Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #62, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

The Noncrossing Bond Poset of a Graph

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Abstract. Given a graph G with vertices labeled by $\{1, 2 \ldots, n\}$, the bonds of G are in natural bijection with the set partitions of n. We say a bond is *noncrossing* if its associated partition is noncrossing. Ordering the noncrossing bonds of G by inclusion, one gets a noncrossing analogue of the bond lattice of G called the *noncrossing bond poset*. In this extended abstract we study this poset showing that several properties of the bond lattice have analogues in the noncrossing bond poset.

Keywords: posets, graphs, noncrossing partitions, Möbius function, bond lattice

1 Introduction

The *partition lattice*, Π_n, consists of set partitions of $[n] := \{1, 2, \ldots, n\}$ ordered by refinement. It has several nice combinatorial properties including being supersolvable and shellable. By removing the crossing partitions from the partition lattice, one gets another lattice called the *noncrossing partition lattice*. It too has many nice combinatorial properties. For example, it is supersolvable, rank symmetric, has a Catalan number of elements, and its Möbius value is a Catalan number. See Simion's survey article [6] for more information about the noncrossing partition lattice.

Now suppose that G is a graph on $[n]$. We can think of the bond lattice of G as a subposet of Π_n. We do this by restricting to the set of partitions in Π_n such that for each block B in the partition, the induced subgraph of G with vertex set B is connected. For example, if G is the path $1 - 2 - 3$, the partition $12/3$ is in the bond lattice, but $13/2$ is not since there is no edge between 1 and 3. Using this idea, one can see that Π_n is the bond lattice for the complete graph. In general, the bond lattice of G carries important combinatorial information about the graph. For example, it encodes exactly the same information as the cycle matroid associated to the graph. Moreover, its characteristic polynomial is (essentially) the chromatic polynomial of the graph. See [7, §2] for background on bond lattices. Since the partition lattice is the bond lattice of the complete graph, one can consider the noncrossing partition lattice as a noncrossing version of a bond lattice. It is this idea that is the starting point for our work.

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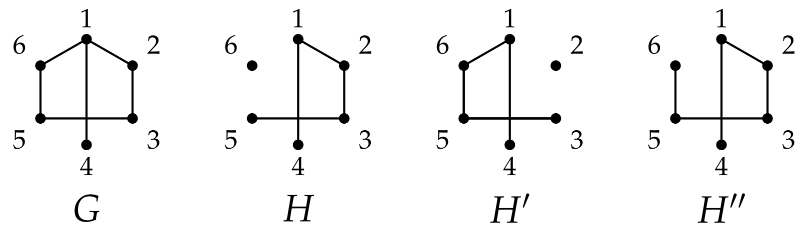

Figure 1: A graph and several subgraphs

In the next section, we explicitly define the noncrossing bond poset of a graph and describe some of its structural properties including necessary and sufficient conditions on the graph which guarantee that its noncrossing bond poset is a lattice. Section 3 is focused on the family of perfectly labeled graphs. We show that in this setting, the noncrossing bond poset is shellable and when it is a lattice, it is supersolvable. This is done using edge labelings. In Section 4, we use Blass and Sagan's work [3] on NBB sets to give a combinatorial interpretation for the Möbius function and the characteristic polynomial of the noncrossing bond poset of a family of graphs. We finish with a brief discussion of an action of the dihedral group.

2 The Structure of the Noncrossing Bond Poset

We assume the reader is familiar with some basic graph theory concepts (see [8, Graph Theory Appendix] for any undefined terms) as well as basic concepts related to posets (see [8, §3] for background and notation). Let G be a graph. For the remainder of this paper, unless otherwise noted, we will assume that the vertex set of G is $[n]$. We will use the notation $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. When we write out edges, we will write them in the form ij where $i < j$. Moreover, we will always draw our graphs so that the vertices lie on a circle with vertex 1 at the top and the remaining vertices appearing in clockwise order around the circle. Edges will always be drawn so that they are the line segments between their endpoints. We will refer to this as the *graphical representation of* G. We say that two edges of G *cross* if their respective line segments intersect in the graphical representation. See Figure 1 for several examples of graphical representations.

