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P-partitions and p-positivity

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Abstract. Using the combinatorics of α-unimodal sets, we establish two new results in the theory of quasisymmetric functions. First, we obtain the expansion of the fundamental basis into quasisymmetric power sums. Secondly, we prove that generating functions of reverse P-partitions expand positively into quasisymmetric power sums. Consequently any nonnegative linear combination of such functions is p-positive whenever it is symmetric. We apply this method to derive positivity results for chromatic quasisymmetric functions and unicellular LLT polynomials.

Keywords: posets, symmetric functions, unimodality

1 Introduction

Whenever a new family of symmetric functions is discovered, one of the most logical first steps to take is to expand them in one of the many interesting bases of the space of symmetric functions. This paradigm can be traced from Newton's identities to modern textbooks such as [17]. Of special interest are expansions in which all coefficients are nonnegative integers. Such coefficients frequently encode highly nontrivial combinatorial or algebraic information.

A symmetric function is called p-*positive* if the expansion in the power-sum basis has nonnegative coefficients. There are numerous results in the literature regarding p-positivity, see for example [20, 19, 4, 21, 8, 2]. In particular, the expansion of a symmetric function into power sum symmetric functions can be useful when one is working with plethystic substitution [15], or evaluating certain polynomials at roots of unity [7, 19].

In this extended abstract we provide a uniform method for finding power sum expansions and proving p-positivity in many of the cases mentioned above. Our approach requires a detour to quasisymmetric functions. There is a quasisymmetric extension of p-positivity, namely positivity in a *quasisymmetric power-sum basis*. Quasisymmetric power sums were recently introduced by C. Ballantine et al. [5]. They appear as duals of noncommutative power sums defined by I. Gelfand et al. [9].

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Our first main result, Theorem 2.2, is an expansion of the fundamental quasisymmetric functions $\mathrm{F}_{n,S}$ into quasisymmetric power sums Ψ_α:

$$\mathrm{F}_{n,S}(\mathbf{x}) = \sum_{\alpha} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}(-1)^{|S \setminus S_\alpha|}$$

Here the sum ranges over all compositions α of n such that the set S is α-unimodal, and z_α and S_α denote an integer factor and a set associated to α. All definitions are given in Section 2.

Our second main result, Theorem 3.2, is the following statement: Let P be a finite poset on n elements and $K_P(\mathbf{x})$ be the generating function of order-preserving maps $f : P \to \mathbb{N}^+$, so called *P-partitions*. Then

$$K_P(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \left|\mathcal{L}^*_\alpha(P, w)\right| = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \left|\mathcal{O}^*_\alpha(P)\right|.$$

Here w denotes an arbitrary natural labeling of P, $\mathcal{L}^*_\alpha(P, w)$ consists of certain α-unimodal linear extensions of P, and $\mathcal{O}^*_\alpha(P)$ is a set of certain order-preserving surjections from P onto a chain. These definitions are found in Section 3.

As a consequence, we can produce an expansion into power sums for any *symmetric* function, for which the expansion in Gessel's fundamental basis or in terms of P-partitions is known. Moreover any symmetric function which is a nonnegative linear combination of functions K_P is automatically p-positive. As a bonus it now becomes apparent that p-positivity of certain families of symmetric functions is really a special case of a more general positivity phenomenon, namely Ψ-positivity, that encompasses a larger class of quasisymmetric functions. In Section 4 we give a few examples of the numerous implications of the above formula. The full paper [3] contains additional applications and results. Some exciting recent result related to this extended abstract are due to by R. Liu and M. Weselcouch [14].

2 The Ψ-expansion of fundamental quasisymmetric functions

In this section we briefly introduce the space of quasisymmetric functions and a few relevant bases. For a more thorough background on quasisymmetric functions, we refer the reader to the references [23, 16]. Afterwards, we define α-unimodality and present the expansion of the fundamental basis into quasisymmetric power sums.

