Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #69, 11 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Tropical ideals do not realise all Bergman fans

Jan Draisma*[1] and Felipe Rincón†[2]

[1]*Mathematisches Institut, Universität Bern, Switzerland.*
[2]*School of Mathematical Sciences, Queen Mary University of London, United Kingdom.*

Abstract. Tropical ideals are combinatorial objects that abstract the possible collections of subsets arising as the supports of all polynomials in an ideal. Every tropical ideal has an associated tropical variety: a finite polyhedral complex equipped with positive integral weights on its maximal cells. This leads to the realisability question, ubiquitous in tropical geometry, of which weighted polyhedral complexes arise in this manner. Using work of Las Vergnas on the non-existence of tensor products of matroids, we prove that there is no tropical ideal whose variety is the Bergman fan of the direct sum of the Vámos matroid and the uniform matroid of rank two on three elements, and in which all maximal cones have weight one.

Résumé. Les idéaux tropicaux sont des objets combinatoires qui font abstraction des collections possibles de sous-ensembles constituant les supports de tous les polynômes d'un idéal. Chaque idéal tropical a une variété tropicale associée: un complexe polyédrique fini équipé de pondérations intégrales positives sur ses cellules maximales. Ceci conduit à la question de la réalisabilité, omniprésente en géométrie tropicale, de laquelle des complexes polyhédriques pondérés se présentent de cette manière. En utilisant les travaux de Las Vergnas sur la non-existence de produits tensoriels de matroïdes, nous prouvons qu'il n'existe pas d'idéal tropical dont la variété est l'éventail de Bergman de la somme directe du matroïde de Vámos et du matroïde uniforme de rang deux sur trois éléments, et dans lesquels tous les cônes maximaux ont un poids un.

Keywords: tropical geometry, tropical variety, tropical ideal, realisability, matroid.

1 Introduction

An ideal in a polynomial ring over a field with the trivial valuation gives rise to a polyhedral fan called its tropical variety, by taking all weight vectors whose initial ideals do not contain a monomial. In the middle of this construction sits a *tropical ideal*, obtained by recording the supports of all polynomials in the ideal. This tropical ideal is a

*jan.draisma@math.unibe.ch. JD was partially supported by the Vici grant Stabilisation in Algebra and Geometry from the Netherlands Organisation for Scientific Research (NWO).

†f.rincon@qmul.ac.uk. FR was partially supported by the Research Council of Norway grant 239968/F20.

purely combinatorial object, and it contains more information than the tropical variety itself. For these reasons, tropical ideals, axiomatised in [6], were proposed as the correct combinatorial/algebraic structures on which to build a theory of tropical schemes.

Concretely, if K is an infinite field, a classical ideal $J \subseteq K[x_1, \ldots, x_n]$ gives rise to the tropical ideal

$$I = \mathrm{trop}(J) := \{\mathrm{supp}(F) : F \in J\} \subseteq 2^{\mathbb{N}^n},$$

where the support $\mathrm{supp}(F)$ of a polynomial $F = \sum_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1, \ldots, x_n]$ is given by $\mathrm{supp}(F) := \{\mathbf{u} \in \mathbb{N}^n : c_{\mathbf{u}} \neq 0\}$. A tropical ideal arising in this way is called *realisable* (over the field K).

In general, a (possibly non-realisable) *tropical ideal* in the variables $x_1, \ldots, x_n$ is a non-empty collection I of finite subsets of $\mathbb{N}^n$ satisfying the following conditions:

- If $S, T \in I$ then $S \cup T \in I$.
- If $S \in I$ then $S + \mathbf{e}_i := \{\mathbf{v} + \mathbf{e}_i : \mathbf{v} \in S\} \in I$ for any $1 \leq i \leq n$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denotes the standard basis of $\mathbb{Z}^n$.
- *Monomial elimination axiom:* If $S, T \in I$ and $\mathbf{u} \in S \cap T$ then there is $U \in I$ such that $S \Delta T \subseteq U \subseteq (S \cup T) \setminus \{\mathbf{u}\}$, where Δ denotes symmetric difference.

Tropical ideals are in this way combinatorial abstractions of the possible collections of subsets of $\mathbb{N}^n$ that arise as the supports of all polynomials in a fixed ideal over a field. As described below, the monomial elimination axiom is basically an instance of the cycle elimination axiom for matroids. While 'most' tropical ideals are expected to be non-realisable, there are essentially very few examples known so far of non-realisable tropical ideals—one of them, for instance, can be found in [6, Example 2.8].

