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Affine transitions for involution Stanley symmetric functions

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Abstract. We define a new family of Stanley symmetric functions for affine involutions and study the basic properties of these power series. After classifying the covering relations in the Bruhat order of the affine symmetric group restricted to involutions, we prove an affine transition formula for involution Stanley symmetric functions.

Keywords: Affine permutations, Stanley symmetric functions, involutions, transition equations, Bruhat order

1 Introduction

The notion of the *Stanley symmetric function* F_π of a permutation $\pi \in S_n$ dates to work of Stanley [15] in the 1980s. These symmetric functions are of interest as the stable limits of the *Schubert polynomials* $\mathfrak{S}_\pi$, which represent the cohomology classes of certain orbit closures in the type A flag variety. They are also useful as a tool for studying the enumeration of reduced expressions for permutations.

Several variations of Stanley's construction have been studied in the literature. In 2006, Lam [9] introduced a generalized family of symmetric functions F_π indexed by affine permutations $\pi \in \tilde{S}_n$. These *affine Stanley symmetric functions* represent cohomology classes for the affine Grassmannian [10, §7]. On the other hand, some recent papers by Hamaker, Pawlowski, and the first author [6, 5, 7] study the so-called *involution Stanley symmetric functions* $\hat{F}_z$, which are indexed by self-inverse permutations $z = z^{-1} \in S_n$. Up to a scalar factor, these power series are the stable limits of the *involution Schubert polynomials* $\hat{\mathfrak{S}}_z$ introduced by Wyser and Yong [16] to represent the cohomology classes of orbit closures of the orthogonal group acting on the type A flag variety.

The subject of this article is a family of symmetric functions indexed by affine involutions $z = z^{-1} \in \tilde{S}_n$, generalizing both of the preceding constructions. (For a diagrammatic summary of the relationships between our new family and other kinds of Stanley symmetric functions, see Figure 1.) Our first results concern several equivalent definitions and basic properties of these "affine" involution Stanley symmetric functions. We expect that these power series are related to the geometry of affine analogues of certain symmetric varieties.

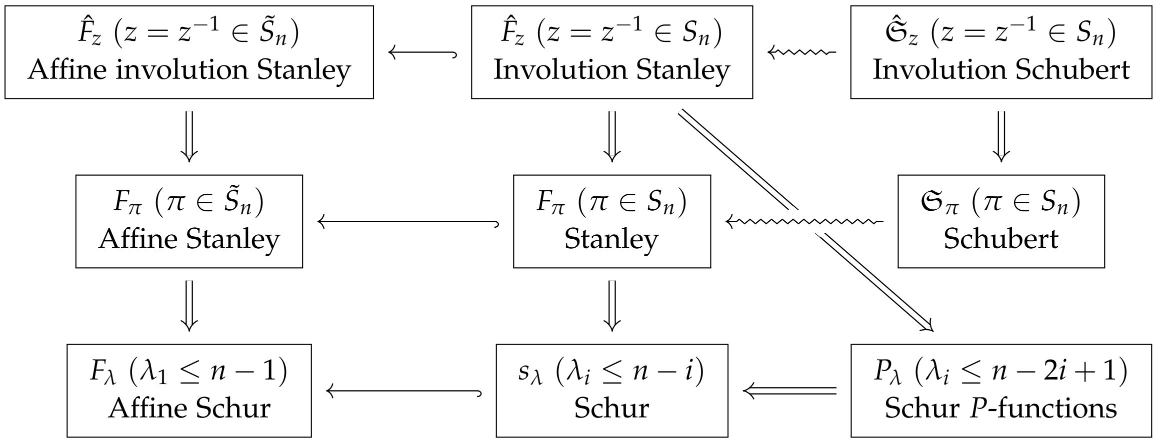


Figure 1: Families of polynomials and symmetric functions of interest. The relationships between each family are indicated as follows: $\hookrightarrow$ means "is a special case of" while $\rightsquigarrow$ means "has stable limit" while $\Rightarrow$ means "expands positively into."

The symmetric functions F_π and $\hat{F}_z$ are noteworthy for their positivity properties. For finite permutations $\pi \in S_n$, the power series F_π is always *Schur positive*, i.e., an $\mathbb{N}$-linear combination of Schur functions s_λ [3]. Similarly, each involution Stanley symmetric function $\hat{F}_z$ is an $\mathbb{N}$-linear combination of *Schur P-functions* [5]. The Stanley symmetric functions F_π indexed by affine permutation $\pi \in \tilde{S}_n$, while not always Schur positive, are at least "affine Schur positive" [10] (see Section 2).

One way to prove these positivity properties is via *transition equations*: certain families of identities relating sums of affine/involution Stanley symmetric functions indexed by Bruhat covers of a given permutation. Lam and Shimozono described transition equations for affine Stanley symmetric functions in [11]. Transition equations for involution Stanley symmetric functions are derived in [4, 5, 7]. Our results in Section 4 show how to extend the latter formulas to the affine case. Formulating these identities is the first step towards studying the positivity properties of affine involution Stanley symmetric functions, which are not yet well understood.

The next section reviews some preliminaries on affine permutations and Stanley symmetric functions. Section 3 introduces our new family of *affine involution Stanley symmetric functions* and surveys their basic properties. We omit most proofs in this extended abstract; for complete arguments, see the full length article [13].

2 Affine permutations

Let n be a positive integer. Write $\mathbb{Z}$ for the set of integers and define $[n] = \{1, 2, \ldots, n\}$.

Definition 2.1. The *affine symmetric group* $\tilde{S}_n$ is the group of bijections $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying $\pi(i+n) = \pi(i) + n$ for all $i \in \mathbb{Z}$ and $\pi(1) + \pi(2) + \cdots + \pi(n) = 1 + 2 + \cdots + n$.

We refer to elements of $\tilde{S}_n$ as *affine permutations*. A *window* for an affine permutation $\pi \in \tilde{S}_n$ is a sequence of the form $[\pi(i+1), \pi(i+2), \ldots, \pi(i+n)]$ where $i \in \mathbb{Z}$. An element $\pi \in \tilde{S}_n$ is uniquely determined by any of its windows, and a sequence of n distinct integers is a window for some $\pi \in \tilde{S}_n$ if and only if the integers represent each congruence class modulo n exactly once.

