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Delta Conjectures

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Abstract. We state the so called generalized Delta conjecture of Haglund, Remmel and Wilson, and we survey the state of the art about this problem. We do the same with our related generalized Delta square conjecture. Both surveys include some of our recent results.

Keywords: Delta conjecture, Delta square conjecture, Macdonald polynomials

1 Introduction

In [13], Haglund, Remmel and Wilson conjectured a combinatorial formula for $\Delta'_{e_{n-k-1}} e_n$ in terms of decorated labelled Dyck paths, which they called *Delta conjecture*, after the so called delta operators Δ'_f introduced by Bergeron, Garsia, Haiman and Tesler [1] for any symmetric function f. In fact in the same article [13] the authors conjectured a combinatorial formula for the more general $\Delta_{h_m}\Delta'_{e_{n-k-1}} e_n$ in terms of decorated partially labelled Dyck paths, which we call *generalized Delta conjecture*.

The special case $m = k = 0$ gives precisely the *Shuffle conjecture* in [12], now a theorem of Carlsson and Mellit [3]. The latter turns out to be a combinatorial formula for the Frobenius characteristic of the $\mathfrak{S}_n$-module of diagonal harmonics studied by Garsia and Haiman in relation to the famous $n!$ *conjecture*, now $n!$ theorem of Haiman [16].

In [17] Loehr and Warrington conjectured a combinatorial formula for $\Delta_{e_n}\omega(p_n) = \nabla\omega(p_n)$ in terms of labelled square paths (ending east), called *square conjecture*. The special case $\langle \cdot, e_n \rangle$ of this conjecture, known as q,t*-square*, has been proved earlier by Can and Loehr in [2]. Recently the full square conjecture has been proved by Sergel in [22] after the breakthrough of Carlsson and Mellit in [3].

In [5], we conjectured a combinatorial formula for $\frac{[n-k]_t}{[n]_t}\Delta_{h_m}\Delta_{e_{n-k}}\omega(p_n)$ in terms of *decorated partially labelled square paths* that we call *generalized Delta square conjecture*. In analogy with the Delta conjecture in [13], we call simply *Delta square conjecture* the special case $m = 0$. Our conjecture extends the square conjecture of Loehr and Warrington [17] (now a theorem [22]), i.e. it reduces to that one for $m = k = 0$. Moreover, it extends the

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generalized Delta conjecture in the sense that on decorated partially labelled Dyck paths gives the same combinatorial statistics. Our conjecture answers a question in [13].

In the present article we sketch the state of the art about the generalized Delta conjecture and the generalized Delta square conjecture. In this survey we highlights some results from our more recent works [5, 7, 6].

2 Combinatorial definitions

In order to state the generalized Delta conjecture and the generalized Delta square conjecture, we need to introduce some combinatorial objects.

Definition 1. A *square path ending east* of size n is a lattice paths going from $(0,0)$ to (n,n) consisting of east or north unit steps, always ending with an east step. The set of such paths is denoted by $\mathsf{SQ^E}(n)$. We call *base diagonal* of a square path the diagonal $y = x + k$ with the smallest value of k that is touched by the path (so that $k \leq 0$). The *shift* of the square path is the non-negative value $-k$. The *breaking point* of the square path is the lowest point in which the path touches the base diagonal. A *Dyck path* is a square path whose breaking point is $(0,0)$.

For example, the path in Figure 1 has shift 3.

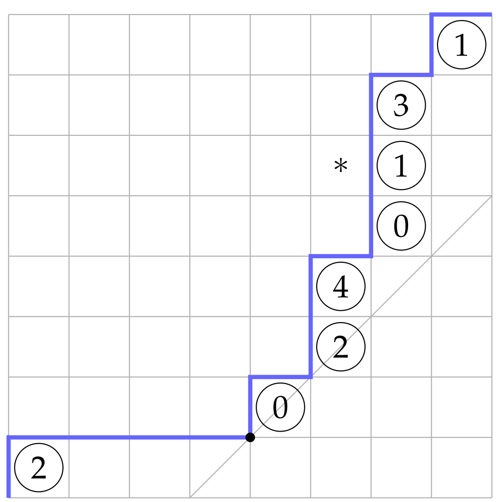

Figure 1: Example of an element in $\mathsf{PLSQ^E}(2,6)^{*1}$ with reading word 241231.