A subgraph of a graph G is called *spanning* if it contains all the vertices of G. Note that when considering spanning subgraphs of G, it is enough to just know the edges which appear in the subgraph. Because of this, we will often make no distinction between subsets of $E(G)$ and spanning subgraphs of G. A subgraph H of a graph G is called *induced* if whenever u and v are vertices in H and uv is an edge of G, uv is an edge of H. We say a spanning subgraph of G is a *bond* if every connected component of the subgraph is induced. As an example, consider the graph G in Figure 1. The subgraphs H and H' are bonds, but H'' is not since it is missing the edge 16. To each bond H, one can associate a set partition, $\pi(H)$, so that i and j are in the same block of $\pi(H)$ if and only if i and j are in the same connected component of H. For example, for the bond H in Figure 1, we have that $\pi(H) = 12345/6$. A partition $\pi = B_1/B_2/\cdots/B_k$ is called *crossing* if there exists $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$ and $a < b < c < d$. For example, the partition $1248/56/37$ is crossing since we can pick $2, 4 \in 1248$ and $3, 7 \in 37$. A partition is *noncrossing* if it is not crossing. We say the bond H is *crossing* (resp. *noncrossing*) if $\pi(H)$ is crossing (resp. noncrossing). It is not hard to verify the following lemma.

Lemma 1. *A bond H is crossing if and only if there exists edges in H in different connected components which cross in the graphical representation of H.*

Note that this lemma implies that it is possible for edges to cross in the graphical representation of a noncrossing bond as long as these edges are in the same connected component. For example, the bond H in Figure 1 is noncrossing since it corresponds to $12345/6$, but it has crossing edges, namely 14 and 35.

The *bond lattice* of G, denoted by L_G, is the collection of bonds of G ordered by inclusion. Removing all the bonds in L_G which are crossing gives us our main object of study.

Definition 2. *Let G be a graph. The* noncrossing bond poset, *denoted by $\mathcal{NC}_G$, is the collection of noncrossing bonds of G ordered by inclusion.*

See Figure 3 for an example of a graph and its noncrossing bond poset. We wish to emphasize that $\mathcal{NC}_G$ need not be a lattice. For example, let G be the graph with vertex set $[4]$ and edges 13 and 24. It is easy to see that $\mathcal{NC}_G$ has no maximum element. The issue is that the graph itself is a crossing bond and so does not appear in $\mathcal{NC}_G$. From this example, one might conjecture that we always get a meet semi-lattice. Unfortunately, this is not the case. For example, consider the graph G in Figure 1. The bonds H and H' are noncrossing and so appear in $\mathcal{NC}_G$. Since the bonds are ordered by containment, if $X \leq H, H'$ then $X \subseteq H \cap H'$. However, $H \cap H' = \{14, 35\}$, which is crossing and thus not in $\mathcal{NC}_G$. Since $\{14\}$ and $\{35\}$ are noncrossing bonds, H and H' do not have a meet in $\mathcal{NC}_G$.

Definition 3. *Let G be a graph. We say G is* crossing closed *if whenever $a, b, c, d \in V(G)$ with $a < b < c < d$ and $ac, bd \in E(G)$, there exists a unique minimum (with respect to inclusion) induced connected component containing a, b, c, d.*

Note that the graph G in Figure 1 is not crossing closed since 14 and 35 are both contained in the minimal induced connected components with vertices $1, 2, 3, 4, 5$ and with vertices $1, 3, 4, 5, 6$. On the other hand, any tree is crossing closed. It turns out that not being crossing closed is the only obstruction to $\mathcal{NC}_G$ being a lattice.

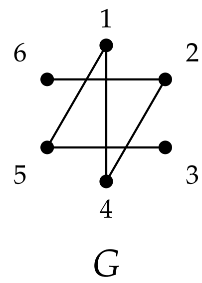

Figure 2: A graph whose noncrossing bond poset is not graded.