Let s_i for $i \in \mathbb{Z}$ be the unique element of $\tilde{S}_n$ that interchanges i and $i+1$ while fixing every integer $j \notin \{i, i+1\} + n\mathbb{Z}$. One has $s_i = s_{i+n}$ for all $i \in \mathbb{Z}$, and $\{s_1, s_2, \ldots, s_n\}$ generates the group $\tilde{S}_n$. With respect to this generating set, $\tilde{S}_n$ is the affine Coxeter group of type $\tilde{A}_{n-1}$. The parabolic subgroup $S_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle \subset \tilde{S}_n$ is the finite Coxeter group of type A_{n-1}; its elements are the permutations $\pi \in \tilde{S}_n$ with $\pi([n]) = [n]$.

A *reduced expression* for $\pi \in \tilde{S}_n$ is a minimal-length factorization $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$. The *length* of $\pi \in \tilde{S}_n$, denoted $\ell(\pi)$, is the number of factors in any of its reduced expressions. The value of $\ell(\pi)$ is also the number of equivalence classes in the inversion set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i < j \text{ and } \pi(i) > \pi(j)\}$ under the relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ with $(a,b) \sim (a',b')$ if and only if $a - a' = b - b' \in n\mathbb{Z}$.

Definition 2.2. A reduced expression $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$ for an affine permutation is *cyclically decreasing* if $s_{i_j+1} \neq s_{i_k}$ for all $1 \leq j < k \leq l$. An element $\pi \in \tilde{S}_n$ is cyclically decreasing if it has a cyclically decreasing reduced expression.

Definition 2.3 (Lam [9])**.** The *(affine) Stanley symmetric function* of an element $\pi \in \tilde{S}_n$ is the sum $F_\pi = \sum_{\pi = \pi^1 \pi^2 \cdots} x_1^{\ell(\pi^1)} x_2^{\ell(\pi^2)} \cdots \in \mathbb{Z}[[x_1, x_2, \ldots]]$ over all factorizations $\pi = \pi^1 \pi^2 \cdots$ of π into countably many (possibly empty) cyclically decreasing factors $\pi^i \in \tilde{S}_n$ such that $\ell(\pi) = \ell(\pi^1) + \ell(\pi^2) + \ldots$.

Example 2.4. Suppose $n = 4$ so that $s_1 = s_5$. There are two reduced expressions for $[0, 3, 6, 1] = s_1 s_2 s_4 s_3 = s_1 s_4 s_2 s_3 \in \tilde{S}_4$. The length-additive factorizations of this element into nontrivial cyclically decreasing factors are $(s_1)(s_2)(s_4)(s_3)$ and $(s_1)(s_4)(s_2)(s_3)$ and $(s_1 s_4)(s_2)(s_3)$ and $(s_1)(s_2 s_4)(s_3) = (s_1)(s_4 s_2)(s_3)$ and $(s_1)(s_2)(s_4 s_3)$, so we have $F_{[0,3,1,6]} = 2m_{1^4} + m_{21^2}$, where m_λ denotes the usual monomial symmetric function of a partition.

One can motivate the definition of F_π using the theory of *combinatorial coalgebras* [1]. Define a combinatorial coalgebra (C, ζ) to be a graded, connected $\mathbb{Q}$-coalgebra C with a linear map $\zeta : C \to \mathbb{Q}$. A morphism of combinatorial coalgebras $(C, \zeta) \to (C', \zeta')$ is a morphism of graded coalgebras $\phi : C \to C'$ satisfying $\zeta = \zeta' \circ \phi$. For $\pi \in \tilde{S}_n$, we write $\pi \doteq \pi' \pi''$ to indicate that $\pi', \pi'' \in \tilde{S}_n$, $\pi = \pi' \pi''$, and $\ell(\pi) = \ell(\pi') + \ell(\pi'')$.

Proposition 2.5. The graded vector space $\mathbb{Q}\tilde{S}_n$, in which $\pi \in \tilde{S}_n$ is homogeneous of degree $\ell(\pi)$, is a graded, connected coalgebra with coproduct $\Delta(\pi) = \sum_{\pi \overset{\bullet}{=} \pi'\pi''} \pi' \otimes \pi''$.

Let $\mathsf{QSym} \subset \mathbb{Q}[[x_1, x_2, \dots]]$ denote the graded, connected Hopf algebra of quasi-symmetric functions over $\mathbb{Q}$, and write ζ_{QSym} for the algebra homomorphism $\mathsf{QSym} \to \mathbb{Q}$ which sets $x_1 = 1$ and $x_2 = x_3 = \cdots = 0$. Define $\zeta_{\mathsf{CD}} : \mathbb{Q}\tilde{S}_n \to \mathbb{Q}$ to be the linear map with $\zeta_{\mathsf{CD}}(\pi) = 1$ if $\pi \in \tilde{S}_n$ is cyclically decreasing and with $\zeta_{\mathsf{CD}}(\pi) = 0$ otherwise.

Proposition 2.6. The linear map with $\pi \mapsto F_\pi$ for $\pi \in \tilde{S}_n$ is the unique morphism of combinatorial coalgebras $(\mathbb{Q}\tilde{S}_n, \zeta_{\mathsf{CD}}) \to (\mathsf{QSym}, \zeta_{\mathsf{QSym}})$.

Corollary 2.7 (Lam [9, Theorem 12])**.** If $\pi \in \tilde{S}_n$ then $\Delta(F_\pi) = \sum_{\pi \overset{\bullet}{=} \pi'\pi''} F_{\pi'} \otimes F_{\pi''}$.

Let $\mathsf{Sym} \subset \mathsf{QSym}$ denote the Hopf subalgebra of symmetric functions over $\mathbb{Q}$. Let Par^n be the set of partitions with all parts less than n and $\mathsf{Sym}^{(n)} = \mathbb{Q}\text{-span}\{m_\lambda : \lambda \in \mathsf{Par}^n\}$.

Theorem 2.8 (Lam [9, Theorem 6])**.** If $\pi \in \tilde{S}_n$ then $F_\pi \in \mathsf{Sym}^{(n)} \subset \mathsf{Sym}$.

