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Counting partitions inside a rectangle

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Abstract. We consider the number of partitions of n whose Young diagrams fit inside an $m \times \ell$ rectangle; equivalently, we study the coefficients of the q-binomial coefficient $\binom{m+\ell}{m}_q$. We obtain sharp asymptotics throughout the regime $\ell = \Theta(m)$ and $n = \Theta(m^2)$ where a limit shape exists. Previously, sharp asymptotics were derived by Takács only in the regime where $|n - \ell m/2| = O(\sqrt{\ell m(\ell + m)})$ using a local central limit theorem. Our approach is to solve a related large deviation problem: we describe the tilted measure that produces configurations whose bounding rectangle has the given aspect ratio and is filled to the given proportion. Our results are sufficiently sharp to yield the first asymptotic estimates on the consecutive differences of these numbers when n is increased by one and m, ℓ remain the same, hence quantifying and significantly refining Sylvester's unimodality theorem.

Résumé. Nous trouvons asymptotique pour le nombre de partitions de n dont les tableaux de Young s'inscrivent dans un rectangle $m \times \ell$; également compté par le coefficient q-binomial $\binom{m+\ell}{m}_q$. Notre technique consiste á utiliser un théoréme central limite local, et nous affinions le théoréme de l'unimodalité de Sylvester en donnant des estimations asymptotiques à des différences consécutives.

Keywords: partitions, q-binomial coefficients, large deviations, local CLT, unimodality

1 Introduction

A partition λ of n is a sequence of weakly decreasing nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ whose sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is equal to n. They are at the heart of much of Enumerative and Algebraic Combinatorics, and connect with Representation Theory (specifically S_n and GL_n) and the theory of special functions. Their study bloomed with the remarkable discoveries of Ramanujan and started the field of partition theory [1].

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The number of partitions of n, typically denoted by $p(n)$ but here unconventionally by N_n to avoid confusion with probabilities, was implicitly determined by Euler via the generating function

$$\sum_{n=0}^{\infty} N_n q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

There is no exact explicit formula for the numbers N_n. The asymptotic formula

$$N_n := \#\{\lambda \vdash n\} \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \tag{1.1}$$

of Hardy and Ramanujan [3], is considered to be the beginning of the use of complex variable methods for asymptotic enumeration of partitions (the so-called circle method).

Here we obtain asymptotic formulas similar to (1.1) for the number of partitions λ of n whose Young diagram fits inside an $m \times \ell$ rectangle, denoted

$$N_n(\ell, m) := \#\{\lambda \vdash n : \lambda_1 \le \ell, \quad \text{length}(\lambda) \le m\}.$$

The generating function of $N_n(\ell, m)$ is the, also ubiquitous, q-binomial coefficient

$$\binom{\ell+m}{m}_q = \frac{\prod_{i=1}^{\ell+m}(1-q^i)}{\prod_{i=1}^{\ell}(1-q^i)\prod_{i=1}^{m}(1-q^i)} = \sum_{n=0}^{\ell m} N_n(\ell, m) q^n.$$

Notably, the numbers $N_n(\ell, m)$ form a symmetric unimodal sequence

$$1 = N_0(\ell, m) \le N_1(\ell, m) \le \cdots \le N_{\lfloor m\ell/2 \rfloor}(\ell, m) \ge \cdots \ge N_{m\ell}(\ell, m) = 1,$$

a fact conjectured by Cayley in 1856 and proven by Sylvester in 1878 via the representation theory of sl_2 [7] and followed by other algebraic and combinatorial proofs (see the next section and the journal version [5] for full bibliography and history of the problem). Over the last one hundred forty years, no previous asymptotic methods have been able to prove this unimodality. As a consequence of our refined asymptotics of $N_n(\ell, m)$ we derive not just the unimodality, but the asymptotics of the difference $N_n(\ell, m) - N_{n-1}(\ell, m)$.

2 The asymptotics: main results

Our main result is an asymptotic formula for $N_n(\ell, m)$ in the regime $\ell/m \to A$ and $n/m^2 \to B$ for any fixed $A > B > 0$, i.e., when the portion of the $m \times \ell$ rectangle that is filled approaches a value that is neither zero nor 1. Then there is a limit shape, whose existence also follows from our methods (see Section 4). By "asymptotic formula" we mean a formula giving $N_n(\ell, m)$ up to a factor of $1 + o(1)$; such asymptotic equivalence is denoted with the symbol $\sim$. By the symmetry $N_n(\ell, m) = N_{m\ell-n}(\ell, m)$ it suffices to consider only the case $A \geq 2B > 0$.