2.1 Quasisymmetric functions

The *monomial quasisymmetric function* M_α, where α is a composition with ℓ parts, is defined as

$$\mathrm{M}_\alpha(\mathbf{x}) := \sum_{i_1 < i_2 < \cdots < i_\ell} \mathbf{x}_{i_1}^{\alpha_1} \mathbf{x}_{i_2}^{\alpha_2} \cdots \mathbf{x}_{i_\ell}^{\alpha_\ell}.$$

The functions M_α constitute a basis for the space of homogeneous quasisymmetric functions of degree n as α ranges over all compositions of n.

Given a composition $\alpha \vDash n$ with ℓ parts, let

$$S_\alpha := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}.$$

The map $\alpha \mapsto S_\alpha$ defines the usual bijection between compositions of n and subsets of $[n-1]$. We let $\alpha \leq \beta$ denote the fact that α is a *refinement* of β, that is, β can be obtained from α by adding consecutive parts of α. Whenever $\alpha \leq \beta$, we can illustrate this relationship with bars between parts of α, such that parts between bars add to parts of β. For example

$$112|341|21|34|2 \quad \text{corresponds to} \quad \alpha = 11234121342, \quad \beta = 48372.$$

Finally, given a permutation $\sigma \in \mathfrak{S}_n$ and compositions $\alpha \leq \beta$, we can partition σ into α-subwords and β-blocks of words. For example,

$$\alpha = 3132, \quad \beta = 45, \quad \sigma = 926345817 \text{ is partitioned as } (926)(3)|(458)(17), \tag{2.1}$$

where the first β-block is $(926)(3)$, consisting of two α-subwords.

The *fundamental quasisymmetric function* $\mathrm{F}_{n,S}$ can be defined in two equivalent ways:

$$\mathrm{F}_{n,S}(\mathbf{x}) := \sum_{\substack{j_1 \leq j_2 \leq \cdots \leq j_n \\ i \in S \Rightarrow j_i < j_{i+1}}} \mathbf{x}_{j_1} \cdots \mathbf{x}_{j_n} \qquad \text{or} \qquad \mathrm{F}_\alpha(\mathbf{x}) := \sum_{\beta \leq \alpha} \mathrm{M}_\beta(\mathbf{x}),$$

where $\mathrm{F}_{n,S_\alpha}(\mathbf{x})$ and $\mathrm{F}_\alpha(\mathbf{x})$ are equal for all compositions $\alpha \vDash n$.

Given a pair of compositions of n, $\alpha \leq \beta$, related by

$$\alpha_{11}\alpha_{12} \ldots \alpha_{1,i_1} | \alpha_{21}\alpha_{22} \ldots \alpha_{2,i_2} | \cdots | \alpha_{k1}\alpha_{k2} \ldots \alpha_{k,i_k}$$

let

$$\pi(\alpha, \beta) := \prod_{j=1}^{k} (\alpha_{j1})(\alpha_{j1} + \alpha_{j2}) \cdots (\alpha_{j1} + \alpha_{j2} + \cdots + \alpha_{j,i_j}).$$

The *quasisymmetric power sum* Ψ_α is defined[1] as

$$\Psi_\alpha(\mathbf{x}) := z_\alpha \sum_{\beta \geq \alpha} \frac{1}{\pi(\alpha,\beta)} \mathrm{M}_\beta(\mathbf{x}). \tag{2.2}$$

For example, $\Psi_{231} = \frac{1}{10}\mathrm{M}_6 + \frac{1}{4}\mathrm{M}_{24} + \frac{3}{5}\mathrm{M}_{51} + \mathrm{M}_{231}$. It was shown in [5, Thm. 3.11] that quasisymmetric power sums refine the usual power sums as

$$p_\lambda(\mathbf{x}) = \sum_{\alpha \sim \lambda} \Psi_\alpha(\mathbf{x})$$

where the sum is taken over all compositions α that are a permutation of λ.