The *code* of an affine permutation $\pi \in \tilde{S}_n$ is the sequence $c(\pi) = (c_1, c_2, \dots, c_n)$ where c_i is the number of integers $j \in \mathbb{Z}$ with $i < j$ and $\pi(i) > \pi(j)$. If $\pi(i)$ is minimal among $\pi(1), \pi(2), \dots, \pi(n)$, then $c_i = 0$. An integer $i \in \mathbb{Z}$ is a *descent* of π if $\pi(i) > \pi(i+1)$, i.e., if $\ell(\pi s_i) = \ell(\pi) - 1$. If $i \in [n]$ is a descent of π then $c(\pi s_i) = (c_1, \dots, c_{i+1}, c_i - 1, \dots, c_n)$, interpreting indices cyclically. By induction $|c(\pi)| = c_1 + c_2 + \dots c_n = \ell(\pi)$, and the map $\pi \mapsto c(\pi)$ is a bijection $\tilde{S}_n \to \mathbb{N}^n - \mathbb{P}^n$.

The *shape* $\lambda(\pi)$ of $\pi \in \tilde{S}_n$ is the transpose of the partition that sorts $c(\pi^{-1})$. Write $<$ for the *dominance order* on partitions.

Theorem 2.9 (Lam [9, Theorem 13])**.** If $\pi \in \tilde{S}_n$ then $F_\pi \in m_{\lambda(\pi)} + \sum_{\mu < \lambda(\pi)} \mathbb{N} m_\nu$.

Since $\lambda : \tilde{S}_n \to \mathsf{Par}^n$ is surjective, this implies that $\mathbb{Q}\text{-span}\{F_\pi : \pi \in \tilde{S}_n\} = \mathsf{Sym}^{(n)}$.

Example 2.10. Suppose $n = 4$ and $\pi = [-3, 4, 3, 6] \in \tilde{S}_4$ so that $w^{-1} = [5, 0, 3, 2]$. Then $c(\pi) = (0, 2, 1, 2)$ and $c(\pi^{-1}) = (4, 0, 1, 0)$ so $\lambda(\pi) = (2, 1, 1, 1)$ and $\lambda(\pi^{-1}) = (3, 2)$, and we have $F_\pi = m_{21^3} + 4m_{1^5}$ and $F_{\pi^{-1}} = m_{32} + 2m_{31^2} + 2m_{2^21} + 3m_{21^3} + 4m_{1^5}$.

Let $\mathrm{Des}_R(\pi) = \{s_i : i \in \mathbb{Z} \text{ is a descent of } \pi\}$ and $\mathrm{Des}_L(\pi) = \mathrm{Des}_R(\pi^{-1})$ for $\pi \in \tilde{S}_n$. An affine permutation $\pi \in \tilde{S}_n$ is *Grassmannian* if $\pi^{-1}(1) < \pi^{-1}(2) < \cdots < \pi^{-1}(n)$.

Definition 2.11. The *affine Schur function* F_λ indexed by $\lambda \in \mathsf{Par}^n$ is the Stanley symmetric function $F_\lambda = F_\pi$ where $\pi \in \tilde{S}_n$ is the unique Grassmannian element of shape λ.

Lam has shown that the symmetric functions F_π are *affine Schur positive*:

Theorem 2.12 (Lam [10])**.** It holds that $\mathbb{N}\text{-span}\{F_\pi : \pi \in \tilde{S}_n\} = \mathbb{N}\text{-span}\{F_\lambda : \lambda \in \mathsf{Par}^n\}$.

As mentioned in the introduction, each F_π for $\pi \in S_n \subset \tilde{S}_n$ is Schur positive [3]. This stronger property does not hold for all affine Schur functions.

One can refine Theorem 2.9. Write $w \mapsto w^*$ for the automorphism of $\tilde{S}_n$ with $s_i \mapsto s_i^* := s_{n-i}$ for $i \in \mathbb{Z}$. If $\lambda \in \mathsf{Par}^n$ then there exists a unique Grassmannian permutation $\pi \in \tilde{S}_n$ with $\lambda = \lambda(\pi)$, and one defines $\lambda^* = \lambda(\pi^*)$. Let $\lambda'(\pi) = \lambda(\pi^{-1})^*$ for $\pi \in \tilde{S}_n$. Define $<^*$ to be the partial order on Par^n with $\lambda <^* \mu$ if and only if $\mu^* < \lambda^*$.

Theorem 2.13 (Lam [9, Theorem 21])**.** If $\pi \in \tilde{S}_n$ then $F_\pi \in F_{\lambda'(\pi)} + \sum_{\lambda'(\pi) <^* \mu} \mathbb{N} F_\mu$ and $F_\pi \in F_{\lambda(\pi)} + \sum_{\mu < \lambda(\pi)} \mathbb{N} F_\mu$.

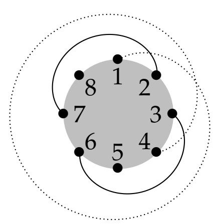
The affine Schur functions form a basis for $\mathsf{Sym}^{(n)}$, so there exists a unique linear involution $\omega^+ : \mathsf{Sym}^{(n)} \to \mathsf{Sym}^{(n)}$ with $\omega^+(F_\lambda) = F_{\lambda^*}$ for all $\lambda \in \mathsf{Par}^n$. This map can be defined directly in terms of the usual bases of symmetric functions; see [9, §9].

Theorem 2.14 (Lam [9, Theorem 15])**.** If $\pi \in \tilde{S}_n$ then $\omega^+(F_\pi) = F_{\pi^*} = F_{\pi^{-1}}$.

3 Affine involutions

For integers $i < j \not\equiv i \pmod{n}$, let $t_{ij} \in \tilde{S}_n$ be the affine permutation interchanging i and j while fixing all integers $k \notin \{i, j\} + n\mathbb{Z}$. Such permutations are precisely the *reflections* in $\tilde{S}_n$, i.e., the elements conjugate to s_i for some $i \in \mathbb{Z}$.

Let $\tilde{I}_n = \{z \in \tilde{S}_n : z = z^{-1}\}$ be the set of involutions in $\tilde{S}_n$. When possible, we represent elements of this set via *winding diagrams* like the following:



Here, the numbers $1, 2, \ldots, n$ are arranged in order around a circle. A curve that connects i and j by traveling m times clockwise around the vertex 1 corresponds to the reflection $t_{i,j+mn} \in \tilde{S}_n$, and the winding diagram represents the commuting product of these reflections. The given diagram represents $z = t_{1,12} \cdot t_{3,6} \cdot t_{7,10} \in \tilde{I}_8$.