Given $A \geq 2B > 0$ we define three quantities c, d and Δ. Consider the equations

$$A = \int_0^1 \frac{1}{1 - e^{-c-td}} dt - 1 = \frac{1}{d} \log\left(\frac{e^{c+d} - 1}{e^c - 1}\right) - 1\,, \tag{2.1}$$

$$B = \int_0^1 \frac{t}{1 - e^{-c-td}} dt - \frac{1}{2} = \frac{d\log(1-e^{-c-d}) + \operatorname{dilog}(1-e^{-c}) - \operatorname{dilog}(1-e^{-c-d})}{d^2}\,, \tag{2.2}$$

using the dilogarithm function $\operatorname{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt = \sum_{k=1}^{\infty} \frac{(1-x)^k}{k^2}$ for $|x - 1| < 1$. The quantities c, d are the unique solution to the system of Equations (2.1), (2.2):

Lemma 1. *For any $A > 0$ and $B \in (0, A/2)$ there exist unique $c, d > 0$ satisfying Equations (2.1) and (2.2). Moreover, for a fixed A, when B decreases from $A/2$ to 0 then d increases strictly from 0 to ∞ and c decreases strictly from $\log\left(\frac{A+1}{A}\right)$ to 1. When $B > 0$ is fixed and A goes to ∞ then c goes to 0 and d goes to the root of $d^2 = B\left(d\log(1 - e^{-d}) - \operatorname{dilog}(1 - e^{-d})\right)$.*

The quantity Δ, which is seen to be strictly positive, is defined by

$$\Delta = \frac{2Be^c(e^d - 1) + 2A(e^c - 1) - 1}{d^2(e^{d+c} - 1)(e^c - 1)} - \frac{A^2}{d^2}\,. \tag{2.3}$$

Theorem 2. *Given m, ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define c, d and Δ as above. Let K be any compact subset of $\{(x, y) : x \geq 2y > 0\}$. As $m \to \infty$ with ℓ and n varying so that (A, B) remains in K,*

$$N_n(\ell, m) \sim \frac{e^{m\left[cA + 2dB - \log(1 - e^{-c-d})\right]}}{2\pi m^2 \sqrt{\Delta\,(1 - e^{-c})\,(1 - e^{-c-d})}}\,, \tag{2.4}$$

where c and d vary in a Lipschitz manner with $(A, B) \in K$.

Remark. In the special case $B = A/2$, the parameters take on the elementary values

$$d = 0\,, \qquad c = \log\left(\frac{A+1}{A}\right)\,, \qquad \text{and} \qquad \Delta = \frac{A^2(A+1)^2}{12}\,.$$

In this case the exponent and leading constant are the limits as $d \to 0$, giving

$$N_{Am^2/2}(Am, m) \sim \frac{\sqrt{3}}{A\pi m^2} \left[\frac{(A+1)^{A+1}}{A^A}\right]^m\,.$$

In the special case when $A \to \infty$, so that the restriction on the size of the parts is removed, one has $c = 0$ and d is a solution to an explicit equation given in Lemma 1. In

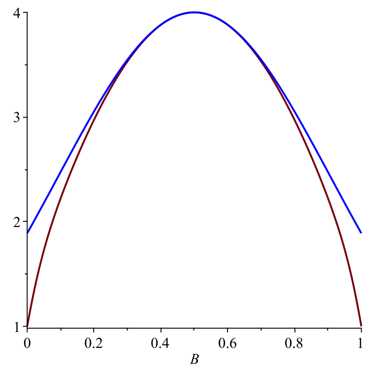


Figure 1: Exponential growth of $N_{Bm^2}(m, m)$ predicted by Takács' formula (blue, above) compared to the actual exponential growth given by Theorem 2 (red, below).

this case the result matches the one obtained first by Szekeres using complex analysis, then by Canfield using a recursion, and most recently by Romik using probabilistic methods based on Fristedt's ensemble. These works and others are explained in [5].

