known factorization results in $\mathfrak{S}_n$

Theorem 1.2 is a special case of a more general result for factorizations of an n-cycle into k factors. To state this theorem we need to define a number that counts certain tuples of sets. Given a positive integer k and nonnegative integers n and $p_1,\ldots,p_k$, define

$$M^n_{p_1,\ldots,p_k} := \sum_{t=0}^{\min(p_i)}(-1)^t\binom{n}{t}\prod_{i=1}^{k}\binom{n-t}{p_i-t} = [x_1^{p_1}\cdots x_k^{p_k}]\Big((1+x_1)\cdots(1+x_k)-x_1\cdots x_k\Big)^n, \tag{2.1}$$

which counts n-tuples $(S_1, \ldots, S_n)$ of proper subsets $S_i \subsetneq [k]$ such that exactly p_j of the sets contain j. From this interpretation, it is easy to see that $M^n_{p_1,p_2}$ is given by a multinomial coefficient $\binom{n}{p_1;p_2;n-p_1-p_2}$ and that the $M^n_{\mathbf{p}}$ satisfy the following recurrence.

Proposition 2.1. *One has* $M^n_{p_1,\ldots,p_k} = \sum_{\varnothing \neq S \subseteq [k]} M^{n-1}_{\mathbf{p}-\mathbf{1}+\mathbf{e}_S}$ *where* $\mathbf{1}$ *is the all-ones vector and* $\mathbf{e}_S$ *denotes the indicator vector for the set* S.

Let c be a fixed n-cycle in $\mathfrak{S}_n$, and for integers $r_1, \ldots, r_k$ let $a_{r_1,\ldots,r_k}$ be the number of k-tuples $(\pi_1, \ldots, \pi_k)$ of elements in $\mathfrak{S}_n$ such that π_i has r_i cycles for $i = 1, \ldots, k$, and $\pi_1 \cdots \pi_k = c$.

Theorem 2.2 (Jackson [7])**.** *One has*

$$\frac{1}{(n!)^{k-1}} \cdot \sum_{r_1,\ldots,r_k \geq 1} a_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \sum_{p_1,\ldots,p_k \geq 1} M^{n-1}_{p_1-1,\ldots,p_k-1} \frac{(x_1)_{p_1}}{p_1!} \cdots \frac{(x_k)_{p_k}}{p_k!}. \tag{2.2}$$

Jackson's proof used the Frobenius approach with irreducible characters. Bijective proofs of the case $k = 2$ were given by Schaeffer–Vassilieva [12], Chapuy–Féray–Fusy [4], and Bernardi [1]. Bernardi–Morales [2, 3] extended Bernardi's approach to give a combinatorial proof of Jackson's formula for all k in terms of maps. These combinatorial proofs use an interpretation of the change of basis in (2.2) that we describe now.

Let $\mathcal{C}_{p_1,\ldots,p_k}$ be the set of factorizations in $\mathfrak{S}_n$ of the fixed n-cycle c as a product $\pi_1 \cdots \pi_k$ such that for each i, the cycles of π_i are colored with the colors in $[p_i]$, and every color is used at least once. Let $c_{p_1,\ldots,p_k}$ be the number of such factorizations.

Proposition 2.3. *With* $a_{r_1,\ldots,r_k}$ *and* $c_{p_1,\ldots,p_k}$ *as above, one has*

$$\sum_{r_1,\ldots,r_k \geq 1} a_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \sum_{p_1,\ldots,p_k \geq 1} c_{p_1,\ldots,p_k} \binom{x_1}{p_1} \cdots \binom{x_k}{p_k}. \tag{2.3}$$

Proof. Let each x_i be a nonnegative integer. The LHS above counts factorizations of the cycle $(12 \cdots n)$ as a product $c = \pi_1 \cdots \pi_k$, where for $i = 1, \ldots, k$, each cycle of π_i is colored with a color in $[x_i]$. These colored factorizations are also counted by the RHS above: for $p_1, \ldots, p_k \in \mathbb{Z}_{>0}$ and $i = 1, \ldots, k$, choose p_i colors from x_i colors available and a colored factorization $\mathcal{C}_{p_1,\ldots,p_k}$ where exactly p_i colors are used in the factor π_i. $\square$