There exists a unique associative product $\circ : \tilde{S}_n \times \tilde{S}_n \to \tilde{S}_n$ with $s_i \circ s_i = s_i$ for $i \in \mathbb{Z}$ and $\pi' \circ \pi'' = \pi$ whenever $\pi \doteq \pi'\pi''$, and it holds that $\tilde{I}_n = \{\pi^{-1} \circ \pi : \pi \in \tilde{S}_n\}$. The set $\mathcal{A}_{\mathsf{Hecke}}(z) := \{\pi \in \tilde{S}_n : \pi^{-1} \circ \pi = z\}$ is therefore nonempty and finite, since $\ell(\pi) \leq \ell(\pi^{-1} \circ \pi)$ for all $\pi \in \tilde{S}_n$. Let $\mathcal{A}(z)$ be the subset of minimal-length permutations in $\mathcal{A}_{\mathsf{Hecke}}(z)$. We refer to elements of $\mathcal{A}(z)$ and $\mathcal{A}_{\mathsf{Hecke}}(z)$ as *atoms* and *Hecke atoms* for z.

Definition 3.1. The *(affine) involution Stanley symmetric function* of $z \in \tilde{I}_n$ is $\hat{F}_z = \sum_{\pi \in \mathcal{A}(z)} F_\pi$.

This is an affine generalization of the symmetric functions studied in [4, 6, 5, 7], which are defined by the same formula but with z restricted to the set $I_n := \tilde{I}_n \cap S_n$.

One can describe $\mathcal{A}_{\mathsf{Hecke}}(z)$ and $\mathcal{A}(z)$ more concretely. Suppose $a_1, a_2, \dots, a_N \in \mathbb{Z}$ represent all congruence classes modulo n at least once. Define $[[a_1, a_2, \dots, a_N]] \in \tilde{S}_n$ to be the affine permutation with a window given by reading the sequence $[a_1, a_2, \dots, a_N]$ left to right and omitting a_j whenever $a_i \equiv a_j \pmod{n}$ for some $i < j$. For example, if $n = 5$ then $[[1, 3, 0, 1, 2, -1, 4, 8]] = [1, 3, 0, 2, -1] = [3, 0, 2, -1, 6] = [0, 2, -1, 6, 8] \in \tilde{S}_5$. Let $z \in \tilde{I}_n$. Write $a_1 < a_2 < \cdots < a_l$ for the numbers $a \in [n]$ with $a \leq z(a)$ and define

$$\alpha_{\min}(z) = [[z(a_1), a_1, z(a_2), a_2, \dots, z(a_l), a_l]]^{-1} \in \tilde{S}_n.$$

Next write $b_1 < b_2 < \cdots < b_l$ for the numbers $b \in [n]$ with $z(b) \leq b$ and define

$$\alpha_{\max}(z) = [[b_1, z(b_1), b_2, z(b_2), \dots, b_l, z(b_l)]]^{-1} \in \tilde{S}_n.$$

Let $\approx$ be the weakest equivalence relation on $\tilde{S}_n$ that has $u^{-1} \approx v^{-1} \approx w^{-1}$ whenever $u, v, w \in \tilde{S}_n$ have windows that are identical except in three consecutive positions where $u = [\cdots\ c, b, a\ \cdots]$, $v = [\cdots\ c, a, b\ \cdots]$, and $w = [\cdots\ b, c, a\ \cdots]$ for some $a < b < c$.

Theorem 3.2. If $z \in \tilde{I}_n$ then $\mathcal{A}_{\mathsf{Hecke}}(z) = \{\pi \in \tilde{S}_n : \pi \approx \alpha_{\min}(z)\}$.

Let $\prec$ be the weakest relation on $\tilde{S}_n$ that has $v^{-1} \prec w^{-1}$ whenever v and w have windows that are identical except in three consecutive positions where $v = [\cdots\ c, a, b\ \cdots]$ and $w = [\cdots\ b, c, a\ \cdots]$ for some $a < b < c$.

Theorem 3.3 ([12, Theorem 6.14]). Let $z \in \tilde{I}_n$. Restricted to $\mathcal{A}(z)$, the relation $\prec$ is a graded partial order and $\mathcal{A}(z) = \{\pi \in \tilde{S}_n : \alpha_{\min}(z) \preceq \pi\} = \{\pi \in \tilde{S}_n : \pi \preceq \alpha_{\max}(z)\}$.

Example 3.4. Suppose $n = 4$ and $z = t_{3,8} = [1, 2, 8, -1] \in \tilde{I}_4$. The elements of $\mathcal{A}(z)$ are

$$\begin{aligned}\alpha_{\min}(z) &= [1,2,8,3]^{-1} = [2,3,5,0] = [2,8,3,5]^{-1}\\ &\prec [2,5,8,3]^{-1} = [0,3,6,1] = [5,8,3,6]^{-1}\\ &\prec [5,6,8,3]^{-1} = [0,1,7,2] = [1,2,4,-1]^{-1} = \alpha_{\max}(z).\end{aligned}$$

The elements of $\mathcal{A}_{\mathsf{Hecke}}(z) - \mathcal{A}(z)$ are $[2,8,5,3]^{-1}$ and $[5,8,6,3]^{-1}$. Both $[2,3,5,0] = s_4 s_1 s_2 s_3$ and $[0,1,2,7] = s_2 s_1 s_4 s_3$ have a single reduced expression, and it holds that $F_{[2,3,0,5]} = m_{1^4}$ and $F_{[0,1,2,7]} = m_{1^4} + m_{21^2} + m_{2^2} + m_{31}$. We saw in Example 2.4 that $F_{[0,3,1,6]} = 2m_{1^4} + m_{21^2}$. Therefore $\hat{F}_z = \hat{F}_{[1,2,8,-1]} = 4m_{1^4} + 2m_{21^2} + m_{2^2} + m_{31}$.

