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Homomorphisms on noncommutative symmetric functions and permutation enumeration

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Abstract. We present a unifying framework for deriving permutation enumeration formulas involving restrictions on descent compositions or permutation statistics describable in terms of descent compositions: these formulas can be proven by first lifting them to the algebra of noncommutative symmetric functions and then applying an appropriate homomorphism. We give several applications of this method including rederivations and extensions of classical results in the literature as well as new results.

Keywords: permutation statistics, noncommutative symmetric functions, descents, inversions, peaks, shuffle-compatibility

1 Introduction

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \dots, n\}$. Let $|\pi|$ be the length of π, so that $|\pi| = n$ whenever $\pi \in \mathfrak{S}_n$. We say that $i \in [n-1]$ is a *descent* of $\pi \in \mathfrak{S}_n$ if $\pi_i > \pi_{i+1}$.

Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences—or equivalently, maximal consecutive subsequences containing no descents—which we call *increasing runs*. For example, the descents of $\pi = 85712643$ are 1, 3, 6, and 7, and the increasing runs of π are 8, 57, 126, 4, and 3. The composition of n whose parts correspond to the increasing run lengths of π is called the *descent composition* of π and is denoted $\operatorname{Comp}(\pi)$. Thus, $\operatorname{Comp}(\pi) = (1, 2, 3, 1, 1)$ for $\pi = 85712643$.

There are many classical results in the literature that count permutations with various restrictions on descent compositions. For example, David and Barton [4] showed that

$$\left[\sum_{n=0}^{\infty}\left(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!}\right)\right]^{-1} \tag{1.1}$$

is the exponential generating function for permutations in which every increasing run has length less than m. Many classical permutation statistics can also be characterized in

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terms of their descent composition. For example, the number of descents of a permutation is exactly one less than its number of increasing runs.

In a series of recent papers [11, 21, 22, 10], the present authors show that permutation enumeration formulas involving restrictions on descent compositions (or involving permutation statistics that can be described in terms of the descent composition) can be proven under a unifying framework by deriving a lifting of the formula in the algebra **Sym** of noncommutative symmetric functions and then applying an appropriate homomorphism. We use this very general method to prove a myriad of new formulas as well as to provide a new derivation of David and Barton's formula (1.1) and of other formulas previously obtained by Carlitz [1], Chebikin [3], Elizalde [5], Elizalde–Noy [6], Petersen [14, 15], Remmel [17], Stanley [18, 19], and Stembridge [20].

The purpose of this extended abstract is to outline this method of permutation enumeration, summarize some of the work done in the papers [11, 21, 22, 10], and to describe new developments in this domain which utilize the theory of shuffle-compatible permutation statistics. We begin in Section 2 by introducing preliminary definitions and ideas from permutation enumeration and the basic theory of noncommutative symmetric functions, including a key reciprocity formula from the first author's Ph.D. dissertation [8] which is used to obtain many noncommutative symmetric function formulas necessary for our applications. In Section 3, we define three homomorphisms on noncommutative symmetric functions and demonstrate several applications of our method using these homomorphisms, including a rederivation of (1.1). Finally, in Section 4, we explain how each "shuffle-compatible descent statistic" induces a new homomorphism that can be used to count permutations by the corresponding "inverse statistic", and present a few applications of these homomorphisms for the descent and peak number statistics.

2 Preliminaries

2.1 Permutation enumeration

We let $\operatorname{des}(\pi)$ denote the number of descents of π. The distribution of the descent number statistic is given by the nth *Eulerian polynomial* $A_n(t)$ defined by

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{des}(\pi)+1}$$

for $n \geq 1$ and by $A_0(t) := 1$.[1] A related permutation statistic is the number of inversions; an *inversion* of a permutation $\pi \in \mathfrak{S}_n$ is a pair of indices (i, j) with $1 \leq i < j \leq n$ such that $\pi_i > \pi_j$. Then the number of inversions of π is denoted $\operatorname{inv}(\pi)$. For example, the

[1]For all families of statistic-encoding polynomials defined in this extended abstract, we take the convention that the 0th polynomial is always defined to be 1.

inversions of $\pi = 1432$ are $(2,3)$, $(2,4)$, and $(3,4)$, so $\operatorname{inv}(\pi) = 3$. The joint distribution of the inversion number and descent number is given by the nth *q-Eulerian polynomial* $A_n(q,t)$ defined by

$$A_n(q,t) \coloneqq \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}(\pi)} t^{\operatorname{des}(\pi)+1}.$$