3 The group $G(d, 1, n)$

The conjugacy classes in $G(d, 1, n)$ are uniquely determined by the following combination of data: first, one needs to know the cycle type of the underlying permutation (*i.e.*, when we take the projection $G(d, 1, n) \twoheadrightarrow \mathfrak{S}_n$ whose kernel is exactly the diagonal matrices), and then for each cycle one needs to know the product of the roots of unity that appear in it. Equivalently, for each power $0, \ldots, d-1$, which we call the *weight*, we have a partition (possibly empty) recording the lengths of the cycles whose product of elements is that power of $\exp(2\pi i/d)$. Thus, conjugacy classes in $G(d,1,n)$ are unambiguously indexed by tuples $(\lambda^{(0)}, \ldots, \lambda^{(d-1)})$ of partitions of total size n. The following proposition is straightforward.

Proposition 3.1. *The fixed space dimension of an element w in G whose conjugacy class is indexed by $(\lambda^{(0)}, \ldots, \lambda^{(d-1)})$ is equal to $\ell(\lambda^{(0)})$, the number of cycles of weight 0 in w.*

We are now prepared to state our main enumerative theorem for $G(d,1,n)$.

Theorem 3.2. *For $d > 1$, let $G = G(d,1,n)$, so that $|G| = d^n n!$, let c be a fixed Coxeter element in G, and let $a^{(d)}_{r_1,\ldots,r_k}$ be the number of factorizations of c as a product of k elements of G with fixed space dimensions $r_1, \ldots, r_k$, respectively. Then*

$$\frac{1}{|G|^{k-1}} \sum_{0 \le r_1,\ldots,r_k \le n} a^{(d)}_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \sum_{0 \le p_1,\ldots,p_k \le n} M^n_{p_1,\ldots,p_k} \frac{(x_1-1)^{(d)}_{p_1}}{d^{p_1} p_1!} \cdots \frac{(x_k-1)^{(d)}_{p_k}}{d^{p_k} p_k!},$$

where $M^n_{p_1,\ldots,p_k}$ is defined in (2.1).

The case of two factors $k = 2$ follows immediately as a corollary.

Proof of Theorem 1.3. Substitute $k = 2$ in Theorem 3.2 and use the fact that $M^n_{p_1,p_2} = \binom{n}{p_1,p_2,n-p_1-p_2}$. □

Combinatorial proof of Theorem 3.2. It is enough to show that if we define constants $C_{p_1,\ldots,p_k}$ by

$$\sum_{r_1,\ldots,r_k} a^{(d)}_{r_1,\ldots,r_k} (x_1 d + 1)^{r_1} \cdots (x_k d + 1)^{r_k} = \sum_{p_1,\ldots,p_k} C_{p_1,\ldots,p_k} \binom{x_1}{p_1} \cdots \binom{x_k}{p_k},$$

then $C_{p_1,\ldots,p_k} = (d^n n!)^{k-1} M^n_{p_1,\ldots,p_k}$. To prove this polynomial identity, it suffices to prove that it is valid when each of the x_i is a nonnegative integer. In this case, the LHS counts factorizations of c as a product $c = u_1 \cdots u_k$, where for $i = 1, \ldots, k$, each cycle of weight 0 in u_i is colored with a color in $X_i := \{0_i, 1_i, 2_i, \ldots, (x_i d)_i\}$, and the cycles u_i of weights other than 0 are colored with the color 0_i. (The subscripts i on the colors are to emphasize that, for example, the allowable colors for cycles of u_1 are not the same as the allowable colors for cycles of u_2.)