There is an analogue of Proposition 2.6 which motivates Definition 3.1. Define $\ell'(\pi)$ to be n minus the number of orbits of $\pi \in \tilde{S}_n$ acting on $\mathbb{Z}/n\mathbb{Z}$. The map $\ell' : \tilde{S}_n \to \mathbb{N}$ is constant on conjugacy classes, and if the congruence classes $i + n\mathbb{Z}$ and $i + 1 + n\mathbb{Z}$ belong to distinct orbits under $\pi \in \tilde{S}_n$ then $\ell'(ws_i) = \ell'(w) + 1$. Define $\hat{\ell}(z) = \frac{1}{2}(\ell(z) + \ell'(z))$ for $z \in \tilde{I}_n$. One can show that $\hat{\ell}(z) = \ell(\pi)$ for any $\pi \in \mathcal{A}(z)$. Give $\mathbb{Q}\tilde{S}_n$ the coalgebra structure from Proposition 2.5 and write Δ for its coproduct.

Proposition 3.5. The graded vector space $\mathbb{Q}\tilde{I}_n$, in which $z \in \tilde{I}_n$ is homogeneous of degree $\hat{\ell}(z)$, is a graded right comodule for $\mathbb{Q}\tilde{S}_n$ with coproduct $\hat{\Delta} : \mathbb{Q}\tilde{I}_n \to \mathbb{Q}\tilde{I}_n \otimes \mathbb{Q}\tilde{S}_n$ given by

$$\hat{\Delta}(z) = \sum_{\substack{(y,\pi)\in\tilde{I}_n\times\tilde{S}_n \\ \hat{\ell}(z)=\hat{\ell}(y)+\ell(\pi) \\ z=\pi^{-1}\circ y\circ\pi}} y \otimes \pi \qquad \text{for } z \in \tilde{I}_n.$$

Let $\mathfrak{F}$ be the coalgebra morphism $\mathbb{Q}\tilde{S}_n \to \mathsf{Sym}$ with $\pi \mapsto F_\pi$ for $\pi \in \tilde{S}_n$. The graded vector space $\mathbb{Q}\tilde{I}_n$ is then a graded right QSym-comodule with respect to the coproduct $(\mathrm{id} \otimes \mathfrak{F}) \circ \hat{\Delta}$. The coalgebra QSym is automatically a right comodule for itself.

Proposition 3.6. The linear map with $z \mapsto \hat{F}_z$ for $z \in \tilde{I}_n$ is the unique morphism of graded right QSym-comodules $\mathbb{Q}\tilde{I}_n \to \mathsf{QSym}$ satisfying $1 \mapsto 1 \in \mathbb{Q}[[x_1, x_2, \dots]]$.

Corollary 3.7. If $z \in \tilde{I}_n$ then $\Delta(\hat{F}_z) = \sum \hat{F}_y \otimes F_\pi$ where the sum is over all pairs $(y, \pi) \in \tilde{I}_n \times \tilde{S}_n$ with $\hat{\ell}(z) = \hat{\ell}(y) + \hat{\ell}(\pi)$ and $z = \pi^{-1} \circ y \circ \pi$.

The notions of codes, shapes, and so forth for affine permutations have analogues for involutions. These definitions are affine generalizations of constructions from [6, 5]. To start, the *involution code* of $z \in \tilde{I}_n$ is the sequence $\hat{c}(z) = (c_1, c_2, \dots, c_n)$ where c_i is the number of integers $j \in \mathbb{Z}$ with $i < j$ and $\pi(i) > \pi(j)$ and $i \geq \pi(j)$. An integer $i \in \mathbb{Z}$ is a *visible descent* of $z \in \tilde{I}_n$ if $z(i) > z(i+1)$ and $i \geq z(i+1)$. Let $\mathrm{Des}_V(z) = \{s_i : i \in \mathbb{Z} \text{ is a visible descent of } z\}$.

Lemma 3.8. If $z \in \tilde{I}_n$ then $\mathrm{Des}_V(z) = \mathrm{Des}_R(\alpha_{\min}(z))$ and $\hat{c}(z) = c(\alpha_{\min}(z))$.

Thus, every involution in $\tilde{I}_n - \{1\}$ has at least one visible descent.

Corollary 3.9. The involution code is an injective map $\hat{c} : \tilde{I}_n \to \mathbb{N}^n - \mathbb{P}^n$.

Corollary 3.10. Suppose $z \in \tilde{I}_n$ and $\hat{c}(z) = (c_1, c_2, \dots, c_n)$. Then $\hat{\ell}(z) = c_1 + c_2 + \cdots + c_n$, and $i \in \mathbb{Z}$ is a visible descent of z if and only if $c_i > c_{i+1}$, interpreting indices modulo n.

Corollary 3.11. For $z \in \tilde{I}_n$, the following are equivalent: (a) $\mathrm{Des}_V(z) \subset \{s_n\}$; (b) $\hat{c}(z)$ is weakly increasing; and (c) $\alpha_{\min}(z)^{-1}$ is Grassmannian.

This corollary suggests the property $\mathrm{Des}_V(z) \subset \{s_n\}$ as a natural definition for the "involution" analogue for a Grassmannian permutation. However, the functions $\hat{F}_z$ indexed by $z \in \tilde{I}_n$ with this property fail to span $\mathbb{Q}\text{-span}\{\hat{F}_z : z \in \tilde{I}_n\}$, although they are linearly independent.

The *(involution) shape* $\mu(z)$ of $z \in \tilde{I}_n$ is the transpose of the partition that sorts $\hat{c}(z)$. The maps $\hat{c} : \tilde{I}_n \to \mathbb{N}^n - \mathbb{P}^n$ and $\mu : \tilde{I}_n \to \mathsf{Par}^n$ are not surjective, and it is an open problem to characterize their images. By results in [5, §4], the shape map μ restricts to a bijection from $I_n = \tilde{I}_n \cap S_n$ to the set of strict partitions contained in $(n-1, n-3, n-5, \dots)$. However, $\mu(z)$ is not strict for all $z \in \tilde{I}_n$.

Recall the definitions of $\lambda(\pi)$, $\lambda'(\pi)$, and λ^* from the discussion before Theorem 2.13.

Lemma 3.12. If $z \in \tilde{I}_n$ then $\mu(z) = \lambda(\alpha_{\max}(z))$ and $\mu(z)^* = \lambda'(\alpha_{\min}(z))$.