Definition 2. A *partially labelled square path ending east* is a square path ending east whose vertical steps are labelled with (not necessarily distinct) non-negative integers such that the labels appearing in each column are strictly increasing bottom to top, there is at least one nonzero label labelling a vertical step starting from the base diagonal, and if the path starts with a vertical step, this first step's label is nonzero. The set of partially labelled square paths ending east with m zero labels and n nonzero labels is denoted by $\mathsf{PLSQ}^{\mathsf{E}}(m,n)$. The subset of the Dyck paths, called *partially labelled Dyck paths*, is denoted by $\mathsf{PLD}(m,n)$.

Definition 3. Let P be a (partially labelled) square path ending east of size $n+m$. We define its *area word* to be the sequence of integers $a(P) = (a_1(P), a_2(P), \cdots, a_{n+m}(P))$ where the i-th vertical step of the path starts from the diagonal $y = x + a_i(P)$. For example the path in Figure 1 has area word $(0, -3, -3, -2, -2, -1, 0, 0)$.

Definition 4. We define for each $P \in \mathsf{PLSQ}^{\mathsf{E}}(m,n)$ a monomial in the variables $x_1, x_2, \ldots$: we set

$$x^P := \prod_{i=1}^{n+m} x_{l_i(P)}$$

where $l_i(P)$ is the label of the i-th vertical step of P (the first being at the bottom), where we conventionally set $x_0 = 1$. The fact that x_0 does not appear in the monomial explains the word *partially*.

Definition 5. The *rises* of a square path ending east P are the indices

$$\mathsf{Rise}(P) := \{2 \leq i \leq n+m \mid a_i(P) > a_{i-1}(P)\},$$

or the vertical steps that are directly preceded by another vertical step. Taking a subset $\mathsf{DRise}(P) \subseteq \mathsf{Rise}(P)$ and decorating the corresponding vertical steps with a $*$, we obtain a *decorated square path*, and we will refer to these vertical steps as *decorated rises*.

Definition 6. Given a partially labelled square path, we call *zero valleys* its vertical steps with label 0 (which are necessarily preceded by a horizontal step, hence the name valleys).

Definition 7. Given a partially labelled square path D, we define its *reading word* $\sigma(D)$ as the sequence of nonzero labels, read starting from the base diagonal going bottom left to top right, then moving to the next diagonal again going bottom left to top right, and so on.

The set of partially labelled decorated square paths ending east with m zero labels, n nonzero labels and k decorated rises is denoted by $\mathsf{PLSQ}^{\mathsf{E}}(m,n)^{*k}$. The subset of the Dyck paths is denoted by $\mathsf{PLD}(m,n)^{*k}$. See Figure 1 and Figure 2 for an example.

We define two statistics on this set that reduce to the same statistics as defined in [17] when $m = k = 0$.

Definition 8. Let $P \in \mathsf{PLSQ}^{\mathsf{E}}(m,n)^{*k}$ and s be its shift. Define

$$\mathsf{area}(P) := \sum_{i \notin \mathsf{DRise}(P)} (a_i(P) + s).$$

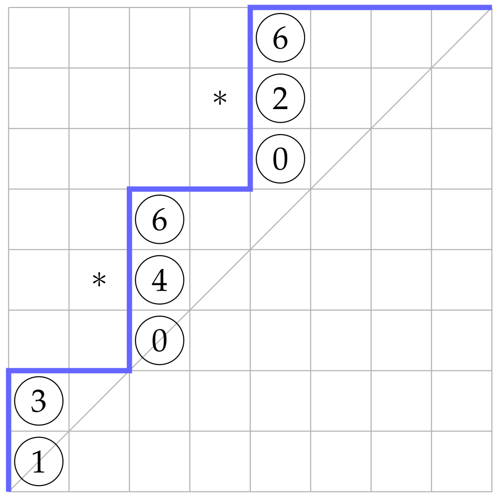

Figure 2: Example of an element in $\mathsf{PLD}(2,6)^{*2}$ with reading word 134626.