Takács [8] observed that for typical partitions of size $m\ell/2$, the gaps between part sizes behave like independent geometric random variables with mean A. Counting the partitions is therefore equivalent to computing the probability that these $m + 1$ geometric random variables will sum precisely to ℓ and that the area of the ensuing Young diagram will be precisely n. A local central limit theorem (LCLT) immediately yields a sharp asymptotic estimate. However, when $|B - A/2| \gg m^{-1/2} \log m$ the error in this asymptotic is much bigger than the main term of the Gaussian estimate provided by the LCLT. We refer the reader to the full version [5] for the history of this problem (from Erdös and Lehner, through Szekeres, Mann and Whitney, Takács, Romik, and Richmond) and the limitations of previous methods (most notably the circle method and CLT).

We circumvent this limitation on the use of the LCLT using a technique from the theory of large deviations. Specifically, we employ a so-called tilted measure for which maximum likelihood occurs at any desired pair (A, B). The tilted measure replaces the IID geometric random variables giving us the part sizes of λ by still independent but no longer identically distributed geometric random variables, where the parameter $1 - p_i$ for the ith variable varies in a log-linear manner. The restrictions given by A and B are linear conditions on the part sizes, and the choice of parameters p_i ensures that the maximum likely outcome occurs at the given restrictions. The refined LCLT then gives precise asymptotics.

Remark. In [4], the authors, independently and concurrently with our paper, used the generating function for q-binomial coefficients and a saddle point analysis to derive the asymptotics for $N_n(m, \ell)$ in the cases when $m, \ell \geq 4\sqrt{n}$, corresponding to $B \leq \min\{1, A^2\}/16$ in our notation. Those authors express their result using the root of a hypergeometric identity similar to (2.2), however their methods give weaker error bounds and consequently cannot answer questions of unimodality.

Unimodality

Sylvester's proof [7] of unimodality of $N_n(\ell, m)$ in n, and most subsequent proofs (notably of Stanley via the hard Lefshetz theorem and later the linear algebra paradigm, refined by Proctor) are algebraic, viewing $N_n(\ell, m)$ as dimensions of certain vector spaces, or their differences as multiplicities of representations. While there are also purely combinatorial proofs of unimodality, notably O'Hara's and the more abstract one of Pouzet and Rosenberg, they do not give the desired symmetric chain decomposition of the subposet of the partition lattice. These methods do not give ways of estimating the asymptotic size of the coefficients or their difference. It is now known that $N_n(\ell, m)$ is strictly unimodal (first shown by Pak and Panova, later also by Zanello and others), and the following lower bound on the consecutive difference was obtained in [6, Theorem 1.2] using a connection between integer partitions and Kronecker coefficients g of the symmetric group $S_{m\ell}$:

$$g((m\ell - n, n), (m^\ell), (m^\ell)) = N_n(\ell, m) - N_{n-1}(\ell, m) \geq 0.004 \frac{2^{\sqrt{s}}}{s^{9/4}}, \tag{2.5}$$

where $n \leq \ell m/2$ and $s = \min\{2n, \ell^2, m^2\}$. In particular, when $\ell = m$ we have $s = 2n$.

Any sharp asymptotics of the difference appears to be out of reach of these algebraic methods, however as a consequence of Theorem 2 we obtain the following estimate.

Theorem 3. *Given m, ℓ and n, let $A := \ell/m$ and $B := n/m^2$ and define d as above. Suppose m, ℓ and n go to infinity so that (A, B) remains in a compact subset of $\{(x, y) : x \geq 2y > 0\}$ and*

$$m^{-1} |n - lm/2| \to \infty.$$

Then for the consecutive difference of N_n and via (2.5) for the Kronecker coefficient of a two-row partition and two rectangles we have

$$g((m\ell - n - 1, n + 1), m^\ell, m^\ell) = N_{n+1}(\ell, m) - N_n(\ell, m) \sim \frac{d}{m} N_n(\ell, m).$$

Remark. The condition $m^{-1} |n - lm/2| \to \infty$ is equivalent to $m\, |A - B/2| \to \infty$ and also to $d \notin O(m^{-1})$. It is automatically satisfied whenever (A, B) is in a compact subset of $\{(x, y) : x > 2y > 0\}$.