Since $*$ is involution, this implies that $\lambda(\alpha_{\max}(z)) = \lambda(\alpha_{\min}(z)^{-1})$.

Theorem 3.13. If $z \in \tilde{I}_n$ then $\hat{F}_z \in m_{\mu(z)} + \sum_{\nu < \mu(z)} \mathbb{N} m_\nu \subset \mathsf{Sym}^{(n)}$.

Theorem 3.14. If $z \in \tilde{I}_n$ then $\hat{F}_z \in \left(F_{\mu(z)^*} + \sum_{\mu(z)^* <^* \nu} \mathbb{N} F_\nu \right) \cap \left(F_{\mu(z)} + \sum_{\nu < \mu(z)} \mathbb{N} F_\nu \right)$.

Example 3.15. Again let $z = t_{3,8} = [1,2,8,-1] \in \tilde{S}_4$, so that we have

$$\alpha_{\min}(z) = [2,3,5,0] \quad \text{and} \quad \alpha_{\max}(z) = [1,2,4,-1]^{-1}.$$

Then $\hat{c}(z) = c(\alpha_{\min}(z)) = (1,1,2,0)$ and $c(\alpha_{\max}(z)^{-1}) = (1,2,0,1)$ so

$$\mu(z) = \lambda(\alpha_{\max}(z)) = (3,1) \quad \text{and} \quad \mu(z)^* = \lambda'(\alpha_{\min}(z)) = \lambda(\alpha_{\min}(z)^{-1})^* = (3,1)^*.$$

The Grassmannian permutation $\pi \in \tilde{S}_4$ with $\lambda(\pi) = (3,1)$ is $\pi = [-3,3,4,6]^{-1}$. Since $\pi^* = [-1,1,2,8]^{-1}$ has shape $\lambda(\pi^*) = (1,1,1,1)$, we have $\mu(z)^* = (1,1,1,1)$. This agrees with Theorems 3.13 and 3.14 since $\hat{F}_z = m_{1^4} + m_{21^2} + m_{2^2} + m_{31} = F_{1^4} + F_{21^2} + F_{31}$.

Some basic questions about involution Stanley symmetric functions are still unresolved. The subspace generated by the functions $\hat{F}_z$ as z ranges over the involutions in the finite group $S_n \subset \tilde{S}_n$ is well-understood: this is precisely the span of the Schur P-functions P_μ indexed by strict partitions μ contained in the "staircase" $(n-1, n-3, n-5, \dots)$ [5, Corollary 5.22].

By contrast, it is an open problem to identify a basis for $\mathbb{Q}$-span$\{\hat{F}_z : z \in \tilde{I}_n\} \subset \mathsf{Sym}^{(n)}$. Computer calculations indicate that no subset of $\{\hat{F}_z : z \in \tilde{I}_n\}$ gives a positive basis for this space, that is, a basis in which every $\hat{F}_z$ expands with positive coefficients. The question of how to define the "Grassmannian" elements of $\tilde{I}_n$ is subtler than for $\tilde{S}_n$.

Finally, note that there are obvious "left-handed" versions of Propositions 2.5 and 3.5. These statements would suggest $\omega^+(\hat{F}_z) = \sum_{\pi \in \mathcal{A}(z)} F_{\pi^{-1}}$ instead of $\hat{F}_z$ as the natural symmetric function corresponding to $z \in \tilde{I}_n$. Computations support the following conjecture, which implies that the choice of left- or right-handed convention is immaterial.

Conjecture 3.16. If $z \in \tilde{I}_n$ then $\omega^+(\hat{F}_z) = \hat{F}_z$, that is, $\sum_{\pi \in \mathcal{A}(z)} F_{\pi^{-1}} = \sum_{\pi \in \mathcal{A}(z)} F_\pi$.

4 Transition equations

Given elements $\pi, \sigma \in \tilde{S}_n$, we write $\pi \lessdot \sigma$ if $\ell(\sigma) = \ell(\pi) + 1$ and $\sigma = \pi t_{ij}$ for some $i < j \not\equiv i \pmod{n}$. The transitive closure of $\lessdot$, denoted $\leq$, is the *Bruhat order* on $\tilde{S}_n$. The relation $\pi \lessdot \pi t_{ij}$ is equivalent to the following more explicit condition:

Lemma 4.1 ([2, Proposition 8.3.6])**.** Fix $\pi \in \tilde{S}_n$ and integers $i < j \not\equiv i \pmod{n}$. One has $\pi \lessdot \pi t_{ij}$ if and only if $\pi(i) < \pi(j)$ and no $e \in \mathbb{Z}$ has $i < e < j$ and $\pi(i) < \pi(e) < \pi(j)$.

For $\pi \in \tilde{S}_n$ and $r \in \mathbb{Z}$ define the sets

$$\begin{aligned} \Psi_r^-(\pi) &= \{\sigma \in \tilde{S}_n : \pi \lessdot \sigma = \pi t_{ir} \text{ for some integer } i < r \text{ with } i \notin r + n\mathbb{Z}\}, \\ \Psi_r^+(\pi) &= \{\sigma \in \tilde{S}_n : \pi \lessdot \sigma = \pi t_{rj} \text{ for some integer } j > r \text{ with } j \notin r + n\mathbb{Z}\}. \end{aligned} \tag{4.1}$$

Lam and Shimozono [11, Theorem 7] proved the following *transition formula* for F_π:

Theorem 4.2 ([11])**.** If $\pi \in \tilde{S}_n$ and $r \in \mathbb{Z}$ then $\sum_{\sigma \in \Psi_r^-(\pi)} F_\sigma = \sum_{\sigma \in \Psi_r^+(\pi)} F_\sigma$.

Lam and Shimozono originally hoped to use this result to give a direct, algebraic proof of Theorem 2.12, but an argument along these lines remains to be found [11, §3.3]. The affine transition formula has found other applications, however; see, e.g., [14].