We say that π is an *alternating* permutation if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. It is well known that the number of alternating permutations in $\mathfrak{S}_n$ is the nth Euler number E_n defined by $\sum_{n=0}^{\infty} E_n x^n/n! = \sec x + \tan x$. In [3], Chebikin introduced a variant of the notion of descents which is closely related to alternating permutations and the Euler numbers: $i \in [n-1]$ is called an *alternating descent* of π if i is odd and $\pi_i > \pi_{i+1}$ or if i is even and $\pi_i < \pi_{i+1}$. We define an *alternating run* of π to be a maximal consecutive subsequence of π containing no alternating descents. For example, the alternating runs of the permutation 3421675 are 342, 1, and 675.

The notions of alternating descents and alternating runs give rise to an "alternating analogue" for nearly every concept defined in terms of descents. For example, the alternating descent composition and the alternating descent number altdes are defined in the obvious way. The distribution of the alternating descent number over $\mathfrak{S}_n$ is given by the nth *alternating Eulerian polynomial* defined by

$$\hat{A}_n(t) \coloneqq \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{altdes}(\pi)+1}.$$

2.2 Noncommutative symmetric functions

Let $\mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$ be the $\mathbb{Q}$-algebra of formal power series in countably many noncommuting variables $X_1, X_2, \ldots$. Consider the *ribbon functions* $\mathbf{r}_L \in \mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$ defined as follows. For a composition $L = (L_1, \ldots, L_k)$, let

$$\mathbf{r}_L \coloneqq \sum_{i_1,\ldots i_n} X_{i_1} X_{i_2} \cdots X_{i_n}$$

where the sum is over all $i_1, \ldots, i_n$ satisfying

$$\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k}.$$

Let **Sym** denote the vector space with basis $\{\mathbf{r}_L\}$; then **Sym** is a graded $\mathbb{Q}$-subalgebra of $\mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$. The elements of **Sym** are called *noncommutative symmetric functions* and were introduced in the seminal paper [7] of Gelfand et al.

In particular, define the noncommutative symmetric function $\mathbf{h}_n$ by

$$\mathbf{h}_n := \mathbf{r}_{(n)} = \sum_{i_1 \leq \cdots \leq i_n} X_{i_1} X_{i_2} \cdots X_{i_n}$$

for $n \geq 1$ and let $\mathbf{h}_0 := 1$. These are noncommutative versions of the complete symmetric functions h_n. The $\mathbf{h}_n$ are algebraically independent and generate **Sym**, so to define a $\mathbb{Q}$-algebra homomorphism on **Sym**, it suffices to define the map on the $\mathbf{h}_n$.

In what follows, we will make use of a reciprocity theorem relating the $\mathbf{h}_n$ and $\mathbf{r}_L$. This theorem was first proved in an equivalent form by the first author [8, Theorem 5.2], and we call this the "run theorem" because it can be used, as we do here, to prove many formulas counting permutations with restrictions on increasing runs.

Theorem 1 (Run theorem)**.** *Suppose that the sequences $\{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ are related by*

$$\sum_{n=0}^{\infty} v_n x^n = \left(\sum_{n=0}^{\infty} w_n x^n \right)^{-1}$$

where $v_0 = w_0 = 1$. Then

$$\sum_L w_L \mathbf{r}_L = \left(\sum_{n=0}^{\infty} v_n \mathbf{h}_n \right)^{-1}$$

where the sum on the left is over all compositions L, and w_L is defined by $w_L := w_{L_1} w_{L_2} \cdots w_{L_k}$ where $L = (L_1, L_2, \ldots, L_k)$.

3 Our method

3.1 Set-up: three homomorphisms

Many results in the next two sections are obtained by applying certain homomorphisms to various identities involving noncommutative symmetric functions. The simplest of these homomorphisms is the map $\Phi\colon \mathbf{Sym} \to \mathbb{Q}[[x]]$ defined by $\Phi(\mathbf{h}_n) = x^n/n!$. We now give an alternating analogue and a q-analogue of Φ. Define the homomorphism $\hat{\Phi}\colon \mathbf{Sym} \to \mathbb{Q}[[x]]$ by $\hat{\Phi}(\mathbf{h}_n) = E_n x^n/n!$ and define the homomorphism $\Phi_q\colon \mathbf{Sym} \to \mathbb{Q}[[q,x]]$ by $\Phi_q(\mathbf{h}_n) = x^n/[n]_q!$.