3 The proofs via a discretized analogue to Theorem 2

We will explain our argument with a probability model by which the proof of Theorem 2 is reduced to a local central limit theorem. We start with the simple case when $B \approx A/2$ as in [8]. Let $\{\lambda_j : 1 \le j \le m\}$ denote the parts, in decreasing size, of a partition of n into at most m parts of size at most ℓ, padded with zeros at the end if necessary. Defining $\lambda_0 := \ell$ and $\lambda_{m+1} := 0$, the *gaps* $x_i := \lambda_i - \lambda_{i+1}$ satisfy the following two identities (see Figure 2),

$$\sum_{i=0}^{m} x_i = \ell\,; \qquad \sum_{i=0}^{m} i x_i = n\,. \tag{3.1}$$

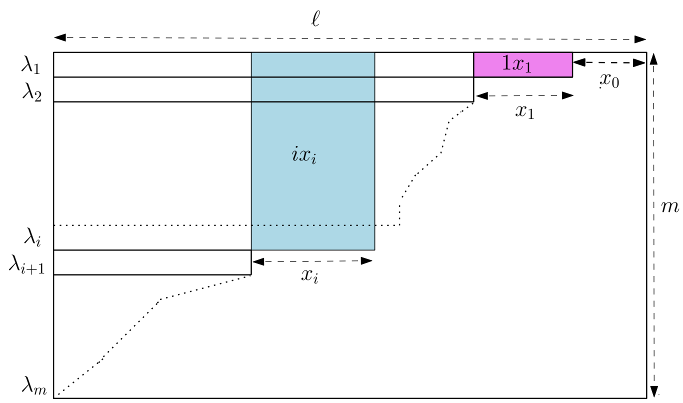


Figure 2: The total area n of a partition is composed of rectangles of area jx_j

By the *reduced geometric distribution with parameter p* we mean the random variable X with $\mathbb{P}(X = k) = p \cdot q^k$ where $q := 1 - p$. In [8], Takács sets $\{X_j : 0 \le j \le m\}$ to be a collection of independent reduced geometric random variables with parameter $p = 1/2$. This distribution has the crucial property that for any set of values $x_0, \ldots, x_m$, the probability $\mathbb{P}(X_j = x_j : j = 0, \ldots, m)$ depends only on the sum $\ell := \sum_{j=1}^{m} x_j$ and is equal to $p^{m+1} q^{\ell}$. Let $P(\ell)$ denote the sum of $\mathbb{P}(\mathbf{X} = \mathbf{x})$ over all $(m+1)$-vectors $\mathbf{x}$ with coordinate sum ℓ. If $N(\ell)$ denotes the number of such vectors summing to ℓ, we see immediately that $1 \ge P(\ell) = p^{m+1} q^{\ell} N(\ell)$, hence $N(\ell) \le [p^{m+1} q^{\ell}]^{-1}$. This inequality is good because $P(\ell)$ is not that small: it is of order $m^{-1/2}$. Now let $N(\ell, n)$ count those vectors satisfying both identities in (3.1) and $P(\ell, n)$ be the probability that the geometric variables lie in this set. Takács gave a sharp asymptotic estimate of $P(\ell, n) \sim cm^{-2}$ when $n = m^2/2 + O(m^{3/2})$, thereby showing that $N(\ell, n) \sim cm^{-2}[p^{m+1} q^{\ell}]^{-1}$.

Central limit theorems do not provide a sharp estimate when $|n - m^2/2| \gg m^{3/2}$. However, because the constraints on the vectors counted by $N(\ell, n)$ are linear, the theory of large deviations [2] implies that $P(\ell, n)$ is well estimated by "tilting" the independent laws of the $\{X_j\}$ so that one is in the central limit regime of the tilted laws. Having solved for the correct tilt, we may dispense with the large deviations theory and prove the estimates directly. Tilting preserves the reduced geometric family, altering only the parameters; it turns out that the correct tilt makes $q_j := 1 - p_j$ log-linear.