More visually, the area is the number of whole squares between the path and the base diagonal and not contained in rows containing a decorated rise.

For example, the path in Figure 1 has area 11.

Definition 9. Let $P \in \mathsf{PLSQ}^{\mathsf{E}}(m,n)$. For $1 \leq i < j \leq n+m$, we say that the pair (i,j) is an *inversion* if

- either $a_i(P) = a_j(P)$ and $l_i(P) < l_j(P)$ (*primary inversion*),
- or $a_i(P) = a_j(P) + 1$ and $l_i(P) > l_j(P)$ (*secondary inversion*),

where $l_i(P)$ denotes the label of the vertical step in the i-th row.

Then we define

$$\begin{aligned}\mathsf{dinv}(P) := \#\{0 \leq i < j \leq n+m \mid (i,j) \text{ is an inversion}\} \\ + \#\{0 \leq i \leq m+n \mid a_i(P) < 0 \text{ and } l_i(P) \neq 0\}.\end{aligned}$$

This second term is referred to as *bonus dinv*.

For example, the path in Figure 1 has dinv 6: 2 primary inversions, i.e. $(1,7)$ and $(2,3)$, 1 secondary inversion, i.e. $(1,6)$, and 3 bonus dinv, coming from the rows 3, 4 and 6. Notice that Dyck paths coincide with the square paths with no bonus dinv.

3 Symmetric functions

In this short section we limit ourself to recall the definitions needed to state the Delta conjectures, in particular the definition of the Delta operators. For the missing notation we refer to [8, Section 1] or [10].

We denote by Λ the algebra over the field $\mathbb{Q}(q,t)$ of symmetric functions in the variables $x_1, x_2, \dots$. We denote by e_n, h_n and p_n the *elementary*, *complete homogeneous* and *power symmetric function* of degree n, respectively. We denote by ω the involution of Λ defined by $\omega(e_n) := h_n$ for all n.

Also, for any partition μ, we denote by $s_\mu \in \Lambda$ the corresponding *Schur function*. It is well-known that the symmetric functions $\{s_\mu\}_\mu$ form a basis of Λ. The *Hall scalar product* on Λ, denoted $\langle\, , \rangle$, can be defined by stating that the Schur functions are an orthonormal basis.

Let $\widetilde{H}_\mu \in \Lambda$ denote the *(modified) Macdonald polynomial* indexed by the partition μ. As the polynomials $\{\widetilde{H}_\mu\}_\mu$ form a basis of Λ, given a symmetric function $f \in \Lambda$, we can define the *Delta operators* Δ_f and Δ'_f on Λ by setting

$$\Delta_f \widetilde{H}_\mu := f[B_\mu(q,t)]\widetilde{H}_\mu \quad \text{and} \quad \Delta'_f \widetilde{H}_\mu := f[B_\mu(q,t)-1]\widetilde{H}_\mu, \quad \text{for all } \mu, \tag{3.1}$$

where $B_\mu(q,t) = \sum_{c\in\mu} q^{a'_\mu(c)} t^{l'_\mu(c)}$ ($a'_\mu(c)$ and $l'_\mu(c)$ are the coarm and coleg of c in μ, respectively) and the square brackets denote the *plethystic* substitution: if $X = x_1 + x_2 + \dots$ is a sum of monomials, then $f[X] := f(x_1, x_2, \dots)$.

For example, if $\mu = (3,2)$ and $f = e_4$, then $B_{(3,2)}(q,t) = 1 + q + q^2 + t + qt$, so that $f[B_\mu(q,t)] = e_4(1,q,q^2,t,qt) = q^4t^2 + q^4t + q^3t^2 + q^3t + q^2t^2$ and $f[B_\mu(q,t)-1] = e_4(q,q^2,t,qt) = q^4t^2$.

Finally, we recall here the notation for the *t-analogue* of $n \in \mathbb{N}$:

$$[n]_t := 1 + t + t^2 + \dots + t^{n-1}.$$

4 The generalized Delta conjecture

4.1 Statement

Definition 10. We define a formal series in the variables $\underline{x} = (x_1, x_2, \dots)$ and coefficients in $\mathbb{N}[q,t]$

$$\mathsf{PLD}_{\underline{x},q,t}(m,n)^{*k} := \sum_{D \in \mathsf{PLD}(m,n)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D.$$

The following conjecture is stated by Haglund, Remmel and Wilson in [13].