For our applications, we need to determine the effect of these homomorphisms on the ribbon functions. Let $\beta(L)$ be the number of permutations with descent composition L, let $\hat{\beta}(L)$ be the number of permutations with alternating descent composition L, and let

$$\beta_q(L) := \sum_{\mathrm{Comp}(\pi)=L} q^{\mathrm{inv}(\pi)}$$

be the polynomial counting permutations with descent composition L by inversion number.

Lemma 2. *Let L be a composition of n. We have*

$$\Phi(\mathbf{r}_L) = \beta(L)\frac{x^n}{n!}, \quad \hat{\Phi}(\mathbf{r}_L) = \hat{\beta}(L)\frac{x^n}{n!}, \quad \text{and} \quad \Phi_q(\mathbf{r}_L) = \beta_q(L)\frac{x^n}{[n]_q!}.$$

See [11, 22] for a proof of Lemma 2.

3.2 Example: David and Barton's formula

The homomorphisms Φ, $\hat{\Phi}$, and Φ_q give us a general principle that whenever we have an exponential generating function that counts permutations with a restriction on increasing run lengths, there is an analogous exponential generating function—obtained by replacing $x^n/n!$ by $E_n x^n/n!$—for counting permutations with the same restriction on alternating run lengths, as well as an analogous q-exponential generating function—obtained by replacing $x^n/n!$ by $x^n/[n]_q!$—for counting permutations with the same restriction on increasing run lengths but also keeping track of the inversion number.

As a first example, let us show how we can derive David and Barton's formula (1.1) as well as an alternating analogue and a q-analogue of (1.1) using these three homomorphisms. First we need the following lemma, which is easily proven using the run theorem by taking $w_n = 1$ for all $n < m$ and $w_n = 0$ for all $n \geq m$.

Lemma 3. *Let m be a positive integer. Then*

$$\sum_L \mathbf{r}_L = \Big(\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \Big)^{-1} \tag{3.1}$$

where the sum on the left is over all compositions L with all parts less than m.

Observe that applying the homomorphism Φ to (3.1) recovers David and Barton's formula (1.1). Applying $\hat{\Phi}$ to (3.1) instead, we obtain an alternating analogue of (1.1).

Theorem 4. *Let m be a positive integer, and let $a_{m,n}$ denote the number of permutations in $\mathfrak{S}_n$ with all alternating runs having length less than m. Then*

$$\sum_{n=0}^{\infty} a_{m,n}\frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \Big(E_{mn}\frac{x^{mn}}{(mn)!} - E_{mn+1}\frac{x^{mn+1}}{(mn+1)!} \Big) \right]^{-1}.$$

And, applying Φ_q to (3.1) yields a q-analogue of (1.1). Let $\mathrm{Av}_n(\underline{12\cdots m})$ denote the set of n-permutations with all increasing runs having length less than m.[2]

[2]This notation comes from the literature on permutation pattern avoidance; permutations with all increasing runs having length less than m are precisely the permutations avoiding the monotone consecutive pattern $\underline{12\cdots m}$.

Theorem 5. *Let m be a positive integer. Then*

$$\sum_{n=0}^{\infty} \sum_{\pi \in \mathrm{Av}_n(\underline{12\cdots m})} q^{\mathrm{inv}(\pi)} \frac{x^n}{[n]_q!} = \left[\sum_{n=0}^{\infty} \left(\frac{x^{mn}}{[mn]_q!} - \frac{x^{mn+1}}{[mn+1]_q!} \right) \right]^{-1}.$$

Theorem 4 is new to our approach, whereas Theorem 5 was also recently proven by Elizalde [5] using a q-analogue of the Goulden–Jackson cluster method in the context of consecutive pattern avoidance.

3.3 Example: A generalization of David and Barton's formula

The next main result is motivated by the following question: Since

$$\left[\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right]^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

do the coefficients of reciprocals of the partial sums of $\sum_{n=0}^{\infty} (-1)^n x^n/n!$ have a combinatorial interpretation? There are negative coefficients for partial sums with an odd number of terms, but for those with an even number of terms the reciprocals have all non-negative coefficients, and here we give a combinatorial interpretation for these coefficients. Related results can be found in [10].