Theorem 4. *Let G be a graph. Then $\mathcal{NC}_G$ is a lattice if and only if G is crossing closed. Moreover, if G is crossing closed and $H, H' \in \mathcal{NC}_G$, then $H \wedge H' = H \cap H'$.*

In addition to $\mathcal{NC}_G$ not necessarily being a lattice, NC_G need not be graded. Consider the graph in Figure 2. The bond corresponding to the partition $1/26/35/4$ is noncrossing, but the only element of $\mathcal{NC}_G$ which covers $1/26/35/4$ is 123456. It follows that $1/2/3/4/5/6 \lessdot 1/26/3/4/5 \lessdot 1/26/35/4 \lessdot 123456$ is a maximal chain in $\mathcal{NC}_G$. However, there is another maximal chain $1/2/3/4/5/6 \lessdot 14/2/3/5/6 \lessdot 124/3/5/6 \lessdot 1246/3/5 \lessdot 12456/3 \lessdot 123456$ and so $\mathcal{NC}_G$ is not graded.

3 Perfectly Labeled Graphs

Definition 5. *Let G be a graph. We say G is* perfectly labeled *if whenever $ik, jk \in E(G)$ with $i < j < k$, $ij \in E(G)$.*

It is well-known that a graph can be perfectly labeled if and only if it is chordal (see, for example, [7, Corollary 4.10]). However, not every labeling of a chordal graph gives rise to a perfectly labeled graph. The distinction between perfectly labeled and chordal is immaterial to the structure of the bond lattice since the lattice does not depend on the labeling of the vertex set. However, in the case for the noncrossing bond poset, the structure of the poset does depend on the labeling of the graph. Because of this, we focus on perfectly labeled graphs as opposed to just chordal graphs. The next lemma will be useful in this section.

Lemma 6. *Let G be a perfectly labeled graph. Suppose that $H \leq H'$ in $\mathcal{NC}_G$. Moreover, suppose that $B_1, B_2, \ldots, B_k$ where $\min B_1 < \min B_2 < \cdots < \min B_k$ are the connected components of H that are merged together to get H'. Then merging B_1 and B_2 in H creates a noncrossing bond of G.*

Proof. Let $3 \leq i \leq k$. If merging B_1 and B_2 crossed with some B_i, then there exists $a, c \in B_1 \cup B_2$ and $b, d \in B_i$ or $a, c \in B_i$ and $b, d \in B_1 \cup B_2$ with $a < b < c < d$. If

$a, c \in B_1 \cup B_2$ and $b, d \in B_i$, then $\min B_1 < \min B_2 < b < c < d$ which implies either B_1 and B_i cross or B_2 and B_i cross. Neither is possible since H is noncrossing. A similar argument shows that it is not possible that there exists $a, c \in B_i$ and $b, d \in B_1 \cup B_2$ with $a < b < c < d$. Moreover, merging B_1 and B_2 cannot cross any other connected components of G since that would mean that H' was crossing. Thus, it suffices to show that merging B_1 and B_2 in H forms a bond of G. In [4], Hallam, Martin, and Sagan showed that every connected graph which is perfectly labeled contains an increasing spanning tree (i.e. a spanning tree where the labels of the vertices along any path from the smallest vertex to any other vertex are increasing). It follows that in H' there is an increasing path from $\min B_1$ to $\min B_2$. Except for $\min B_2$, this path must only contain vertices from B_1 and so there is an edge between B_1 and B_2. It follows that merging B_1 and B_2 gives a bond of G. □

As we saw earlier, the noncrossing bond poset may not be graded. However, if G is perfectly labeled and G is a noncrossing bond of itself (for example if G is connected), then one can use the previous lemma to show that $\mathcal{NC}_G$ is graded. Throughout the rest of the abstract, we will use $cc(G)$ to denote the number of connected components of G.

Proposition 7. *Let G be a perfectly labeled graph. If $H, H' \in \mathcal{NC}_G$ and $H \lessdot H'$ then exactly two connected components of H merge together to get H' and thus $cc(H) = cc(H') + 1$. Additionally, if G is a noncrossing bond of itself, then $\mathcal{NC}_G$ is graded and for $H \in \mathcal{NC}_G$, the rank of H is given by $\rho(H) = n - cc(H)$.*

It is well-known that a graph G is chordal if and only if the bond lattice of G is supersolvable (see [7, §4] for details). We will now show an analogue of this result for the noncrossing bond poset. To do this, we give an S_n EL-labeling of the noncrossing bond poset of perfectly labeled graphs. We briefly review some material about edge labelings of posets. For more information about edge labelings and their implications see [8] and [9].