Generalising the notion of tropical variety for a classical ideal, the *variety* of an arbitrary tropical ideal I is defined to be

$$V(I) := \{\mathbf{x} \in \mathbb{R}^n : \forall S \in I, \text{ the minimum } \min_{\mathbf{u} \in S}(\mathbf{x} \cdot \mathbf{u}) \text{ is attained by at least two terms}\}.$$

Tropical ideals turn out to have very nice algebraic and geometric properties. It was proved in [6] that tropical ideals, while not finitely generated as ideals–nor in any sense that we know of!—have a rational Hilbert series, satisfy the ascending chain condition and the weak Nullstellensatz, and have varieties that are finite weighted polyhedral fans. This leads to the following realisability question.

Question 1.1. Which weighted polyhedral fans are the variety of some tropical ideal?

When the tropical ideal records the supports of the polynomials in a classical prime ideal J, then the tropical variety is a pure-dimensional and balanced polyhedral fan [7, Theorem 3.3.5]. Conversely, the question of which balanced polyhedral complexes are realised by classical ideals has received much attention, especially in the case of curves (see, e.g., [12, 3, 2]). But for general tropical ideals, very little is known about Question 1.1: no natural algebraic criterion that ensures that the variety is pure-dimensional is known, nor has their top-dimensional part been proved to be balanced. In fact, until recently we had no intuition as to whether tropical ideals are flexible enough that they can realise basically any balanced polyhedral fan, or rather more rigid, like algebraic varieties. In view of the following theorem, we now lean towards the latter intuition.

Theorem 4.2. There exists no tropical ideal whose tropical variety is the Bergman fan of the direct sum of the Vámos matroid V_8 and the uniform matroid $U_{2,3}$ of rank two on three elements, with all maximal cones having weight 1.

We believe that this theorem marks the beginning of an interesting research programme, which, in addition to the pureness and balancing questions mentioned above, asks which tropical ideals define algebraic matroids on the set of variables, and which matroids are, in this sense, tropically algebraic—see Problem 3.5 and Question 3.6.

2 Definitions and basic results

In this section we review the basic definitions of tropical linear spaces and tropical ideals. We will work in the general context of the full tropical semifield (corresponding to fields with a general non-Archimedean valuation), but all these notions can be specialised to the case of the Boolean semifield, as it was presented in the introduction.

Consider the tropical semifield $(\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with $\oplus :=$ min and $\odot := +$. Let R be a sub-semifield of $\overline{\mathbb{R}}$. The example most relevant to us is the Boolean semifield $\mathbb{B} := \{0, \infty\}$, which is not only a sub-semifield but also a quotient of $\overline{\mathbb{R}}$.

Definition 2.1. Let N be a finite set. A set $L \subseteq R^N$ is a **tropical linear space** if it is an R-submodule (i.e., $(\infty, \ldots, \infty) \in L$ and $f, g \in L, c \in R \Rightarrow (c \odot f) \oplus g \in L$) and if, moreover, L satisfies the following elimination axiom: for $i \in N$ and $f, g \in L$ with $f_i = g_i \neq \infty$, there exists an $h \in L$ with $h_i = \infty$ and $h_j \geq f_j \oplus g_j$ for all $j \in N$, with equality whenever $f_j \neq g_j$. The $\overline{\mathbb{R}}$-submodule $L_{\overline{\mathbb{R}}}$ of $\overline{\mathbb{R}}^N$ generated by L is a tropical linear space in $\overline{\mathbb{R}}^N$, and has the structure of a finite polyhedral complex; we denote its dimension as such by $\dim L$.

If K is a field equipped with a non-Archimedean valuation onto R and if $V \subseteq K^N$ is a linear subspace, then the image of V under the coordinate-wise valuation is a tropical linear space in R^N, but not all tropical linear spaces arise in this manner. Tropical linear spaces are well-studied objects in tropical geometry and matroid theory: the definition above is equivalent to that of [11], except that we allow some coordinates to be ∞. A tropical linear space L gives rise to a matroid $M(L)$ in which the independent sets are those subsets $A \subseteq N$ for which $L \cap (R^A \times \{\infty\}^{N \setminus A}) = \{\infty\}^N$, and L is the set of **vectors**

(R-linear combinations of valuated circuits) of a **valuated matroid** on $M(L)$ [9]. With this setup, $\dim L = |N| - \mathrm{rk}(M(L))$. We will freely alternate between these different characterisations of tropical linear spaces.