Theorem 6. *Let k be a positive integer, and let $b_{k,n}$ be the number of permutations in $\mathfrak{S}_n$ in which every increasing run has length congruent to 0 or 1 modulo $2k$. Then*

$$\sum_{n=0}^{\infty} b_{k,n} \frac{x^n}{n!} = \left[\sum_{n=0}^{2k-1} (-1)^n \frac{x^n}{n!} \right]^{-1}.$$

We first state a lemma, which can be obtained via the run theorem by taking $w_n = 1$ whenever n is congruent to $0, 1, \ldots,$ or $m-1$ modulo km and taking $w_n = 0$ otherwise.

Lemma 7. *Let m and k be positive integers. Then*

$$\sum_{L} \mathbf{r}_L = \left(\sum_{n=0}^{k-1} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right)^{-1} \tag{3.2}$$

where the sum on the right is over all compositions L where every part is congruent to $0, 1, \ldots,$ or $m-1$ modulo km.

Applying the homomorphism Φ to (3.2), we obtain the following.

Theorem 8. *Let m and k be positive integers, and let $c_{k,m,n}$ be the number of permutations in $\mathfrak{S}_n$ in which every increasing run has length congruent to $0, 1, \ldots,$ or $m-1$ modulo km. Then*

$$\sum_{n=0}^{\infty} c_{k,m,n} \frac{x^n}{n!} = \left[\sum_{n=0}^{k-1} \left(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!} \right) \right]^{-1}. \tag{3.3}$$

Note that Theorem 6 follows by setting $m = 2$ in (3.3) and we can recover David and Barton's formula by taking the limit as $k \to \infty$ in (3.3). We can also obtain an alternating analogue and a q-analogue of this formula, but we omit these results here.

3.4 Example: Eulerian polynomials

Lastly, we demonstrate how the homomorphisms Φ, $\hat{\Phi}$, and Φ_q can be used to recover several results relating to the polynomials $A_n(t)$, $\hat{A}_n(t)$, and $A_n(q,t)$ defined in Section 2.1. As before, we begin with a noncommutative symmetric function formula; this one can be obtained from the run theorem by setting $w_n = t$ for all $n \geq 1$.

Lemma 9. *Let m be a positive integer. Then*

$$\sum_{L} t^{l(L)} \mathbf{r}_L = (1-t) \left(1 - t \sum_{n=0}^{\infty} (1-t)^n \mathbf{h}_n \right)^{-1} \tag{3.4}$$

where the sum on the left is over all compositions L, and $l(L)$ denotes the number of parts of L.

Applying Φ to (3.4) yields the classical exponential generating function

$$\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1-t}{1 - te^{(1-t)x}}$$

for Eulerian polynomials. Applying $\hat{\Phi}$ instead yields the exponential generating function

$$\sum_{n=0}^{\infty} \hat{A}_n(t) \frac{x^n}{n!} = \frac{1-t}{1 - t(\sec((1-t)x) + \tan((1-t)x))}$$

for alternating Eulerian polynomials, proved earlier in an equivalent form by Chebikin [3]. Finally, applying Φ_q yields Stanley's [18] well-known q-exponential generating function

$$\sum_{n=0}^{\infty} A_n(q,t) \frac{x^n}{[n]_q!} = \frac{1-t}{1 - t \exp_q((1-t)x)}$$

for q-Eulerian polynomials, where $\exp_q(x) := \sum_{n=0}^{\infty} x^n / [n]_q!$.

4 Homomorphisms arising from shuffle-compatibility

4.1 Quasisymmetric functions and shuffle-compatibility

Let $x_1, x_2, \ldots$ be commuting variables. Then QSym is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}[[x_1, x_2, \ldots]]$ with basis given by the *fundamental quasisymmetric functions* $\{F_L\}_{L \vDash n}$, where

$$F_L := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}(L)}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

and $\mathrm{Des}(L)$ is the descent set of any permutation with descent composition L. The elements of QSym are called *quasisymmetric functions* and were introduced by the first author [9]. The algebra QSym of quasisymmetric functions is closely related to the algebra **Sym** of noncommutative symmetric functions; for instance, they are dual as Hopf algebras. We will make use of a natural homomorphism $\tau\colon \mathbf{Sym} \to \mathrm{QSym}$ given by

$$\tau(\mathbf{r}_L) = \sum_{\mathrm{Comp}(\pi)=L} F_{\mathrm{Comp}(\pi^{-1})}.$$

This homomorphism is obtained by composing the canonical inclusion from **Sym** to the Malvenuto–Reutenauer algebra with the canonical projection from the Malvenuto-Reutenauer algebra to QSym; see [13, Section 8] for details.