Let ω be the involution on quasisymmetric functions that sends $\mathrm{F}_{n,S}$ to $\mathrm{F}_{[n-1]\setminus(n-S)}$. This extends the classical involution on symmetric functions, for which $\omega \mathrm{h}_\lambda = \mathrm{e}_\lambda$, and $\omega(\mathrm{p}_\lambda) = (-1)^{|\lambda|-\ell(\lambda)}\mathrm{p}_\lambda$. Then (see [5, Sec. 4]) we have that $\omega\left(\Psi_\alpha\right) = (-1)^{|\alpha|-\ell(\alpha)}\Psi_{\alpha^r}$, where α^r denotes the reverse of α.

2.2 The Ψ-expansion of the fundamental basis

A word $\sigma_1\sigma_2\cdots\sigma_n$ is said to be *unimodal* if there is a $k \in [n]$ such that

$$\sigma_1 > \cdots > \sigma_k < \cdots < \sigma_n.$$

Given a permutation $\sigma \in \mathfrak{S}_n$ and a composition $\alpha \vDash n$, we can partition σ into nonoverlapping subwords with sizes given by α. A permutation σ is *α-unimodal* if each subword determined by α is unimodal. Finally, a subset of $[n-1]$ is said to be *α-unimodal* if it is the descent set of some α-unimodal permutation in $\mathfrak{S}_n$. There are various equivalent characterizations of α-unimodal sets, and their properties are an interesting topic in itself.

Example 2.1. *Let $\alpha = 3513$. Then the permutation $\sigma = 7,2,3,12,9,8,6,11,4,1,5,10$ is α-unimodal as the the four subwords*

$$7,2,3 \quad | \quad 12,9,8,6,11 \quad | \quad 4 \quad | \quad 1,5,10$$

are unimodal. Hence the set $\mathrm{DES}(\sigma) = \{1,4,5,6,8,9\}$ is α-unimodal.

The first main result is the following theorem:

Theorem 2.2. *Let $n \in \mathbb{N}$ and $S \subseteq [n-1]$. Then*

$$\mathrm{F}_{n,S}(\mathbf{x}) = \sum_\alpha \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}(-1)^{|S\setminus S_\alpha|}, \tag{2.3}$$

where the sum ranges over all compositions α of n such that S is α-unimodal.

[1] Here, z_α is the standard quantity $\prod_{i\geq 1} i^{m_i} m_i!$, with m_i being the number of parts of α equal to i.

Sketch of proof. Expand both sides in the monomial basis and compare coefficients of $\mathrm{M}_\beta(\mathbf{x})$. It then suffices to prove that

$$\sum_{\alpha\in R(\beta,\gamma)} \frac{(-1)^{|S_\gamma\setminus S_\alpha|}}{\pi(\alpha,\beta)} = \begin{cases} 1 & \text{if } \beta\leq\gamma, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

where $R(\beta,\gamma)$ is the set of all compositions $\alpha\leq\beta$ such that S_γ is α-unimodal.

We now need some additional terminology from [5]. Let σ be a permutation and $\alpha\leq\beta$ be compositions. Consider the partitioning of σ into α-subwords and β-blocks of subwords according to (2.1). We say $\sigma\in\mathfrak{S}_n$ is *consistent with* $\alpha\leq\beta$ if the following conditions are satisfied:

(i) In each α-subword the maximum appears in last position.

(ii) The subwords in each β-block are sorted increasingly with respect to their maxima.

Let $\textsc{Cons}(\alpha,\beta)$ denote the set of permutations $\sigma\in\mathfrak{S}_n$ that are consistent with $\alpha\leq\beta$. For example, the permutation $(4)(38)|(7)|(56)(219)$ lies in $\textsc{Cons}(12123,3155)$, but the permutations $(4)(38)|(7)|(65)(219)$ and $(4)(38)|(7)|(59)(216)$ do not.