Within the color set X_i, a *d-strip* is any of the following collections of d consecutive colors: $\{1_i, \ldots, d_i\}$, $\{(d+1)_i, \ldots, (2d)_i\}$, $\ldots$, $\{((x_i-1)d+1)_i, \ldots, (x_i d)_i\}$. Thus, for each i, there are exactly x_i d-strips in X_i (and the color 0_i does not belong to any d-strip). We use these d-strips to divide the set of colored factorizations counted by the LHS into disjoint subsets, as follows: for $p_1, \ldots, p_k \in \mathbb{Z}_{\ge 0}$, consider the colored factorizations in which colors from exactly p_i of the d-strips in X_i are used to color cycles of the factor u_i. The number of ways to choose the p_i d-strips is $\binom{x_i}{p_i}$; consequently, the coefficient $C_{p_1,\ldots,p_k}$ on the RHS counts factorizations of c as a product $c = u_1 \cdots u_k$ where for $i = 1, \ldots, k$, each cycle of weight 0 in u_i is colored with colors from a prescribed set of p_i d-strips in X_i, or with the color 0_i, in such a way that at least one color is used from each of the prescribed d-strips, and the cycles u_i of weights other than 0 are colored with the color 0_i. We now wish to relate this number to certain factorizations in the symmetric group.

We claim there is the following connection between the Cs (the numbers of colored factorizations in $G(d,1,n)$) and the cs (the numbers of colored factorizations in $\mathfrak{S}_n$), with proof given below.

Lemma 3.3. *For any* $\mathbf{p} = (p_1, \ldots, p_k)$ *in* $(\mathbb{Z}_{\geq 0})^k$*, we have* $C_{\mathbf{p}} = d^{(k-1)n} \sum_{\varnothing \neq S \subseteq [k]} c_{\mathbf{p}+\mathbf{e}_S}$ *where* $\mathbf{e}_S$ *is the indicator vector for* S*.*

We have by Jackson's Theorem 2.2 that $c_{\mathbf{p}+\mathbf{e_S}} = (n!)^{k-1} \cdot M^{n-1}_{\mathbf{p}-\mathbf{1}+\mathbf{e}_S}$ where $\mathbf{1}$ is the all-ones vector. Then Theorem 3.2 follows by Lemma 3.3 and Proposition 2.1. □

Proof of Lemma 3.3. Given a colored factorization $c = u_1 \cdots u_k$ of the Coxeter element c for $G(d,1,n)$ of the relevant kind, we associate to it a colored factorization $(1 \cdots n) = \pi_1 \cdots \pi_k$ in $\mathfrak{S}_n$, as follows: π_i is the projection of u_i in $\mathfrak{S}_n$; if a cycle of u_i is colored with a color in the d-strip $\{((a-1)d+1)_i, \ldots, (ad)_i\}$, then the corresponding cycle of π_i is colored with color a; if a cycle of u_i is colored with color 0_i, then the corresponding cycle of π_i is colored with color 0. Thus, in the resulting colored factorization of $(1 \cdots n)$, the ith factor is colored in either p_i or $p_i + 1$ colors, with every color appearing. Let $S \subseteq [k]$ denote the set of indices i such that π_i is colored in $p_i + 1$ colors (rather than p_i); equivalently, it is the set of indices i such that some cycle of u_i is colored with color 0_i.

First, we observe that $S \neq \varnothing$: the product $c = u_1 \cdots u_k$ has total weight nonzero, so at least one of the factors u_i has a cycle with weight different from zero. Such a cycle is colored with the special color 0_i, and so at least for this value of i we have $i \in S$. Thus the collection of underlying factorizations is in the disjoint union of pieces $\mathcal{C}_{\mathbf{p}+\mathbf{e}_S}$ for nonempty sets $S \subseteq [k]$.

Second, we must consider how many preimages each factorization in $\mathcal{C}_{\mathbf{p}+\mathbf{e}_S}$ has under this map. To choose a preimage, we must assign weights to the entries of each factor π_i in such a way that the product of the resulting factors u_i really is the Coxeter element c, *and* so that in each u_i, any factor of nonzero weight was originally colored by the color 0_i; and we must choose one of d colors from a d-strip for each of the cycles in u_i that corresponds to a cycle in π_i of nonzero color.