Recently, the present authors initiated the study of shuffle-compatible permutation statistics [12]. Roughly speaking, a permutation statistic st is said to be shuffle-compatible if the distribution of st over the "shuffles" of two permutations π and σ depend only on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, and the lengths of π and σ. We will not need the precise definition of a shuffle-compatible permutation statistic here, but we note that the descent number is shuffle-compatible whereas the alternating descent number and inversion number are not. Other classical permutation statistics known to be shuffle-compatible include the major index, comajor index, and peak number.

A permutation statistic is called a *descent statistic* if it depends only on the descent composition. For example, the descent number, alternating descent number, major index, comajor index, and peak number are all descent statistics, but the inversion number inv is not. Given a descent statistic st, we write $\mathrm{st}(L)$ to mean the value of st on any permutation with descent composition L. Thus every descent statistic st induces an equivalence relation on the set of compositions in the following way: we say that compositions J and K are st-*equivalent* if $|J| = |K|$ and $\mathrm{st}(J) = \mathrm{st}(K)$. For example, the compositions $J = (3,1,4)$ and $K = (2,4,2)$ are des-equivalent.

The following is one of our main theorems in [12], relating the study of shuffle-compatible descent statistics to the theory of quasisymmetric functions.

Theorem 10. *A descent statistic* st *is shuffle-compatible if and only if there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{st}} : \mathrm{QSym} \to \mathcal{A}_{\mathrm{st}}$*, where* $\mathcal{A}_{\mathrm{st}}$ *is a* $\mathbb{Q}$*-algebra with basis* $\{a_\alpha\}$ *indexed by* st*-equivalence classes* α *of compositions, such that* $\phi_{\mathrm{st}}(F_L) = a_\alpha$ *whenever* $L \in \alpha$*.*

Here, we note that $\mathcal{A}_{\mathrm{st}}$ is called the *shuffle algebra* of st and the multiplication of its basis elements a_α encode the distribution of st over shuffles of permutations.

Given a shuffle-compatible descent statistic st, let us define the homomorphism Φ_{st} by $\Phi_{\mathrm{st}} := \phi_{\mathrm{st}} \circ \tau$. As with Φ, $\hat{\Phi}$, and Φ_q, we can use the homomorphisms Φ_{st} under the same framework to produce analogous permutation enumeration formulas for the "inverse statistic" corresponding to st. We shall illustrate this with Φ_{des}, the homomorphism arising from the shuffle-compatibility of the descent number.

4.2 Counting permutations by inverse descents

The *Hadamard product* $*$ on formal power series in t is given by

$$\Big(\sum_{n=0}^{\infty} a_n t^n\Big) * \Big(\sum_{n=0}^{\infty} b_n t^n\Big) = \sum_{n=0}^{\infty} a_n b_n t^n.$$

Let $\mathbb{Q}[[t*, x]]$ denote the $\mathbb{Q}$-algebra of formal power series in t and x, where the multiplication is Hadamard product in t. The shuffle algebra of the descent number is a subalgebra of $\mathbb{Q}[[t*, x]]$, and we can describe $\Phi_{\mathrm{des}} \colon \mathbf{Sym} \to \mathbb{Q}[[t*, x]]$ by

$$\Phi_{\mathrm{des}}(\mathbf{r}_L) = \sum_{\mathrm{Comp}(\pi)=L} \frac{t^{\mathrm{ides}(\pi)+1}}{(1-t)^{n+1}} x^n,$$

where ides is the *inverse descent number* statistic defined by $\mathrm{ides}(\pi) := \mathrm{des}(\pi^{-1})$. In particular, $\Phi_{\mathrm{des}}(\mathbf{h}_n) = \Phi_{\mathrm{des}}(\mathbf{r}_{(n)}) = tx^n/(1-t)^{n+1}$.