Using the hook-length formula for forests [12, Chap. 5.1.4, Ex. 20], one can show that $\textsc{Cons}(\alpha,\beta)\pi(\alpha,\beta)=n!$. To prove (2.4) it therefore suffices to show that

$$\sum_{\alpha\in R(\beta,\gamma)} |\textsc{Cons}(\alpha,\beta)|\,(-1)^{|S_\gamma\setminus S_\alpha|} = \begin{cases} n! & \text{if } \beta\leq\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

This can be done by finding a bijection in the case $\beta\leq\gamma$, and a sign-reversing involution otherwise. □

Theorem 2.2 implies two previously known results that also involve α-unimodal sets. The first one is due to C. Athanasiadis and gives the p-expansion of a symmetric function for which the fundamental expansion is known.

Corollary 2.3. ([4, Prop. 3.2], see also [1, Prop. 9.3]). *Let $n\in\mathbb{N}$ and $X=\sum_{S\subseteq[n-1]}c_S F_{n,S}$ be a symmetric function. Then*

$$X(\mathbf{x}) = \sum_{\lambda\vdash n}\frac{p_\lambda(\mathbf{x})}{z_\lambda}\sum_{S\text{ is }\lambda\text{-unimodal}}(-1)^{|S\setminus S_\lambda|}c_S.$$

The second corollary is due to Y. Roichman, and gives a p-expansion of the Schur functions.

Corollary 2.4. ([18, Thm. 4]) *Let λ be a partition of n. Then*

$$s_\lambda(\mathbf{x}) = \sum_{\mu\vdash n}\frac{p_\mu(\mathbf{x})}{z_\mu}\sum_{\substack{T\in\mathrm{SYT}(\lambda)\\ \mathrm{DES}(T)\text{ is }\mu\text{-unimodal}}}(-1)^{|\mathrm{DES}(T)\setminus S_\mu|}.$$

3 The Ψ-expansion of reverse P-partitions

In this section we combine α-unimodality and naturally labeled posets. We then present a positive expansion of generating functions of P-partitions into quasisymmetric power sums.

3.1 Posets and α-unimodality

We use the same terminology for posets as [24]. A *labeled poset* (P, w) consists of a (finite) poset P together with a bijection $w : P \to [n]$. A labeled poset is *natural* if w is order-preserving, that is, $x \leq_P y$ implies $w(x) \leq w(y)$ for all $x, y \in P$. The *Jordan–Hölder set* of a labeled poset is defined as

$$\mathcal{L}(P, w) := \{\sigma \in \mathfrak{S}_n : \sigma^{-1} \circ w \text{ is order-preserving}\}.$$

Let $\alpha \vDash n$ be a composition with ℓ parts. Let

$$\mathcal{L}_\alpha(P, w) := \{\sigma \in \mathcal{L}(P, w) : \sigma \text{ is } \alpha\text{-unimodal}\}.$$

Moreover for $\sigma \in \mathfrak{S}_n$ divide P into subposets

$$P_i^\alpha(\sigma) := \{x \in P : \alpha_1 + \cdots + \alpha_{i-1} < \sigma^{-1} \circ w(x) \leq \alpha_1 + \cdots + \alpha_i\}.$$

Let $\mathcal{L}_\alpha^*(P, w)$ denote the set of all permutations $\sigma \in \mathcal{L}_\alpha(P, w)$ such that the subposet $P_i^\alpha(\sigma)$ has a unique minimal element for all $i \in [\ell]$. For example, let $P = \{x, y, z\}$ be the poset with a single relation $x <_P y$, and define a labeling $w : P \to [3]$ by $w(x) = 1$, $w(y) = 2$ and $w(z) = 3$. Then $\mathcal{L}(P, w) = \{123, 132, 312\}$. We have $\mathcal{L}_3(P, w) = \{123, 312\}$ and $\mathcal{L}_{21}(P, w) = \mathcal{L}(P, w)$. Moreover $\mathcal{L}_3^*(P, w) = \emptyset$ and $\mathcal{L}_{21}^*(P, w) = \{123\}$.