Let P be a graded poset. An *edge labeling* of P is a function $\lambda : \mathcal{E}(P) \to \Lambda$ where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of P and Λ is a set of labels which is partially ordered. We note here that although the labels are allowed to be partially ordered, in this extended abstract they will always be totally ordered. Now suppose that P is a graded poset with edge labeling λ. Let $\mathbf{c} : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$ be a saturated chain in P. We say $\mathbf{c}$ is *increasing* if

$$\lambda(x_0 \lessdot x_1) < \lambda(x_1 \lessdot x_2) < \cdots < \lambda(x_{k-1} \lessdot x_k).$$

Moreover, we say $\mathbf{c}$ is *decreasing* if

$$\lambda(x_0 \lessdot x_1) \geq \lambda(x_1 \lessdot x_2) \geq \cdots \geq \lambda(x_{k-1} \lessdot x_k).$$

Let λ be an edge labeling of P. We say λ is an *EL-labeling* if every interval has a unique increasing maximal chain and this chain precedes every other maximal chain in

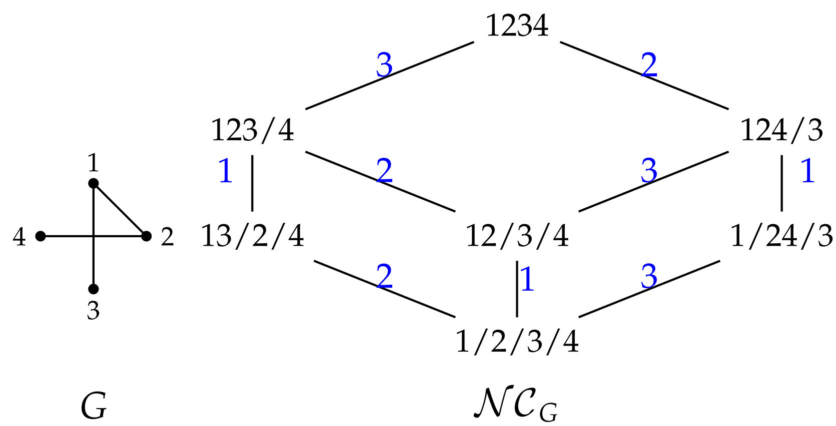

Figure 3: A graph and its noncrossing bond poset with its edges labeled in blue.

the interval in lexicographic order. The edge labeling in Figure 3 is an EL-labeling. It also has the property that every maximal chain is labeled by a permutation of $[3]$. Such labelings have a special name. We say an EL-labeling of P is an S_n *EL-labeling* if every maximal chain of P is labeled by a permutation of $[n]$ where n is the rank of P. We note that the condition that the unique maximal chain in each interval is lexicographically first is automatically implied if the maximal chains are labeled by permutations and thus, we only need to check that each interval has a unique increasing maximal chain.

Björner [1] and Björner and Wachs [2] showed that there are several nice topological consequences of a poset having an EL-labeling. For example, they showed that given a poset with an EL-labeling, the order complex of P is shellable and has the homotopy type of a wedge of spheres. In addition to these properties, there are special properties that a lattice with an S_n EL-labeling possesses. In particular, McNamara [5] showed that if L is a graded lattice then L is supersolvable if and only if it has an S_n EL-labeling. We will show that if G is perfectly labeled and connected, then it has an S_n EL-labeling.