With c_m, d_m to be specified later, let

$$q_j := e^{-c_m - jd_m/m}, \quad p_j := 1 - q_j, \quad L_m := \sum_{j=0}^{m} \log p_j.$$

Let $\mathbb{P}_m$ be a probability law making the random variables $\{X_j : 0 \le j \le m\}$ independent reduced geometrics with respective parameters p_j. Define random variables S_m and T_m by

$$S_m := \sum_{i=0}^{m} X_i \,; \qquad T_m := \sum_{i=1}^{m} iX_i \,, \tag{3.2}$$

corresponding to the unique partition λ satisfying $X_j = \lambda_j - \lambda_{j+1}$. We first prove a result similar to Theorem 2, except that the parameters c and d that solve integral Equations (2.1) and (2.2) are replaced by c_m and d_m satisfying the discrete summation equations (3.3) below. These equations say that $\mathbb{E}S_m = \ell$ and $\mathbb{E}T_m = n$. Writing this out, using $\mathbb{E}X_j = 1/p_j - 1 = 1/\left(1 - e^{-c_m - d_m j/m}\right) - 1$, gives

$$\ell = \sum_{j=0}^{m} \frac{1}{1 - e^{-c_m - d_m j/m}} - (m+1)\,, \qquad n = m\sum_{j=0}^{m} \frac{j/m}{1 - e^{-c_m - d_m j/m}} - \frac{m(m+1)}{2}\,. \tag{3.3}$$

Let M_m denote the covariance matrix for (S_m, T_m). The entries may be computed from the basic identity $\mathrm{Var}\,(X_j) = q_j/p_j^2$, resulting in

$$\mathrm{Var}\,(S_m) = \sum_{j=0}^{m} \frac{e^{-c_m - d_m j/m}}{\left(1 - e^{-c_m - d_m j/m}\right)^2} \tag{3.4}$$

$$\mathrm{Cov}\,(S_m, T_m) = \sum_{j=0}^{m} j \frac{e^{-c_m - d_m j/m}}{\left(1 - e^{-c_m - d_m j/m}\right)^2} \tag{3.5}$$

$$\mathrm{Var}\,(T_m) = \sum_{j=0}^{m} j^2 \frac{e^{-c_m - d_m j/m}}{\left(1 - e^{-c_m - d_m j/m}\right)^2}\,. \tag{3.6}$$

Theorem 4 (discretized analogue)**.** *Let c_m and d_m satisfy equations (3.3). Define α_m, β_m and γ_m to be the normalized entries of the covariance matrix*

$$\alpha_m := m^{-1}\mathrm{Var}\,(S_m)\,; \qquad \beta_m := m^{-2}\mathrm{Cov}\,(S_m, T_m)\,; \qquad \gamma_m := m^{-3}\mathrm{Var}\,(T_m)\,,$$

which are $O(1)$ as $m \to \infty$. Again, let $A := \ell/m$ and $B := n/m^2$ and $\Delta_m := \alpha_m\gamma_m - \beta_m^2$. Then

$$N_n(\ell, m) \sim \frac{1}{2\pi m^2 \sqrt{\Delta_m}} \exp\left\{ m\left(-\frac{L_m}{m} + c_m A + d_m B \right) \right\}\,. \tag{3.7}$$

Proof outline. The atomic probabilities $\mathbb{P}_m(\mathbf{X} = \mathbf{x})$ depend only on S_m and T_m as

$$\log \mathbb{P}_m(\mathbf{X} = \mathbf{x}) = \sum_{j=0}^{m} (\log p_j + x_j \log q_j)$$
$$= L_m - \sum_{j=0}^{m} \left(c_m + j\frac{d_m}{m} \right) x_j = L_m - c_m \left(\sum_{j=0}^{m} x_j \right) - \frac{d_m}{m} \left(\sum_{j=0}^{m} j x_j \right).$$

In particular, for any $\mathbf{x}$ satisfying (3.1),

$$\log \mathbb{P}_m(\mathbf{X} = \mathbf{x}) = L_m - c_m \ell - \frac{d_m}{m} n\,. \tag{3.8}$$

The following are equivalent: (i) the vector $\mathbf{X}$ satisfies the identities (3.1); (ii) the pair (S_m, T_m) is equal to (ℓ, n); (iii) the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ defined by $\lambda_j - \lambda_{j+1} = X_j$ for $2 \le j \le m-1$, together with $\lambda_1 = \ell - X_0$ and $\lambda_m = X_m$, is a partition of n fitting inside a $m \times \ell$ rectangle. Thus, setting $p_m(\ell, n) := \mathbb{P}_m \left[(S_m, T_m) = (\ell, n) \right]$ we have

$$N_n(\ell, m) = \frac{p_m(\ell, n)}{\mathbb{P}_m(\mathbf{X} = \mathbf{x})} = p_m(\ell, n) \exp \left[m \left(-\frac{L_m}{m} + c_m A + d_m B \right) \right]. \tag{3.9}$$

Comparing (3.7) to (3.9), the proof is completed by an application of the following LCLT, for which a proof is sketched below. It is stated for an arbitrary sequence of parameters $p_0, \ldots, p_m$ bounded away from 0 and 1, we use it with $p_j = 1 - e^{-c_m - d_m j/m}$. $\square$

Denote by $M(s,t) := [s\,,\ t]\, M\, [s\,,\ t]^T$ the quadratic form corresponding to a matrix M, then we can state the LCLT and its refinement as follows.