There is an equivalent interpretation of the set $\mathcal{L}_\alpha^*(P, w)$ in terms of order-preserving surjections, which has the advantage of being independent of the labeling w. Let $\mathcal{O}(P)$ denote the set of all order-preserving surjective maps $f : P \to [k]$ for some $k \in \mathbb{N}$. The *type* of such a map is defined as

$$(|f^{-1}(1)|, \ldots, |f^{-1}(k)|)$$

and is a composition of $n = |P|$ with k parts. Let $\mathcal{O}_\alpha(P)$ denote the set of all maps $f \in \mathcal{O}(P)$ with type α. Moreover, let $\mathcal{O}^*(P)$ denote the set of all maps $f \in \mathcal{O}(P)$ such that for all $y, z \in P$ with $f(y) = f(z)$ there exists $x \in P$ with $f(x) = f(y)$ and $x \leq_P y$ and $x \leq_P z$. Finally set $\mathcal{O}_\alpha^*(P) := \mathcal{O}_\alpha(P) \cap \mathcal{O}^*(P)$.

Let α be a composition with ℓ parts, and $\sigma \in \mathcal{L}_\alpha^*(P)$. Define a map $f_\sigma : P \to [\ell]$ by $f(x) = i$ for all $x \in P_i^\alpha(\sigma)$ and all $i \in [\ell]$. It is not too difficult to prove the following:

Proposition 3.1. *Let (P, w) be a naturally labeled poset with n elements, and let α be a composition of n. The the map defined by $\varphi(\sigma) = f_\sigma$ is a bijection $\varphi : \mathcal{L}_\alpha^*(P, w) \to \mathcal{O}_\alpha^*(P)$.*

3.2 P-partitions

In this section we discuss reverse P-partitions of naturally labeled posets. For more background we refer to [24, 23]. Let P be a finite poset. A *reverse P-partition* is an order-preserving map $f : P \to \{1, 2, \dots\}$, that is, $x \leq_P y$ implies $f(x) \leq f(y)$ for all $x, y \in P$. Let $\mathcal{A}^r(P)$ denote the set of reverse P-partitions. The generating function of reverse P partitions is defined as

$$K_P(\mathbf{x}) := \sum_{f \in \mathcal{A}^r(P)} \prod_{x \in P} \mathbf{x}_{f(x)}.$$

We now state the main result of this section.

Theorem 3.2. *Let (P, w) be a naturally labeled poset. Then*

$$K_P(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \left| \mathcal{L}^*_\alpha(P, w) \right| = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \left| \mathcal{O}^*_\alpha(P) \right|. \tag{3.1}$$

In particular, K_P is Ψ-positive for any finite poset P.

Sketch of proof. It is well-known (see [23, Cor. 7.19.5]) that the expansion of reverse P-partitions in into the fundamental basis is given by

$$K_P(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(P,w)} \mathrm{F}_{n,\mathrm{DES}(\sigma)}(\mathbf{x}).$$

From Theorem 2.2 we obtain

$$K_P(\mathbf{x}) = \sum_{\alpha \vDash n} \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha} \sum_{\sigma \in \mathcal{L}_\alpha(P,w)} (-1)^{\mathrm{DES}(\sigma) \setminus S_\alpha}.$$

The proof can be finished by finding a sign-reversing involution on the set $\mathcal{L}_\alpha(P, w) \setminus \mathcal{L}^*_\alpha(P, w)$, which is omitted in this extended abstract. A similar involution was first defined by B. Ellzey in [8, Thm. 4.1]. $\square$

The following is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let $P_1, \dots, P_k$ be finite posets, $a_1, \dots, a_k \in \mathbb{N}$, and*

$$X(\mathbf{x}) = \sum_{i=1}^{k} a_i K_{P_i}(\mathbf{x}).$$

Then X is Ψ-positive, and X is p*-positive whenever X is symmetric.*

4 Consequences

Theorem 3.2 is a highly useful tool in proving p-positivity (or Ψ-positivity) of many families of symmetric or quasisymmetric functions. In this extended abstract we focus on two such families that have received considerable recent attention, the chromatic quasisymmetric functions and certain LLT polynomials.