In order to do this, we consider a too-large set of factorizations in $G(d,1,n)$: we choose a total order on all the entries of all the π_i, in such a way that the last entry chosen belongs to a cycle of color 0_j for some j. (Note that such cycles must exist, since $S \neq \varnothing$.) Then we weight the entries in order, choosing the weights arbitrarily except in two cases: if an element belongs to a cycle of nonzero color and is the last element in its cycle to be weighted, we assign it the unique weight so that the total weight of its cycle is 0; and we choose the weight of the specially selected final element so that the total weight of all elements is 1. (These two exceptions never conflict because the special element was chosen in a cycle of color 0.) The number of ways to perform these choices is $d^{nk-1-\#(\text{cycles not colored } 0)}$. Finally, for each cycle of π_i that is colored some nonzero color, there are d choices for the color in the associated d-strip of the corresponding cycle of the lift u_i of π_i; this contributes a factor of $d^{\#(\text{cycles not colored } 0)}$, for a total of d^{nk-1} lifts.

Each lift is a colored factorization $u_1 \cdots u_k$ in $G(d,1,n)$ of some element c^* of weight 1 whose underlying permutation is the n-cycle $(1 \cdots n)$. The number of such elements is d^{n-1}; they are all conjugate to c by some diagonal matrix a in $G(d,1,n)$. Moreover, since a is a diagonal matrix, conjugating any $w \in G(d,1,n)$ by a preserves the weight of every cycle of w. Consequently, conjugation by a extends to a bijection between factorizations of c and factorizations of c^* that respects the underlying permutation of each factor and the weight of each cycle of each factor. Thus, in particular it gives a bijection between the lifts of $\pi_1 \cdots \pi_k$ that factor c and those that factor c^*. Thus, of the total d^{nk-1} lifts, exactly $\frac{1}{d^{n-1}} \cdot d^{nk-1} = d^{n(k-1)}$ of them are factorizations of c. This completes the proof of the lemma. □

3.1 Specializations and leading terms for factorizations of $G(d,1,n)$

We first remark on two natural specializations of the factorization results for $G(d,1,n)$. Let $F_d(x,y)$ denote the RHS of (1.1). One can check from Theorem 1.3 that $F_d(1,1) = \binom{n}{0;0;n} \cdot 1 = 1$ as expected: by the definition of F_d, we have $F_d(1,1) = \frac{1}{|G|} \cdot |G| = 1$. The next specialization is more interesting.

Proposition 3.4. *One has* $F_d(x,1) = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} x^{\dim \operatorname{fix}(g)} = \dfrac{1}{|G|} \prod_{i=1}^{n} (x + id - 1).$

Proof. By Theorem 1.3, we have

$$F_d(x,1) = \sum_{p \geq 0} \binom{n}{p} \frac{(x-1)_p^{(d)}}{d^p p!} = \binom{n + (x-1)/d}{n} = \frac{1}{|G|} \prod_{i=1}^{n} (x + id - 1),$$

where the second equality is the Vandermonde identity. On the other hand, by the definition of F_d we have

$$F_d(x,1) = \frac{1}{|G|} \sum_{g \in G} x^{\dim \operatorname{fix}(g)},$$

so we recover in this case the general fact that the generating function for fixed space dimension in a finite complex reflection group splits into linear factors, with roots equal to $1 - d_i$ where $\{d_i\}$ are the *degrees* of the group. □

Lastly, we show that the leading term $a_{r,n-r}$ of $|G| \cdot F_d(x,y)$ is given by the type B Narayana number $\binom{n}{r}^2$ A008459 [10, Table 12.3]. In particular, it is independent of d.