By Proposition 7 if $H \lessdot H'$ in $\mathcal{NC}_G$, then two connected components of H merge together in H'. Suppose that B and B' are the connected components which merge while going from H to H'. We define an edge labeling λ on $\mathcal{NC}_G$ by

$$\lambda(H \lessdot H') = \max\{\min B, \min B'\} - 1.$$

See the poset in Figure 3 for an example of this labeling. Björner and Edelman [1] showed that λ gives an EL-labeling of the noncrossing partition lattice (which is also the noncrossing bond poset of the complete graph). As McNamara points out in [5], this labeling is in fact an S_n EL-labeling. It turns out that if G is perfectly labeled, then λ is also an S_n EL-labeling of $\mathcal{NC}_G$. Note that in the hypothesis of the following theorem, we assume G has $n+1$ vertices and is connected. This guarantees that $\mathcal{NC}_G$ has rank n and a $\hat{1}$.

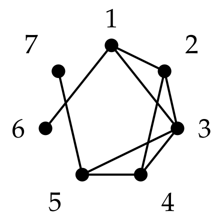

Figure 4: A perfectly labeled graph which is not crossing closed

Theorem 8. *Let G be a perfectly labeled graph on $[n+1]$ which is connected. Then $\mathcal{NC}_G$ has an S_n EL-labeling.*

(Proof Sketch). First, it is clear that the maximal chains are labeled by permutations of $[n]$. Thus it suffices to show that each interval in $\mathcal{NC}_G$ has a unique increasing maximal chain. Since G has $n+1$ vertices, it is a subgraph of the complete graph K_{n+1}. It follows that $\mathcal{NC}_G$ is a subposet of $\mathcal{NC}_{K_{n+1}}$ which is isomorphic to the noncrossing partition lattice. Since it is known that λ is an EL-labeling of the noncrossing partition lattice, it is enough to show that each interval in $\mathcal{NC}_G$ has an increasing maximal chain.

Suppose that $[X,Y]$ is an interval in $\mathcal{NC}_G$ and suppose that $B_1, B_2, \ldots, B_k$ are the connected components of X that will merge together to get Y. Moreover, assume that $\min B_1 < \min B_2 < \cdots < \min B_k$. It is not hard to see that if there is an increasing maximal chain in $[X,Y]$, the first step must be to merge B_1 and B_2. Let Z be obtained by merging B_1 and B_2 in X. We can apply Lemma 6 to see that $Z \in \mathcal{NC}_G$. Now we can use induction to prove that $[Z,Y]$ has an increasing maximal chain which can be concatenated with the label from X to Z to give an increasing maximal chain in $[X,Y]$. □

Corollary 9. *Let G be a connected graph which is perfectly labeled. Then we have the following.*

(a) $\mathcal{NC}_G$ is shellable.

(b) If G is crossing closed, then $\mathcal{NC}_G$ is supersolvable.

We mention here that not every connected perfectly labeled graph is crossing closed (hence the necessity of the hypothesis in part (b) of Corollary 9). The graph in Figure 4 is perfectly labeled, but not crossing closed. This is because there are two minimal induced connected components containing 16 and 57, namely the one containing the vertices $1,3,5,6,7$ and the one containing $1,2,4,5,6,7$.

The reader may be wondering if $\mathcal{NC}_G$ being supersolvable implies that G is chordal since this is the case for the bond lattice of a graph. The graph in Figure 5 shows this is not true. It is a 4-cycle and thus is not chordal. Nevertheless, the noncrossing bond poset is a supersolvable lattice with modular chain $1/2/3/4 \lessdot 12/3/4 \lessdot 12/34 \lessdot 1234$.