4.1 Chromatic quasi-symmetric functions

In [22] R. Stanley introduced the chromatic-symmetric function $X_G(\mathbf{x})$ attached to a finite unlabeled graph G, which generalizes the chromatic polynomial of G. In [22, Cor. 2.7.] he noted that $\omega X_G(\mathbf{x})$ is p-positive. This definition and observation have recently been generalized by several authors.

In [21] J. Shareshian and M. Wachs defined a quasisymmetric q-analogue $X_G(\mathbf{x};q)$ of the chromatic symmetric function $X_G(\mathbf{x})$ that is attached to a finite labeled graph G. The chromatic symmetric function $X_G(\mathbf{x})$ is recovered by setting $q = 1$ and forgetting the labels. In [21, Conj. 7.6] they conjectured a positive p-expansion for $\omega X_G(\mathbf{x};q)$ when G belongs to a very special family of labeled graphs, namely the incomparability graphs of natural unit interval orders. This conjecture was proven by C. Athanasiadis in [4].

B. Ellzey extended the definition of chromatic quasisymmetric functions to finite directed graphs in [8]. We recall this definition now. Let $G = (V, E)$ be a finite directed graph. A *proper vertex coloring* (or just *coloring*) of G is a function $\kappa : V \to \mathbb{N}^+$ such that $\kappa(u) \neq \kappa(v)$ for all directed edges $(u, v) \in E$. An *ascent* of a coloring κ is a directed edge (u, v) in E such that $\kappa(u) < \kappa(v)$. Denote the number of ascents of a coloring κ by $\operatorname{asc}(\kappa)$. The *chromatic quasisymmetric function* indexed by G is defined as

$$X_G(\mathbf{x};q) = \sum_{\kappa} q^{\operatorname{asc}(\kappa)} \prod_{v \in V} \mathbf{x}_{\kappa(v)},$$

where the sum ranges over all colorings of G.

The chromatic quasisymmetric function $X_G(\mathbf{x};q)$ of an undirected labeled graph as defined by J. Shareshian and M. Wachs is equal to the chromatic quasisymmetric function corresponding to the acyclic orientation induced by the labels. In [8, Thm. 4.1] B. Ellzey proved a considerable strengthening of the result of C. Athanasiadis, namely that $\omega X_G(\mathbf{x};q)$ is p-positive for any directed graph G for which $X_G(\mathbf{x};q)$ is a symmetric function. Using the results of Section 3.2 we can improve as follows.

Theorem 4.1. *Let G be a finite directed graph, and*

$$\omega X_G(\mathbf{x};q) = \sum_{\alpha} c^G_\alpha(q) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}.$$

Then $c^G_\alpha(q) \in \mathbb{N}[q]$ for all compositions α.

Proof. It is not difficult to show ([2, 8]) that

$$\omega X_G(\mathbf{x};q) = \sum_{\theta} q^{\mathrm{asc}(\theta)} K_{P(\theta)}(\mathbf{x}),$$

where the sum ranges over all acyclic orientations θ of G, $\mathrm{asc}(\theta)$ denotes the number of edges of G that have the same orientation in G and in θ, and $P(\theta)$ denotes the transitive closure of θ. The claim now follows from Corollary 3.3. □

4.2 LLT polynomials

LLT polynomials can be seen as q-deformations of products of skew Schur functions. They are named after A. Lascoux, B. Leclerc and J.-Y. Thibon who introduced them in [13] using ribbon tableaux. A different combinatorial model for the LLT polynomials was considered in [10], where each k-tuple of skew shapes indexes an LLT polynomial. When each such skew shape consists of a single box, the corresponding LLT polynomial is *unicellular*. Unicellular LLT polynomials have a central role in the proof of the shuffle-conjecture due to E. Carlsson and A. Mellit [6], in which they introduced a combinatorial model for the unicellular LLT polynomials using Dyck paths. In [2] this Dyck path model was extended to certain directed graphs.