Corollary 3.5. *For $d > 1$, let $G = G(d,1,n)$, and let $a_{r,n-r}$ be the number of factorizations of a Coxeter element c in G as a product of two elements u and v with fixed space dimension r and $n-r$ respectively, then $a_{r,n-r} = \binom{n}{r}^2$.*

Proof. The leading coefficients of (1.1) have total degree n. Extracting such coefficients from the RHS gives $a_{r,n-r} = |G|\binom{n}{r;n-r;0}\frac{1}{r!(n-r)!d^n}$ which equals the desired formula. □

4 The subgroup $G(d,d,n)$

The hyperoctahedral group of signed permutations (the Coxeter group of type B_n) has an index-2 subgroup of "even-signed permutations" (the Coxeter group of type D_n), consisting of matrices in which the total number of negative entries is even. Similarly, for every d, the group $G(d,1,n)$ has an irreducible rank-n well generated reflection subgroup: the group $G(d,d,n)$ of $n \times n$ weighted permutation matrices in which the product of the nonzero entries is 1 (equivalently, in which the total weight is 0), of order $|G(d,d,n)| = \frac{|G(d,1,n)|}{d} = d^{n-1}n!$. The Coxeter elements in $G(d,d,n)$ are the elements whose underlying permutation is an $(n-1)$-cycle and the weight of the fixed point is 1 (and so also the weight of the $(n-1)$-cycle is -1).

Most of the results in this section rely on the Frobenius character approach, which is based on the following lemma.

Lemma 4.1 (Frobenius, *e.g.* [13, Ex. 7.67(b)])**.** *Let W be a finite group, g an element of W, and $A_1, \ldots, A_k$ subsets of W that are each closed under conjugation by elements of W. Then the number of factorizations of g as a product $g = t_1 \cdots t_k$, such that for each i the factor t_i is required to lie in the set A_i, is equal to*

$$\frac{1}{|W|}\sum_{\lambda \in \mathrm{Irr}(W)} \dim(\lambda)\chi_\lambda(g^{-1})\widetilde{\chi}_\lambda(z_1)\cdots\widetilde{\chi}_\lambda(z_k),$$

where $\mathrm{Irr}(W)$ is the set of irreducible representations of W, $\dim(\lambda)$ is the dimension of the representation λ, χ_λ is the character associated to λ, $\widetilde{\chi}_\lambda = \frac{\chi_\lambda}{\dim(\lambda)}$ is the normalized character associated to λ, and for $i = 1, \ldots, k$, z_i is the formal sum in the group algebra of elements in A_i.

4.1 Results in $\mathfrak{S}_n$ for factorizations of an $(n-1)$-cycle

If one factors an $(n-1)$-cycle c in $\mathfrak{S}_n$ as a product of other permutations, there are two possibilities: either every factor shares c's fixed point, or some factor acts nontrivially on it. The factorizations in the former case correspond to factorizations of an $(n-1)$-cycle in $\mathfrak{S}_{n-1}$. The factorizations in the latter case have a more elegant description: they are exactly the factorizations in which the factors act transitively on the set $[n]$. The study of transitive factorizations plays an important role in the field of permutation factorizations; it is present already in the late 19th century work of Hurwitz [6].

Theorem 4.2. *Let c be a fixed $(n-1)$-cycle in $\mathfrak{S}_n$. For integers $r_1, \dots, r_k$ let $b_{r_1,\dots,r_k}$ be the number of k-tuples $(u_1, \dots, u_k)$ of elements in $\mathfrak{S}_n$ such that u_i has r_i cycles for $i = 1, \dots, k$, $u_1 \cdots u_k = c$, and $(u_1, \dots, u_k)$ is a transitive factorization. Then*

$$\sum_{r_1,\dots,r_k \geq 1} b_{r_1,\dots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \frac{(n-1)!^k}{n!} \sum_{p_1,\dots,p_k \geq 1} M^n_{p_1,\dots,p_k} \frac{(x_1)_{p_1}}{(p_1-1)!} \cdots \frac{(x_k)_{p_k}}{(p_k-1)!}, \tag{4.1}$$

where $M^n_{p_1,\dots,p_k}$ is as defined in (2.1).