Let

$$M_{m,n}^{\mathrm{ides}}(t) := \sum_{\pi \in \mathrm{Av}_n(\underline{12\cdots m})} t^{\mathrm{ides}(\pi)+1}.$$

By applying Φ_{des} to (3.1), we obtain

$$\sum_{n=0}^{\infty} \frac{M_{m,n}^{\mathrm{ides}}(t)}{(1-t)^{n+1}} x^n = \left[\frac{t}{(1-t)^2} x - \sum_{k=1}^{\infty} \Big(\frac{t}{(1-t)^{mk+1}} x^{mk} - \frac{t}{(1-t)^{mk+2}} x^{mk+1} \Big) \right]^{-1},$$

where the reciprocal is with respect to the Hadamard product in t. After some additional algebraic manipulations, we arrive at the following "Hadamard product-free" formula:

Theorem 11. *Let m be a positive integer and let $\omega := e^{2\pi i/m}$. Then*

$$\sum_{n=0}^{\infty} \frac{M_{m,n}^{\text{ides}}(t)}{(1-t)^{n+1}} x^n = m \sum_{k=0}^{\infty} \left[\sum_{j=1}^{m-1} \frac{1-\omega^{-j}}{(1-\omega^j x)^k} \right]^{-1} t^k$$
$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1} x^{jm} - \binom{k+jm}{k-1} x^{jm+1} \right) \right]^{-1} t^k.$$

This is a new result which refines David and Barton's formula by inverse descents. Note that taking the limit as $m \to \infty$ and extracting coefficients of x^n recovers the classical identity $A_n(t)/(1-t)^{n+1} = \sum_{k=0}^{\infty} k^n t^k$ for Eulerian polynomials.

Let us discuss one more application of the homomorphism Φ_{des}. The nth *two-sided Eulerian polynomial* $A_n(s,t)$ is defined by

$$A_n(s,t) := \sum_{\pi \in \mathfrak{S}_n} s^{\text{des}(\pi)} t^{\text{ides}(\pi)}.$$

The formula

$$\frac{A_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}} = \sum_{j,k=0}^{\infty} \binom{jk+n-1}{n} s^j t^k \tag{4.1}$$

was proven by Carlitz–Roselle–Scoville [2] and by Petersen [16]; applying Φ_{des} to (3.4) leads to a rederivation of this formula.

4.3 Counting permutations by peaks and inverse peaks

Given $\pi \in \mathfrak{S}_n$, we say that i is a *peak* of π if $2 \leq i \leq n-1$ and $\pi_{i-1} < \pi_i > \pi_{i+1}$; let $\text{pk}(\pi)$ be the number of peaks of π. The peak number pk is a shuffle-compatible permutation statistic, so it induces a homomorphism Φ_{pk} which can be used to produce new formulas counting permutations by "inverse peaks". Like the descent number shuffle algebra, the peak number shuffle algebra is a subalgebra of $\mathbb{Q}[[t*, x]]$. We can describe $\Phi_{\text{pk}} \colon \mathbf{Sym} \to \mathbb{Q}[[t*, x]]$ by

$$\Phi_{\text{pk}}(\mathbf{r}_L) = \sum_{\text{Comp}(\pi)=L} \frac{2^{2\,\text{ipk}(\pi)+1} t^{\text{ipk}(\pi)+1} (1+t)^{n-2\,\text{ipk}(\pi)-1}}{(1-t)^{n+1}} x^n,$$

where $\text{ipk}(\pi) := \text{pk}(\pi^{-1})$ is the number of inverse peaks of π.

We end this paper with an example of a new formula which can be obtained via the homomorphism Φ_{pk}. Define the nth *two-sided peak polynomial* $P_n(s,t)$ by

$$P_n(s,t) := \sum_{\pi \in \mathfrak{S}_n} s^{\text{pk}(\pi)} t^{\text{ipk}(\pi)}.$$

Theorem 12. *Let n be a positive integer. Then*

$$\frac{1}{4}\left(\frac{(1+s)(1+t)}{(1-s)(1-t)}\right)^{n+1} P_n\left(\frac{4s}{(1+s)^2}, \frac{4t}{(1+t)^2}\right) = \sum_{i,j=0}^{\infty}\sum_{k=0}^{n}\binom{2ij}{k}\binom{2ij+n-k-1}{n-k}s^i t^j.$$

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