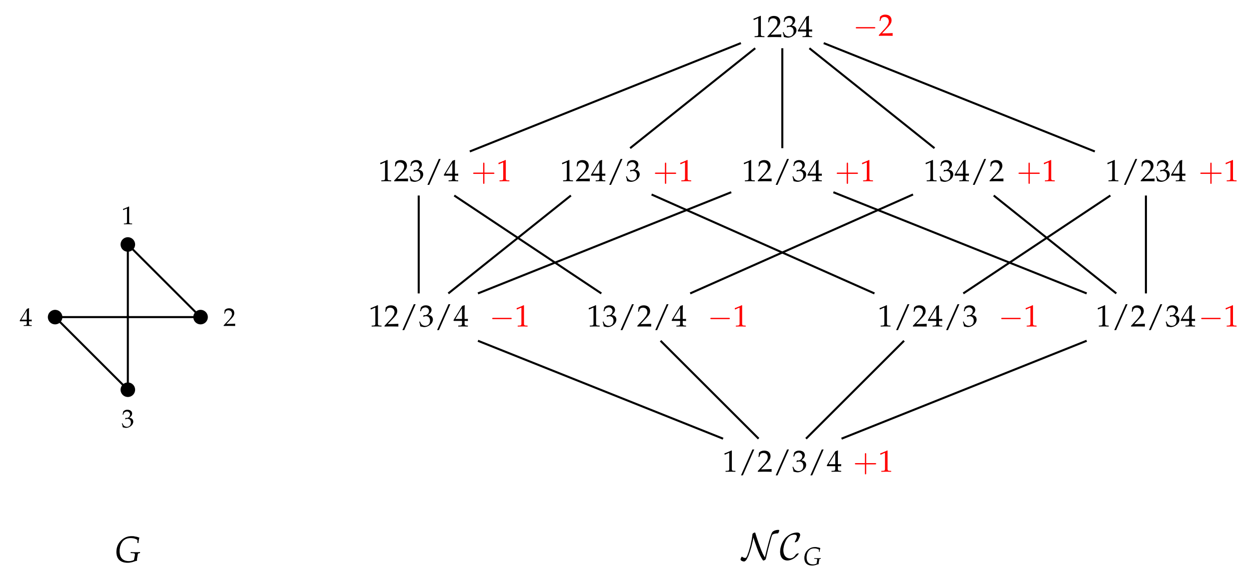

Figure 5: Twisted 4-cycle and its noncrossing bond poset. Möbius values are in red.

4 The Möbius Function and the Characteristic Polynomial

In this section we provide a combinatorial interpretation for the Möbius function and characteristic polynomial of the noncrossing bond poset for a family of graphs. Given a poset P, the (one-variable) Möbius function is recursively defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -\sum_{y<x} \mu(y) & \text{otherwise.} \end{cases}$$

Moreover, if P is graded, then the *characteristic polynomial* of P is given by

$$\chi(P,t) = \sum_{x \in P} \mu(x) t^{\rho(P)-\rho(x)}.$$

See [8] for background on the Möbius function and characteristic polynomial. It is well-known that for a graph G, $\chi(L_G, t)$ is (up to a factor of t) the chromatic polynomial of G (see [7]). In [10], Whitney gave a combinatorial interpretation for the coefficients of the chromatic polynomial in terms of NBC sets. We briefly recall some facts about NBC sets.

Let G be a graph. Put a total order on the edges of G. A *broken circuit* of G is a collection of edges of G obtained by removing the smallest edge of a cycle of G. We say a subset S of $E(G)$ is an *NBC (non-broken circuit) set* if S contains no subsets which are broken circuits. Whitney [10] showed that if

$$\chi(L_G,t) = \sum_{k \geq 0} (-1)^k c_k t^{\rho(L_G)-k}$$

then c_k is the number of NBC sets of G with k edges.

As an example, let G be the 4-cycle given in Figure 5. Order the edges lexicographically. Since the graph is a cycle, every subset of the edge set is an NBC set except for $\{13, 24, 34\}$ and $\{12, 13, 24, 34\}$. It follows that the characteristic polynomial of L_G is given by

$$\chi(L_G, t) = t^3 - 4t^2 + 6t - 3.$$

Now lets compare this with the characteristic polynomial of $\mathcal{NC}_G$. From the Möbius values shown in Figure 5, we see that

$$\chi(\mathcal{NC}_G, t) = t^3 - 4t^2 + 5t - 2.$$

Since the absolute value of the coefficients of $\chi(\mathcal{NC}_G, t)$ are less than the corresponding values in $\chi(L_G, t)$, it is at least plausible that the coefficients of $\chi(\mathcal{NC}_G, t)$ count a subset of the NBC sets of G. It turns out this is the case as we see next.