Conjecture 11 (Generalized Delta)**.** *For $m,n,k \in \mathbb{N}$, $m \geq 0$ and $n > k \geq 0$,*

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \mathsf{PLD}_{\underline{x},q,t}(m,n)^{*k}.$$

4.2 State of the art

The case $m = k = 0$, i.e. the Shuffle conjecture, has been proved by Carlsson and Mellit in [3]. Several special cases have also been proved in the original work of Haglund, Remmel and Wilson [13].

We limit ourselves to summarize in the following table some of the most general special cases that have been proved more recently.

Conditions	Reference
$m = 0$ and $q = 0$	[9]
$m = 0$ and $q = 1$	[21]
$m = 0$ and $\langle \cdot, h_{n-d} h_d \rangle$	[4]
$\langle \cdot, e_{n-d} h_d \rangle$	Section 4.3
$t = 0$ or $q = 0$	Section 4.3

We should mention here also the proof of the so called 4-*variable Catalan conjecture* due to Zabrocki [25], which, combined with the results in [8], gives a "compositional" refinement of the case $m = 0$ and $\langle \cdot, e_{n-d} h_d \rangle$.

For more results on the Delta conjecture, i.e. the case $m = 0$ see [14, 15, 14, 18, 19, 20, 24, 23].

4.3 Our recent results

Definition 12. Given two sequences $(a_1, \dots, a_m)$, $(b_1, \dots, b_n)$ of pairwise distinct elements, their *shuffle* $(a_1, \dots, a_m) \sqcup\!\sqcup (b_1, \dots, b_n)$ is the set of sequences $(c_1, \dots, c_{m+n})$ such that

- $\{c_k \mid 1 \leq k \leq m+n\} = \{a_i, b_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$,
- $c_r = a_i,\ c_s = a_j,\ i < j \implies r < s$,
- $c_r = b_i,\ c_s = b_j,\ i < j \implies r < s$,

i.e. it is the set of sequences obtained by interlacing the elements of the two starting sequences while preserving their relative order.

Definition 13. Given $\mu \vdash n - d$ and $\nu \vdash d$, a μ, ν-*shuffle* is a sequence of numbers from 1 to n in

$$(1, \dots, \mu_1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup (n - \mu_{\ell(\mu)} + 1, \dots, n - d)$$
$$\sqcup\!\sqcup (n - d + \nu_1, \dots, n - d + 1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup (n, \dots, n - \nu_{\ell(\nu)} + 1)$$

i.e. a shuffle of $\ell(\mu)$ increasing sequences of length $\mu_1, \dots, \mu_{\ell(\mu)}$, and $\ell(\nu)$ decreasing sequences of length $\nu_1, \dots, \nu_{\ell(\nu)}$, obtained by picking every time the smallest available positive integers.

It is well known (cf. [10, Chapter 6]) that taking the scalar product of the combinatorial side of any of the Delta conjectures with $e_\mu \cdot h_\nu$ corresponds to taking the subsets of paths whose reading word is a μ, ν-shuffle.

The following theorem is proved in [6]. It extends the case $m = 0$ proved in [8].

Theorem 14. *For $m, n, k, d \in \mathbb{N}$ with $n > k \geq 0$, $n \geq d \geq 0$ and $m \geq 0$*

$$\langle \Delta_{h_m} \Delta'_{e_{n-k-1}} e_n, e_{n-d} h_d \rangle = \sum_{\substack{D \in \mathsf{PLD}(m,n)^{*k} \\ \sigma(D) \text{ is a } (n-d),(d)\text{-shuffle}}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D. \tag{4.1}$$

The following theorem is proved in [7]. It extends the case $m = 0$ first proved in [9], giving a new independent proof of that special case (see also [14] and [5] for alternative proofs).