By modifying the definition of the chromatic quasisymmetric functions slightly, we recover the unicellular LLT polynomials considered in [2]. Let $G = (V, E)$ be a finite directed graph without loops. The *unicellular graph LLT polynomial* indexed by G is defined as

$$\mathrm{G}_G(\mathbf{x};q) = \sum_{\kappa: V \to \mathbb{N}^+} q^{\mathrm{asc}(\kappa)} \prod_{v \in V} \mathbf{x}_{\kappa(v)},$$

where the sum is now taken over all functions (not just *proper* vertex colorings).

The functions $\mathrm{G}_G(\mathbf{x};q)$ are quasisymmetric. In special cases they are known to be symmetric and contain the family of unicellular LLT polynomials. It was observed in [2, 11] that $\omega\mathrm{G}_G(\mathbf{x};q+1)$ is p-positive whenever $\mathrm{G}_G(\mathbf{x};q)$ is a unicellular LLT polynomial. We can now proof a much stronger statement.

Theorem 4.2. *Let G be a finite directed graph without loops, and*

$$\omega\mathrm{G}_G(\mathbf{x};q+1) = \sum_{\alpha} c_{\alpha}^{G}(q) \frac{\Psi_{\alpha}(\mathbf{x})}{z_{\alpha}}.$$

Then $c_{\alpha}^{G}(q) \in \mathbb{N}[q]$ for all compositions α.

Proof. It is known that

$$\omega\mathrm{G}_G(\mathbf{x};q+1) = \sum_{S} q^{|S|} K_{P(S)}(\mathbf{x}),$$

where the sum ranges over all subsets $S \subseteq E$ of (directed) edges of G that do not contain a directed cycle, and $P(S)$ denotes the transitive closure of the edges in S. Again the claim follows from Corollary 3.3. □

If in the model introduced in [10] each skew shape is a vertical strip, the resulting LLT polynomial is called *vertical strip LLT polynomial*. Vertical strip LLT polynomials occur naturally in the study of the delta operator and diagonal harmonics. The family of vertical strip LLT polynomials contains (a version of) modified Hall–Littlewood polynomials. Theorem 4.2 can easily be extended to also include vertical strip LLT polynomials.

5 Open problems

We conclude this extended abstract with two open problems. The first concerns the coefficients in the Ψ-expansion of unicellular graph LLT polynomials, which we know to be polynomials with nonnegative coefficients.

Conjecture 5.1. *Let G be a finite oriented graph (no loops or multiple edges), and let*

$$\omega \mathrm{G}_G(\mathbf{x}; q+1) = \sum_{\alpha} c^G_\alpha(q) \frac{\Psi_\alpha(\mathbf{x})}{z_\alpha}.$$

Then the polynomial $c^G_\alpha(q)$ is unimodal[2] *for all compositions α.*

Conjecture 5.1 has been verified by computer for all oriented graphs with six or fewer vertices. The second open problem concerns the combinatorics of consistent permutations and order-preserving surjections.

Open Problem 5.2. *Let P be a poset with n elements, and let β be a composition of n. Find a bijection*

$$\varphi : \mathfrak{S}_n \times \mathcal{O}_\beta(P) \to \bigcup_{\alpha \leq \beta} \textsc{Cons}(\alpha, \beta) \times \mathcal{O}^*_\alpha(P).$$

The fact that the two sets in Problem 5.2 have the same cardinality is equivalent to Theorem 3.2 and can be obtained by extracting the coefficient of the monomial quasisymmetric function M_β on the left and right hand sides of (3.1). Currently we can give such a bijection only in the simple case where P is a chain. It would be particularly appealing if the solution to Problem 5.2 avoided sign-reversing involutions, and thus led to a purely bijective proof of Theorem 3.2.

[2] Here we use the more common definition of unimodality, that is, $c^G_\alpha(q) = \sum_{i=0}^d a_i q^i$ with $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some k.

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