Lemma 5 (LCLT). *Fix $0 < \delta < 1$ and let $p_0, \ldots, p_m$ be any real numbers in the interval $[\delta, 1-\delta]$. Let $\{X_j\}$ be independent reduced geometrics with respective parameters $\{p_j\}$, $S_m := \sum_{j=0}^m X_j$, and $T_m := \sum_{j=0}^m j X_j$. Let M_m be the covariance matrix for (S_m, T_m), written*

$$M_m = \begin{pmatrix} \alpha_m m & \beta_m m^2 \\ \beta_m m^2 & \gamma_m m^3 \end{pmatrix},$$

Q_m denote the inverse matrix to M_m, and $\Delta_m = m^{-4} \det M_m = \alpha_m \gamma_m - \beta_m^2$. Let μ_m and ν_m denote the respective means $\mathbb{E} S_m$ and $\mathbb{E} T_m$. Then

$$\sup_{a,b \in \mathbb{Z}} m^2 \left| p_m(a,b) - \frac{1}{2\pi (\det M_m)^{1/2}} e^{-\frac{1}{2} Q_m(a - \mu_m, b - \nu_m)} \right| \to 0 \tag{3.10}$$

as $m \to \infty$, uniformly in the parameters $\{p_j\}$ in the allowed range. In particular, if the sequence (a_m, b_m) satisfies $Q_m(a_m - \mu_m, b_m - \nu_m) \to 0$ then

$$p_m(a_m, b_m) = \frac{1}{2\pi \sqrt{\Delta_m} m^2} \left(1 + O\left(m^{-3/2} \right) \right).$$

The following consequence is used to prove Theorem 3.

Corollary 6 (LCLT consecutive differences)**.** *Let $\mathcal{N}(a,b) := \frac{1}{2\pi(\det M)^{1/2}} e^{-\frac{1}{2}Q(a-\mu,b-\nu)}$ be the normal approximation in Equation (3.10). Using the notation of Lemma 5,*

$$\sup_{a,b\in\mathbb{Z}} \left| p(a,b+1) - p(a,b) - \big(\mathcal{N}(a,b+1) - \mathcal{N}(a,b)\big) \right| = O(m^{-4}).$$

Sketch of the proofs of Theorems 2 and 3 and the LCLT Lemma 5

Theorem 2 follows from the discretized Theorem 4 after analyzing c_m, d_m (in particular showing their existence and uniqueness), and their relation to c and d defined in the introduction. Asymptotics follow from careful analysis of the error bounds.

Theorem 3 follows from Equation (3.9) and Corollary 6. Let

$$L_m(x,y) := \sum_{j=0}^{m} \log(1 - e^{-x-yj/m}) \,.$$

Then if $A_m(x,y) = (\partial/\partial x)L_m(x,y)$ and $B_m(x,y) = (\partial/\partial y)L_m(x,y)$, we have c_m and d_m are the solutions to $A_m(c_m,d_m) = \ell$ and $B(c_m,d_m) = n/m$. Let c_m', d_m' be the solutions to $A_m(c_m', d_m') = \ell$ and $B_m(c_m', d_m') = (n+1)/m$, and let $\Delta x = c_m' - c_m = O(m^{-2})$ and $\Delta y = d_m' - d_m = O(m^{-2})$ with these bounds obtained from an analysis of c_m, c, d_m, d. See [5] for details.