We were surprised not to find this statement in the literature. In the case of two factors, there is a simple combinatorial proof. We sketch here the algebraic proof for any number k of factors.

Proof sketch. By the Murnaghan–Nakayama rule, the character values on an $(n-1)$-cycle in $\mathfrak{S}_n$ are usually 0, so the sum in Lemma 4.1 simplifies considerably. We first use this lemma on the generating polynomial of all factorizations of c into k factors (not necessarily transitive). We then subtract the generating polynomial of the non-transitive factorizations. It is not hard to see that this is just the product of $x_1 \cdots x_k$ and the generating polynomial of factorizations of a long cycle in $\mathfrak{S}_{n-1}$ into k permutations. Unexpected cancellations occur when we subtract, and we finally rewrite the expression in terms of the basis $(x_1)_{p_1} \cdots (x_k)_{p_k}$ to obtain the desired formula. □

4.2 Results for transitive factorizations in $G(d,d,n)$

As $G(d,1,n)$ is a wreath product, it and its subgroups carry a natural permutation action: they act on d copies of $[n]$ indexed by dth roots of unity. (Equivalently, they act on $\{z^i e_j : 0 \leq i < d, 1 \leq j \leq n\}$ where z is a primitive dth root of unity and e_j are the standard basis vectors for $\mathbb{C}^n$.) The Coxeter elements for $G(d,1,n)$ act transitively on this set, and consequently every factorization of a Coxeter element in $G(d,1,n)$ is a transitive factorization. However, the same is not true for the subgroup $G(d,d,n)$, where the underlying permutations of the Coxeter elements are $(n-1)$-cycles. In this abstract, we consider only the case of transitive factorizations of a Coxeter element in $G(d,d,n)$. As mentioned in the introduction, our generating function in this case is in terms of the polynomials $P_k^{(d)}(x)$ defined by $P_0^{(d)}(x) = 1$, $P_1^{(d)}(x) = x$, and for $k > 1$ by

$$P_k^{(d)}(x) := \prod_{i=1}^{k} (x - e_i^*) = (x - (k-1)(d-1)) \cdot (x-1)_{k-1}^{(d)} = (x-1)_k^{(d)} + k(x-1)_{k-1}^{(d)}, \tag{4.2}$$

where the e_i^* are the coexponents of the group $G(d,d,k)$.

Theorem 4.3. *For $d > 1$, let $G = G(d,d,n)$ and let $b^{(d)}_{r_1,\ldots,r_k}$ be the number of transitive factorizations of a Coxeter element c in G as a product of k elements of G with fixed space dimensions $r_1, \ldots, r_k$, respectively. Then*

$$\sum_{r_1,\ldots,r_k \geq 0} b^{(d)}_{r_1,\ldots,r_k} x_1^{r_1} \cdots x_k^{r_k} = \frac{|G|^{k-1}}{n^k} \sum_{r_1,\ldots,r_k \geq 1} M^n_{r_1,\ldots,r_k} \frac{P^{(d)}_{r_1}(x_1)}{d^{r_1-1}(r_1-1)!} \cdots \frac{P^{(d)}_{r_k}(x_k)}{d^{r_k-1}(r_k-1)!},$$

where $M^n_{r_1,\ldots,r_k}$ is as defined in (2.1).

Proof sketch. The character values on a Coxeter element are usually zero, so when applying Lemma 4.1 the sum simplifies considerably. The relevant character values are provided in [5]. It is technically but not conceptually challenging to use this to produce a formula for the generating function of all factorizations of the Coxeter element. One then subtracts the generating function for non-transitive factorizations.