Example 4.3. Suppose $n = 4$ and $\pi = [1, 0, 2, 7] \in \tilde{S}_4$. Setting $r = 3$, we have

$$\begin{aligned} \Psi_3^-(\pi) &= \{[2,0,1,7], [1,2,0,7]\} = \{\pi t_{i,3} : i \in \{1,2\}\}, \\ \Psi_3^+(\pi) &= \{[1,0,7,2], [-2,0,5,7], [1,-2,4,7]\} = \{\pi t_{3,j} : j \in \{4,5,6\}\}, \end{aligned}$$

and $F_{[2,0,1,7]} + F_{[1,2,0,7]} = F_{[1,0,7,2]} + F_{[-2,0,5,7]} + F_{[1,-2,4,7]} = F_{2111} + F_{221} + F_{311} + F_{32}$.

Our goal in this section is to prove an analogue of Theorem 4.2 for the symmetric functions $\hat{F}_y$. Write $\lessdot_I$ for the covering relation of the Bruhat order $<$ on $\tilde{S}_n$ restricted to $\tilde{I}_n$, so that $y \lessdot_I z$ for $y, z \in \tilde{I}_n$ if and only if $\{\pi \in \tilde{I}_n : y \leq \pi < z\} = \{y\}$. For each pair of integers $i < j \not\equiv i \pmod{n}$, we introduce associated operators $\tau_{ij}^n : \tilde{I}_n \to \tilde{I}_n$ that will play the role of multiplication by a reflection in the poset $(\tilde{I}_n, <)$. Just as $\pi \lessdot \sigma$ only if $\sigma = \pi t_{ij}$ for some i, j, it will hold that $y \lessdot_I z$ only if $z = \tau_{ij}^n(y)$ for some i, j. To define τ_{ij}^n precisely, we need some auxiliary terminology.

Fix $y \in \tilde{I}_n$ and integers $i < j \not\equiv i \pmod{n}$. Define $\mathcal{G}_{ij}(y)$ to be the graph with vertex set $\{i, j, y(i), y(j)\}$ and edge set $\{\{i, y(i)\}, \{j, y(j)\}\} \setminus \{\{i\}, \{j\}\}$, in which the vertices i and j are colored white and all other vertices are colored black. Let $\sim$ be the equivalence relation on vertex-colored graphs with integer vertices in which $\mathcal{G} \sim \mathcal{H}$ if and only there exists a graph isomorphism $\mathcal{G} \to \mathcal{H}$ which is an order-preserving bijection on vertex sets. Finally, writing $m \in \{2, 3, 4\}$ for the size of $\{i, j, y(i), y(j)\}$, define $\mathcal{D}_{ij}(y)$ to be the unique vertex-colored graph on $\{1, 2, \ldots, m\}$ satisfying $\mathcal{D}_{ij}(y) \sim \mathcal{G}_{ij}(y)$.

There are twenty possibilities for $\mathcal{D}_{ij}(y)$, which we draw by arranging the vertices in order from left to right, using $\circ$ and $\bullet$ for the white and black vertices. For example, if $y, z \in \tilde{I}_n$ are involutions such that $y(i) < j = y(j) < i$ and $i < z(j) < j < z(i)$, then

$$\mathcal{D}_{ij}(y) = \bullet\ \circ\ \circ \qquad \text{and} \qquad \mathcal{D}_{ij}(z) = \circ\ \bullet\ \circ\ \bullet\,.$$

The following slightly rephrases [12, Definition 8.6]:

Definition 4.4. Fix $y \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod n$. Define

$$t_{ii} = t_{jj} = 1, \quad (\circ,\circ) = t_{ij}, \quad (\circ,\bullet) = t_{i,y(j)}, \quad \text{and} \quad (\bullet,\circ) = t_{y(i),j}.$$

Let $\overline{y} \in \tilde{I}_n$ be the affine permutation fixing each integer in the set $\{i, j, y(i), y(j)\} + n\mathbb{Z}$ and acting on all other integers as $k \mapsto y(k)$. Finally, define $\tau^n_{ij}(y) \in \tilde{I}_n$ by

$$\tau^n_{ij}(y) = \begin{cases} (\circ,\circ)\cdot y \cdot (\circ,\circ) & \text{if } \mathcal{D}_{ij}(y) \text{ is } \bullet\ \circ\ \ \circ\ \bullet \text{ or } \circ\ \circ\ \bullet\ \bullet \text{ or } \bullet\ \bullet\ \circ\ \circ \\ (\circ,\bullet)\cdot y \cdot (\circ,\bullet) & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \bullet\ \bullet\ \circ \text{ and } i \not\equiv y(j) \pmod n \\ (\circ,\circ)\cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \bullet\ \bullet\ \circ \text{ and } i \equiv y(j) \pmod n \\ (\circ,\circ)\cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \ \circ \text{ or } \circ\ \ \bullet\ \circ \text{ or } \circ\ \bullet\ \ \circ \text{ or } \circ\ \bullet\ \ \bullet\ \circ \\ (\circ,\bullet)\cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \ \circ\ \bullet \text{ or } \circ\ \bullet\ \ \circ\ \bullet \\ (\bullet,\circ)\cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \bullet\ \circ\ \ \circ \text{ or } \bullet\ \circ\ \ \bullet\ \circ \\ y & \text{otherwise.} \end{cases}$$

The operators τ^n_{ij} are affine analogues of the "covering transformations" studied in [7, 8]. They are related to the Bruhat order on affine involutions by the following theorem.

Theorem 4.5. If $y, z \in \tilde{I}_n$ then the following are equivalent: (a) $y \lessdot_I z$; (b) for each $\sigma \in \mathcal{A}(z)$, some $\pi \in \mathcal{A}(y)$ has $\pi \lessdot \sigma$; and (c) $\hat{\ell}(z) = \hat{\ell}(y) + 1$ and $z = \tau^n_{ij}(y)$ for some i, j.

One always has $y \leq \tau^n_{ij}(y)$ [12, Lemma 8.8], but determining whether $y \lessdot_I \tau^n_{ij}(y)$ can be complicated; see [12, Proposition 8.9]. The following is often useful for this purpose:

Lemma 4.6. Suppose $y \in \tilde{I}_n$ and $i < j \not\equiv i \pmod n$ are such that $y \neq \tau^n_{ij}(y)$. Assume $i \not\equiv y(j) \pmod n$ and either $y(i) \leq i$ or $j \leq y(j)$. Then $y \lessdot_I \tau^n_{ij}(y)$ if and only if $y \lessdot y t_{ij}$.