Taylor expansion for $L_m' := L_m(c_m', d_m')$ around (c_m, d_m) and the L_m partials gives

$$-L_m(c_m', d_m') + (c_m + \Delta x)\ell + (d_m + \Delta y)(n+1)\, m^{-1} = -L_m(c_m,d_m) + c_m\ell + d_m(n+1)\, m^{-1} + O(m^{-3}).$$

Then the consecutive differences from Sylvester's unimodality can be expressed as a sum of terms which can be individually bounded as follows,

$$N_{n+1}(\ell,m) - N_n(\ell,m) = p_m(\ell,n) \exp\left[-L_m + c_m\ell + \frac{d_m}{m} n\right] \left[e^{d_m/m} - 1\right] \tag{3.11}$$

$$+ \left[p_m(\ell,n+1) - p_m(\ell,n)\right] \exp\left[-L_m + c_m\ell + \frac{d_m}{m}(n+1)\right] \tag{3.12}$$

$$+ p_m(\ell,n+1)\left(e^{-L_m' + c_m'\ell + d_m'(n+1)/m} - e^{-L_m + c_m\ell + d_m(n+1)/m}\right). \tag{3.13}$$

Since $d_m = d + O(m^{-1})$, Equation (3.9) gives that line (3.11) equals

$$N_n(\ell,m)\left(\frac{d}{m} + O(m^{-2})\right)$$

as long as $d \notin O(m^{-1})$. This holds when $|A - B/2| \notin O(m^{-1})$ as $d = 0$ when $A = B/2$ and the map taking (A, B) to (c, d) is Lipschitz.

The quantity on line (3.12) is $O(m^{-4} \cdot m^2 N_n(\ell, m)) = O(m^{-2} N_n(\ell, m))$ since

$$\begin{aligned}[p_m(\ell, n+1) - p_m(\ell, n)] &\leq |\mathcal{N}(\ell, n+1) - \mathcal{N}(\ell, n)| + O(m^{-4}) \\ &= O\left(m^{-2} \cdot \left|1 - e^{\frac{1}{2}Q_m(0,1)}\right|\right) + O(m^{-4}) = O(m^{-4}),\end{aligned}$$

by Corollary 6, where Q_m is the inverse of the covariance matrix of (S_m, T_m).

The quantity on line (3.13) is

$$\begin{aligned}p_m(\ell, n+1)\, e^{-L_m + c_m \ell + d_m(n+1)/m}\, \psi_m &= N_n(\ell, m)\, \psi_m\, e^{d_m/m} + O(m^{-4}\, e^{d_m/m}\, e^{-L_m + c_m \ell + d_m n/m}\, \psi_m) \\ &= O(m^{-3} N_n(\ell, m)).\end{aligned}$$

since $p_m(\ell, n+1) = p_m(\ell, n) + O(m^{-4})$, where

$$\psi_m := \exp\left[-L'_m + c'_m \ell + d'_m (n+1)\, m^{-1} - \left(-L_m + c_m \ell + d_m (n+1)\, m^{-1}\right)\right] - 1 = O(m^{-3}).$$

Putting everything together, we reach the desired

$$N_{n+1}(\ell, m) - N_n(\ell, m) = N_n(\ell, m) \left(\frac{d}{m} + O(m^{-2})\right). \qquad \square$$

The proof of the LCLT Lemma 5 consists of several steps. First, assuming that the p_js are arbitrary numbers in some interval $[\delta, 1-\delta]$ we show that the constants $\alpha_m, \beta_m, \gamma_m$ and Δ_m are bounded below and above by positive constants depending only on δ.

We then estimate the characteristic function for a reduced geometric r.v. X_p with parameter p. For each $\delta \in (0, 1/2)$ there is a K such that simultaneously for all $p \in [\delta, 1-\delta]$,

$$\left|\log \mathbb{E} \exp(iuX_p) - \left(i\frac{q}{p}u - \frac{q^2}{2p^2}u^2\right)\right| \leq Ku^3\,.$$

Finally, the proof of Lemma 5 comes from expressing the probability as an integral of the characteristic function $\phi(s, t) := \mathbb{E}e^{i(sS+tT)}$, via the inversion formula, and then estimating the integrand in various regions.