Let S be an NBC set of G. We say that S is a *noncrossing NBC set* if S contains no edges which cross. Returning to our example of the 4-cycle given in Figure 5, we can see that the only NBC sets which have crossing edges are $\{13, 24\}$ and $\{12, 13, 24\}$. Note that this means that for this particular graph the coefficients of $\chi(\mathcal{NC}_G, t)$ are counted by noncrossing NBC sets of G. While this combinatorial interpretation need not hold for all graphs, it does hold for a family of graphs.

Definition 10. *Let G be a graph with a total ordering, $\trianglelefteq$, on the edge set of G. We say that a graph G is* upper crossing closed *with respect to $\trianglelefteq$ if it is crossing closed and whenever ac, bd are crossing edges, then the unique minimum induced connected component containing a, b, c, d contains an edge e such that $e \triangleleft ac, bd$.*

The 4-cycle in Figure 5 with edges ordered lexicographically is upper crossing closed. To see why, note that the unique minimum induced connected component containing the crossing edges $13, 24$ is the entire graph which contains the edge 12 and 12 is lexicographically smaller than 13. For an example of a graph which is not upper crossing closed, consider the graph obtained by removing the edge 12 from the 4-cycle in Figure 5 while keeping the lexicographic ordering on the edge set.

Theorem 11. *Let G be a graph on $[n]$ with total ordering $\trianglelefteq$ on $E(G)$. Suppose G is upper crossing closed with respect to $\trianglelefteq$. Then for $H \in \mathcal{NC}_G$,*

$$\mu(H) = (-1)^{n-cc(H)}(\#\text{ of noncrossing NBC sets of } G \text{ whose join is } H).$$

Moreover, if $\mathcal{NC}_G$ is graded with rank function $\rho(H) = n - cc(H)$ and

$$\chi(\mathcal{NC}_G, t) = \sum_{k \geq 0} (-1)^k c_k t^{n-k},$$

then c_k is the number of noncrossing NBC sets of G with k edges, where the NBC sets are with respect to $\trianglelefteq$.

In order to prove this theorem, we make use of NBB sets introduced by Blass and Sagan [3]. We briefly discuss their results.

Definition 12 ([3])**.** *Let L be a lattice and let $\trianglelefteq$ be a partial order on the atoms of L. A subset S of the atoms of L is* bounded below *if there exists an atom a such that*

(a) $a \lhd s$ *for all* $s \in S$

(b) $a < \bigvee S$

We say subset, S, of the atoms of L is an NBB set for x if S contains no bounded below sets and $\bigvee S = x$.

Theorem 13 ([3])**.** *Let L be a lattice and let $\trianglelefteq$ be a partial order on the atoms of L. Then for all $x \in L$,*

$$\mu(x) = \sum_B (-1)^{|B|}$$

where the sum is over NBB sets for x.

We are now in a position to sketch a proof of Theorem 11.

(Proof Sketch of Theorem 11). First note that the atoms of $\mathcal{NC}_G$ are the edges of G. Order the atoms of L by $\trianglelefteq$. Using the fact that G is upper crossing closed, we can show that a subset of atoms of $\mathcal{NC}_G$ is NBB if and only if it is a noncrossing NBC set of G. Then using Blass and Sagan's result, we have that for each $H \in \mathcal{NC}_G$,

$$\mu(H) = \sum_B (-1)^{|B|}$$

where the sum is over all the noncrossing NBC sets B such that $\bigvee B = H$. Since B is noncrossing, $\bigvee B$ is the same in L_G and $\mathcal{NC}_G$. For a fixed H, all the NBC sets whose join is H in L_G have the same size, namely $n - cc(H)$. It follows that

$$\begin{aligned}
\mu(H) &= \sum_B (-1)^{|B|} \\
&= \sum_B (-1)^{n-cc(H)} \\
&= (-1)^{n-cc(H)} (\text{\# of noncrossing NBC sets of } G \text{ whose join is } H)
\end{aligned}$$

The result now follows. □

In [4], the authors showed that if G is perfectly labeled, then the NBC sets corresponding to the lexicographic order on the edges are exactly the increasing spanning forests of G. Moreover, if G is perfectly labeled and crossing closed, then it is upper crossing closed with respect to the lexicographic order. Finally, by Proposition 7, if G is perfectly labeled $\mathcal{NC}_G$ is graded. Thus, we get the following.