Theorem 15. *For $m, n, k, d \in \mathbb{N}$ with $n > k \geq 0$, $n \geq d \geq 0$ and $m \geq 0$, we have*

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n \Big|_{t=0} = \mathsf{PLD}_{\underline{x},q,0}(m,n)^{*k} \quad \text{and} \quad \Delta_{h_m} \Delta'_{e_{n-k-1}} e_n \Big|_{q=0} = \mathsf{PLD}_{\underline{x},0,t}(m,n)^{*k}. \tag{4.2}$$

5 The generalized Delta square conjecture

5.1 Statement

In analogy with the Delta conjecture, we will refer to the case $m = 0$ of the following conjecture simply as the *Delta square conjecture*.

We formulated the following conjecture in [5].

Conjecture 16 (Generalized Delta square)**.** *For $m, n, k \in \mathbb{N}$, $m \geq 0$ and $n > k \geq 0$,*

$$\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) = \mathsf{PLSQ^E}_{\underline{x},q,t}(m,n)^{*k}.$$

Remark 17. Observe that $\mathsf{PLSQ^E}_{\underline{x},q,t}(m,n)^{*k}$ is a symmetric function. Indeed, consider the expression $\sum_P q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P$ where the sum is taken over all $P \in \mathsf{PLSQ^E}(m,n)^{*k}$ of a fixed *shape*, i.e. a fixed underlying square path with prescribed zero valleys. From this sum we can factor $t^{\mathsf{area}(P)}$, as the area is the same for all such paths P, and $q^{a(P)}$, where $a(P)$ is the contribution to the dinv of the 0 labels and of the negative letters of the area word (the bonus dinv): indeed this contribution does not depend on the nonzero labels, but only on the shape, so it will be the same for all our paths. What we are left with is in fact an LLT polynomial: the argument is essentially the same as in [13, Section 6.2], so we omit it (cf also [10, Remark 6.5]). As it is well-known that the LLT polynomials are symmetric functions (cf. [11, Appendix]), we deduce that also $\mathsf{PLSQ^E}_{\underline{x},q,t}(m,n)^{*k}$ is symmetric.

This conjecture answers a question in [13, Section 8.2].

Remark 18. Notice that the case $m = k = 0$ of the generalized Delta square conjecture reduces precisely to the *square conjecture* of Loehr and Warrington [17], recently proved by Sergel [22] after the breakthrough of Carlsson and Mellit [3].

5.2 State of the art

We limit ourselves to summarize in the following table some of the most general special cases that have been proved more recently.

Conditions	Reference
$m = 0$ and $k = 0$	[22]
$\langle \cdot, e_{n-d}h_d \rangle$	Section 5.3
$q = 0$	Section 5.3

We would like to mention here the early proof by Can and Loehr [2] of the special case $m = k = 0$ and $\langle \cdot, e_n \rangle$.

5.3 Our recent results

We recall that taking a scalar product with $e_\mu \cdot h_\nu$ corresponds to taking the subsets of paths whose reading word is a μ, ν-shuffle.

The following theorem is proved in [5]. It extends the case $m = k = d = 0$ proved by Can and Loehr in [2].

Theorem 19. *For $m, n, k, d \in \mathbb{N}$ with $n > k \geq 0$, $n \geq d \geq 0$ and $m \geq 0$*

$$\left\langle \frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n), e_{n-d}h_d \right\rangle = \sum_{\substack{P \in \mathsf{PLSQ}^{\mathsf{E}}(m,n)^{*k} \\ \sigma(P) \text{ is a } (n-d),(d)\text{-shuffle}}} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P. \tag{5.1}$$

The following theorem is proved in [5].

Theorem 20. *For $m, n, k, d \in \mathbb{N}$ with $n > k \geq 0$, $n \geq d \geq 0$ and $m \geq 0$, we have*

$$\left. \frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} \omega(p_n) \right|_{q=0} = \mathsf{PLSQ}^{\mathsf{E}}{}_{\underline{x},0,t}. \tag{5.2}$$

Observe that the case $t = 0$ of the generalized Delta square conjecture remains open.

6 Relations among Delta conjectures

In [5] we proved the following results.