In order to estimate the error terms in the approximation of $p(a, b)$ for Corollary 6 we consider the partial differences and repeat the approximation arguments for the original LCLT, estimating an expression of the form

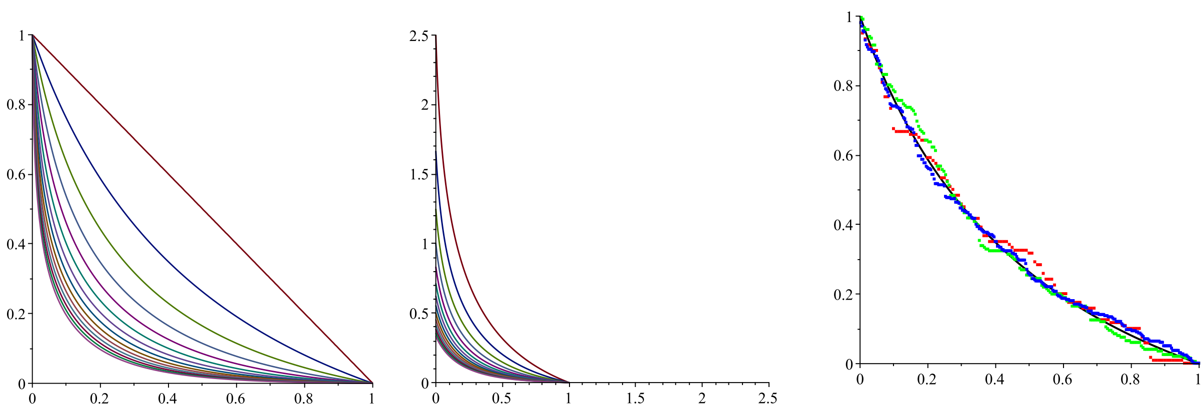
$$\left|p(a, b+1) - p(a, b) - (\mathcal{N}(a, b+1) - \mathcal{N}(a, b))\right| = \int_{[-\pi,\pi]^2} \left|1 - e^{-it}\right| \left|\widehat{\phi}(s, t) - e^{-1/2M(s,t)}\right| ds dt.$$

As in the analysis for the LCLT, we estimate the integrals in different regions where $\widehat{\phi} = \phi e^{-is\mathbb{E}[S]-it\mathbb{E}[T]}$ behaves differently, and ultimately determine that the above integral is $O(m^{-4})$.

4 Limit shape

The limit shape of an unrestricted partition, i.e. the curve which approximates most Young diagrams of $\lambda \vdash n$, was posed as a problem by Vershik and first answered by Szalay and Turan; later Vershik and Yakubovich described the limit shape for singly restricted partitions. The limit shape for partitions inside a rectangle in the regime $m, \ell = \Theta(\sqrt{n})$ was first described by Petrov, where it is identified with a portion of the curve $e^{-x} + e^{-y} = 1$, the limit shape of unrestricted partitions. Fluctuations have also been obtained; see [5] for additional historical details and references.

Here, using the independent random variables X_i we can rederive the limit shape for the partitions of n inside a rectangle. The existence, i.e. the concentration phenomenon, of the limit curve follows from the fact that the maximum discrepancy $\max_{j \le m} \left| \sum_{i=0}^{j} (X_i - 1/p_i) \right|$, conditional on $\sum_{i=0}^{m} X_i = \ell$, is $o(m)$ in probability; see [5] for a rigorous proof.



$(A, B) = (1, 1/k)$
$k = 2, \ldots, 15$

$(A, B) = (5/k, 1/k)$
$k = 2, \ldots, 15$

Limit curve of $(A, B) = (1, 1/3)$ and random partitions of size 120, 201 and 300.

Figure 3: Limit shapes of scaled partitions as $m \to \infty$.

We have $\lambda_i = \ell - (X_0 + X_1 + \cdots + X_{i-1})$ and hence $\mathbb{E}[\lambda_i] = \ell - \sum_{j=0}^{i-1}(1/p_j - 1)$.

Setting $x = i/m$ we approximate the sum by an integral as $m \to \infty$, namely we have

$$y := \mathbb{E}\left[\frac{\lambda_i}{m}\right] = A + x - \int_0^x \frac{1}{1 - e^{-c-td}} dt = A + x - \frac{1}{d} \ln\left(\frac{e^{xd+c} - 1}{e^c - 1}\right),$$

and the limit curve of partitions scaled to the $1 \times A$ rectangle is given by points (x, y) satisfying this equation, or equivalently

$$1 = (1 - e^{-c})e^{d(A-y)} + e^{-c}e^{-dx}, \tag{4.1}$$

when $A > 2B$, and $y = A(1 - x)$ when $A = 2B$ and $d = 0$; see Figure 3.

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