The proof of our transition formula for $\hat{F}_y$ relies on two technical theorems:

Theorem 4.7 (Covering property)**.** Suppose $y, z \in \tilde{I}_n$ and $\pi \in \mathcal{A}(y)$. Fix $i < j \not\equiv i \pmod n$ such that $\pi \lessdot \pi t_{ij}$. Then $\pi t_{ij} \in \mathcal{A}(z)$ if and only if $z = \tau^n_{ij}(y) \neq y$.

Theorem 4.8 (Toggling property)**.** Suppose $y \in \tilde{I}_n$ and $\pi \in \mathcal{A}(y)$. Fix $i < j \not\equiv i \pmod n$ such that $\pi \lessdot \pi t_{ij}$ and $y = \tau^n_{ij}(y)$. There are integers $k < l \not\equiv k \pmod n$ with $k \in \{j, y(j)\}$ and $l \in \{i, y(i)\}$ (which can be described explicitly) such that $\pi \neq \pi t_{ij} t_{kl} \in \mathcal{A}(y)$.

Fix $y \in \tilde{I}_n$ and define $\mathrm{Cyc}(y) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i \leq j = y(i)\}$. For $r \in \mathbb{Z}$, define

$$\begin{aligned} \Phi^-_r(y) &= \left\{ z \in \tilde{I}_n : y \lessdot_I z = \tau^n_{ir}(y) \text{ for some } i < r \text{ with } i \notin \{r, y(r)\} + n\mathbb{Z} \right\}, \\ \Phi^+_r(y) &= \left\{ z \in \tilde{I}_n : y \lessdot_I z = \tau^n_{rj}(y) \text{ for some } j > r \text{ with } j \notin \{r, y(r)\} + n\mathbb{Z} \right\}. \end{aligned} \tag{4.2}$$

The following theorem, which is the main result of this section, is both an involution analogue of Theorem 4.2 and an affine generalization of [5, Theorem 3.10].

Theorem 4.9. If $y \in \tilde{I}_n$ and $(p,q) \in \mathrm{Cyc}(y)$ then $\sum_{z\in\Phi_p^-(y)} \hat{F}_z = \sum_{z\in\Phi_q^+(y)} \hat{F}_z$.

Proof sketch. Lam and Shimozono's transition formula, Theorem 4.2, implies that we have $\sum_{\pi\in\mathcal{A}(y)} \sum_{\sigma\in\Psi_p^-(\pi)\sqcup\Psi_q^-(\pi)} F_\sigma = \sum_{\pi\in\mathcal{A}(y)} \sum_{\sigma\in\Psi_p^+(\pi)\sqcup\Psi_q^+(\pi)} F_\sigma$. It follows by an argument using the covering property, Theorem 4.7, that this identity can be rewritten as $\sum_{z\in\Phi_p^-(y)} \hat{F}_z + (\text{extra terms}) = \sum_{z\in\Phi_q^+(y)} \hat{F}_z + (\text{extra terms})$, where each extra term on the left (respectively, right) has the form $F_{\pi t_{ij}}$ for some $\pi \in \mathcal{A}(y)$ and $i < j \not\equiv i \pmod n$ with $\pi \lessdot \pi t_{ij}$, $y = \tau_{ij}^n(y)$, and $j \in \{p,q\}$ (respectively, $i \in \{p,q\}$). Using the toggling property, Theorem 4.8, it can be shown that exactly the same extra terms appear on both sides. $\square$

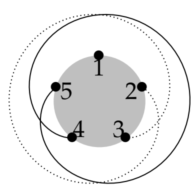
We conclude with two examples.

Example 4.10. Suppose $n = 4$ and $y = t_{3,8} = [1,2,8,-1] \in \tilde{I}_4$. Setting $p = q = 2$, we have

$$\Phi_2^-(y) = \{t_{1,2}t_{3,8}\} = \left\{\tau_{1,2}^4(y)\right\} \quad \text{and} \quad \Phi_2^+(y) = \{t_{2,8},\ t_{2,5}t_{3,8}\} = \left\{\tau_{2,3}^4(y), \tau_{2,5}^4(y)\right\},$$

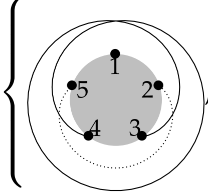
so $\hat{F}_{[2,1,8,-1]} = \hat{F}_{[1,8,3,-2]} + \hat{F}_{[-2,5,8,-1]} = F_{1^5} + F_{21^3} + F_{2^21} + F_{31^2} + F_{32}$.

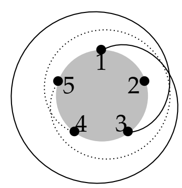
Example 4.11. Suppose $n = 5$ and

$y =$ 

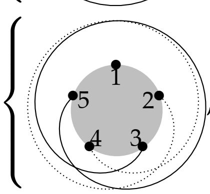
$= t_{2,8}t_{4,10} = [1,8,-3,10,-1] \in \tilde{I}_5.$

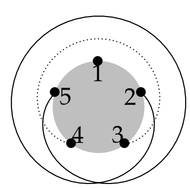
Setting $(p,q) = (2,8)$, we have

$\Phi_2^-(y) = \Bigg\{$ 

, 

$\Bigg\} = \{t_{2,5}t_{4,13},\ t_{1,8}t_{4,10}\} = \left\{\tau_{-1,2}^5(y), \tau_{1,2}^5(y)\right\},$

$\Phi_8^+(y) = \Bigg\{$ 

, 

$\Bigg\} = \{t_{2,9}t_{3,10},\ t_{2,10}t_{4,8}\} = \left\{\tau_{8,9}^5(y), \tau_{8,10}^5(y)\right\},$

so $\hat{F}_{[1,5,-6,13,2]} + \hat{F}_{[8,2,-4,10,-1]} = \hat{F}_{[1,9,10,-3,-2]} + \hat{F}_{[1,10,-1,8,-3]} = F_{21^7} + F_{2^21^5} + F_{2^31^3} + 2F_{2^41} + F_{31^6} + F_{321^4} + 3F_{32^21^2} + F_{32^3} + F_{3^21^3} + 2F_{3^221} + F_{3^3} + F_{421^3} + F_{42^21} + F_{431^2} + F_{432}$.

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