Theorem 14. *Let G be a graph which is perfectly labeled. Suppose G is crossing closed and that*

$$\chi(\mathcal{NC}_G, t) = \sum_{k\geq 0} (-1)^k c_k t^{\rho(\mathcal{NC}_G)-k}.$$

Then c_k is the number of noncrossing increasing spanning forests of G with k edges.

We note that in [3], Blass and Sagan used NBB sets to show that the Möbius function of the noncrossing partition lattice counts noncrossing increasing trees and hence is a Catalan number. The previous theorem generalizes this result since the noncrossing partition lattice is the noncrossing bond poset of the complete graph.

5 An Action of the Dihedral Group

We finish this extended abstract with a brief discussion of an action by the dihedral group. Recall that the bond lattice does not depend on the labeling of the vertices. Thus if G is a graph on $[n]$ and σ is an element of S_n, the symmetric group, then $L_G \cong L_{\sigma(G)}$ where $\sigma(G)$ is the graph obtained by permuting the vertices by σ. However, this is not the case for the noncrossing bond poset. For example, if we apply the permutation $(123)(4)$ to the twisted 4-cycle in Figure 5, we get an untwisted 4-cycle with edges $12, 23, 34, 14$ and the noncrossing bond posets of the two graphs are not isomorphic. Nevertheless, we can apply certain permutations to the graph without changing the structure of the noncrossing bond poset.

Since the graphs we are interested in appear on a circle, we can assume that the vertices all lie on a regular polygon. Thus, if G has n vertices, the dihedral group, D_{2n} acts on the graph. To define this action, we apply the standard geometric action keeping the labels $1, 2, \ldots, n$ in the same position. This action preserves the structure of the noncrossing bond poset.

Proposition 15. *Let G be a graph on $[n]$. If $\sigma \in D_{2n}$, then*

$$\mathcal{NC}_G \cong \mathcal{NC}_{\sigma(G)}.$$

Acknowledgements

The results in this extended abstract had their genesis in the first author's master thesis. We would like to thank Ed Allen, Hugh Howards, and Sarah Mason who made helpful suggestions and comments on that thesis. We would also like to thank the anonymous referees for their helpful comments and suggestions.

References

[1] A. Björner. "Shellable and Cohen-Macaulay partially ordered sets". *Trans. Amer. Math. Soc.* **260**.1 (1980), pp. 159–183. Link.

[2] A. Björner and M. Wachs. "On lexicographically shellable posets". *Trans. Amer. Math. Soc.* **277**.1 (1983), pp. 323–341. Link.

[3] A. Blass and B. E. Sagan. "Möbius functions of lattices". *Adv. Math.* **127**.1 (1997), pp. 94–123. Link.

[4] J. Hallam, J. L. Martin, and B. E. Sagan. "Increasing spanning forests in graphs and simplicial complexes". *Eur. J. Comb.* **76** (2019), pp. 178 –198. Link.

[5] P. McNamara. "EL-labelings, supersolvability and 0-Hecke algebra actions on posets". *J. Combin. Theory Ser. A* **101**.1 (2003), pp. 69–89. Link.

[6] R. Simion. "Noncrossing partitions". *Discrete Math.* **217**.1-3 (2000). Proceedings of FPSAC'97 (Vienna, 1997), pp. 367–409. Link.

[7] R. P. Stanley. "An introduction to hyperplane arrangements". *Geometric Combinatorics*. IAS/Park City Math. Ser. 13. Amer. Math. Soc., Providence, RI, 2007, pp. 389–496.

[8] R. P. Stanley. *Enumerative Combinatorics. Vol. 1*. 2nd ed. Cambridge Studies in Advanced Mathematics 49. Cambridge University Press, Cambridge, 2012.

[9] M. L. Wachs. "Poset topology: tools and applications". *Geometric Combinatorics*. IAS/Park City Math. Ser. 13. Amer. Math. Soc., Providence, RI, 2007, pp. 497–615.

[10] H. Whitney. "A logical expansion in mathematics". *Bull. Amer. Math. Soc.* **38** (1932), pp. 572–579. Link.