As in the case of the $(n-1)$-cycle in the symmetric group, the non-transitive factorizations are exactly those in which the underlying permutations of the factors all fix the element 1, and so (ignoring the fixed point) these factorizations correspond to factorizations of a Coxeter element in $G(d,1,n-1)$. However, there is a wrinkle: in the symmetric group, this correspondence increases the number of cycles (fixed space dimension) of each factor by exactly 1 (the new fixed point); in $G(d,d,n)$, whether the extra cycle increases the fixed space dimension or not depends on whether it has weight 0 (*i.e.*, whether the $(1,1)$ entry of the matrix is 1 or some other root of unity). This requires a second set of technical gymnastics to handle. In the end, there is massive unexpected cancellation between the non-transitive factors and the messier terms of the sum for all factorizations, leaving the claimed formula. □

The case of $k = 2$ factors follows immediately as a corollary.

Proof of Theorem 1.4. Take $k = 2$ in Theorem 4.3 and use the fact that $M^n_{p,q} = \binom{n}{p,q,n-p-q}$. □

5 Rank 2 complex reflection groups

In this section, we record some tantalizing data that suggests that Theorems 3.2 and 4.3 could be particular cases of a more general, uniform statement, along the lines of the Chapuy–Stump result. There are two infinite families of irreducible rank-2 well generated complex reflection groups, the wreath products $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_2$ (of type $G(r,1,2)$) and the dihedral groups (of type $G(r,2,2)$), as well as twelve exceptional groups (Shephard–Todd classes G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, and G_{21}). For any such group

G, one may define $a_{r,s}$ to be the number of factorizations of a given Coxeter element c as a product $c = u \cdot v$ where u has fixed space dimension r and v has fixed space dimension s, and so also one may define the generating function

$$F_G(x,y) = \sum_{r,s} a_{r,s} x^r y^s.$$

Since the matrices act on a space of dimension 2 and the Coxeter elements have fixed space dimension 0, every such generating function has the form

$$F_G(x,y) = \sum_{0 \le r,s \le 2} a_{r,s} x^r y^s = x^2 + y^2 + axy + \text{lower order terms}.$$

Easy arguments in the two infinite families combined with exhaustive computation (performed in Sage [14]) for the exceptional groups reveal the following formula.

Theorem 5.1. *Let G be an irreducible well generated complex reflection group of rank 2. Let $e_1^* = 1$ and e_2^* be the coexponents of G, and let h be the Coxeter number of G (i.e., the multiplicative order of the Coxeter element; equivalently, the largest degree of G). Then one has*

$$F_G(x,y) = (x-1)(x-e_2^*) + k(x-1)(y-1) + (y-1)(y-e_2^*) + 2h(x-1) + 2h(y-1) + |G|,$$

where k is the nontrivial G-Narayana number (i.e., the number of reflections that lie below each Coxeter element of G in the absolute order).

So far, we have been unable to find a correspondingly attractive formula for the rank-3 complex reflection groups, where the correct choice of basis is less clear. In particular, we have not found a natural basis in which the generating function for the group G_{25} (a Shephard group, with abstract presentation $\langle a, b, c \mid a^3 = b^3 = c^3 = 1, aba = bab, bcb = cbc \rangle$) has coefficients that we understand. This seems ripe for further investigation.

6 Rederiving Chapuy–Stump formulas for $G(d,1,n)$ and $G(d,d,n)$

One can rederive the Chapuy–Stump formula (Theorem 1.1) for the groups $G(d,1,n)$ and $G(d,d,n)$ from Theorems 3.2 and 4.3, respectively. In $G(d,d,n)$, we make use of the fact that every *reflection* factorization of a Coxeter element is transitive in our sense. In both cases, this amounts to extracting the coefficient $x_1^{n-1} \cdots x_\ell^{n-1}$ from the generating function for all factorizations. This involves contributions only from terms where the coefficient is $M^n_{n,\ldots,n,n-1,\ldots,n-1} = k! \cdot S(n,k)$ where $S(n,k)$ is a Stirling number of the second kind. In the case of $G(d,1,n)$, this gives an elementary proof of the Chapuy–Stump formula; it would be nice also to have an elementary proof (without characters) for $G(d,d,n)$.

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