Proposition 21. *For $n \in \mathbb{N}$, $n > 0$,*

$$\Delta_{h_m} \Delta'_{e_{n-1}} e_n \Big|_{t=0} = \mathsf{PLD}_{\underline{x},q,0}(m,n)^{*0}. \tag{6.1}$$

Theorem 22. *Given $m, n \in \mathbb{N}$, $m \geq 0$ and $n \geq 1$, we have the two identities*

$$\sum_{s=0}^{n-1} (-t)^s \Delta_{h_m} \Delta'_{e_{n-s-1}} e_n = \Delta_{h_m} \Delta'_{e_{n-1}} e_n \Big|_{t=0} \tag{6.2}$$

and

$$\sum_{s=0}^{n-1} (-t)^s \frac{[n-s]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-s}} \omega(p_n) = \Delta_{h_m} \Delta'_{e_{n-1}} e_n \Big|_{t=0} \quad . \tag{6.3}$$

The last theorem, combined with the Delta conjectures, suggested the following one.

Theorem 23. *Given $m, n \in \mathbb{N}$, $m \geq 0$ and $n \geq 1$, we have the two identities*

$$\sum_{s=0}^{n-1} (-t)^s \mathsf{PLSQ^E}_{\underline{x},q,t}(m,n)^{*s} = \mathsf{PLD}_{\underline{x},q,0}(m,n)^{*0} \tag{6.4}$$

$$\sum_{s=0}^{n-1} (-t)^s \mathsf{PLD}_{\underline{x},q,t}(m,n)^{*s} = \mathsf{PLD}_{\underline{x},q,0}(m,n)^{*0}. \tag{6.5}$$

Proof. Fix $m, n \in \mathbb{N}$, $m \geq 0$ and $n > 0$. Let

$$X := \bigsqcup_{k=0}^{n-1} \mathsf{PLSQ^E}(m,n)^{*k}, \tag{6.6}$$

and define a map $\varphi \colon X \to X$ in the following way: if $P \in X$ has no rises, i.e. no two consecutive vertical steps, then $\varphi(P) := P$; otherwise, consider the first rise encountered by following the path P starting from its breaking point (notice that this rise will always occur before the north-east corner): if the rise is decorated/undecorated, then $\varphi(P)$ is the path obtained from P by undecorating/decorating that rise. Observe that φ is clearly an involution, whose fixed points are the paths $P \in X$ with no rises, i.e. the paths of area 0 with no decorated rises. Notice also that φ restricts to an involution of

$$Y := \bigsqcup_{k=0}^{n-1} \mathsf{PLD}(m,n)^{*k} \subset X. \tag{6.7}$$

For any $P \in X$ we define a *weight* by setting

$$wt(P) := (-t)^{\mathsf{dr}(P)} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} x^P \tag{6.8}$$

where $\mathsf{dr}(P)$ is defined to be the number of decorated rises of P.

Observe that

$$\sum_{P \in X} wt(P) = \sum_{s=0}^{n-1} (-t)^s \mathsf{PLSQ^E}_{\underline{x},q,t}(m,n)^{*s} \tag{6.9}$$

and

$$\sum_{P \in Y} wt(P) = \sum_{s=0}^{n-1} (-t)^s \mathsf{PLD}_{\underline{x},q,t}(m,n)^{*s}. \tag{6.10}$$

Suppose that $P \in X$ is such that $\varphi(P) \neq P$. Notice that the rise occurring in the definition of φ is always at distance 1 from the base diagonal, so undecorating/decorating it when it is decorated/undecorated gives $\mathsf{dr}(\varphi(P)) = \mathsf{dr}(P) \mp 1$, but $\mathsf{area}(\varphi(P)) = \mathsf{area}(P) \pm 1$. Since the decorations of the rises do not affect the dinv, we deduce that $wt(\varphi(P)) = -wt(P)$. This shows that in the sum $\sum_{P \in X} wt(P)$ all the contributions of the P that are not fixed by φ cancel out, leaving the sum over the fixed points of φ, i.e. over the paths with no rises, as we claimed.

The same argument applies to the sum $\sum_{P \in Y} wt(P)$. □

These last results combined give relations among the Delta conjectures. We get immediately the following curious corollary.

Corollary 24. *For fixed $m, n \in \mathbb{N}$, with $m \geq 0$ and $n > 0$, the truth of the generalized Delta (square) conjectures for all values of k in $\{0, 1, \ldots, n-1\}$ except one imply the truth of the missing case.*

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