Set $\mathbb{N} := \{0, 1, 2, \ldots\}$, and let $n \in \mathbb{N}$. Denote by $R[x_1, \ldots, x_n]$ the semiring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in R. We write $\mathrm{Mon_d}$ and $\mathrm{Mon_{\leq d}}$ for the set of monomials in $x_1, \ldots, x_n$ of degree equal to d and at most d, respectively, and we identify a polynomial in $R[x_1, \ldots, x_n]$ of degree at most d with its coefficient vector in $R^{\mathrm{Mon_{\leq d}}}$.

Definition 2.2. A subset $I \subseteq R[x_1, \ldots, x_n]$ is a **tropical ideal** if $x_i \odot I \subseteq I$ for all $i = 1, \ldots, n$ and if for each $d \in \mathbb{N}$ the set $I_{\leq d} := \{f \in I : \deg(f) \leq d\}$ is a tropical linear space in $R^{\mathrm{Mon_{\leq d}}}$. If I is homogeneous, then the latter condition is equivalent to the condition that for each d the set I_d of homogeneous polynomials in I of degree d is a tropical linear space in $R^{\mathrm{Mon_d}}$.

By Definition 2.1, I is a tropical ideal if and only if I is an ideal of $R[x_1, \ldots, x_n]$ satisfying the following monomial elimination axiom:

- For any $f, g \in I_{\leq d}$ and any monomial $\mathbf{x}^{\mathbf{u}}$ for which $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}} \neq \infty$, there exists $h \in I_{\leq d}$ such that $[h]_{\mathbf{x}^{\mathbf{u}}} = \infty$ and $[h]_{\mathbf{x}^{\mathbf{v}}} \geq \min([f]_{\mathbf{x}^{\mathbf{v}}}, [g]_{\mathbf{x}^{\mathbf{v}}})$ for all monomials $\mathbf{x}^{\mathbf{v}}$, with the equality holding whenever $[f]_{\mathbf{x}^{\mathbf{v}}} \neq [g]_{\mathbf{x}^{\mathbf{v}}}$.

Here we use the notation $[f]_{\mathbf{x}^{\mathbf{u}}}$ to denote the coefficient of the monomial $\mathbf{x}^{\mathbf{u}}$ in the tropical polynomial f.

There is a natural notion of tropical ideals living in the Laurent polynomial semiring $R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that we will also use, and if I is a tropical ideal in $R[x_1, \ldots, x_n]$ then the set $I' := \{f/\mathbf{x}^{\mathbf{u}} \mid f \in I, \mathbf{u} \in \mathbb{N}^n\}$ is a tropical ideal in $R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Tropical ideals were introduced by Maclagan and Rincón in [6] as a framework for developing algebraic foundations for tropical geometry. Tropical ideals are much better behaved than general ideals of the polynomial semiring $R[x_1, \ldots, x_n]$, as we explain below.

Definition 2.3. For $\mathbf{w} \in \mathbb{R}^n$ and $f = \bigoplus_{\mathbf{u}} c_{\mathbf{u}} \odot \mathbf{x}^{\mathbf{u}} \in R[x_1, \ldots, x_n]$, define the **initial part** of f relative to $\mathbf{w}$ as

$$\mathrm{in}_{\mathbf{w}}(f) := \bigoplus_{\mathbf{u} \,:\, c_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w} = f(\mathbf{w})} \mathbf{x}^{\mathbf{u}} \quad \in \mathbb{B}[x_1, \ldots, x_n].$$

For a tropical ideal I define the its **initial ideal** relative to $\mathbf{w}$ as

$$\mathrm{in}_{\mathbf{w}} I := \langle \mathrm{in}_{\mathbf{w}} f \mid f \in I \rangle_{\mathbb{B}}.$$

Note that in this paper we only consider weights $\mathbf{w}$ in $\mathbb{R}^n$, not in $\overline{\mathbb{R}}^n$ as in [6]. In other words, we do geometry only inside the tropical torus.

Definition 2.4. The **Hilbert function** of a tropical ideal $I \subseteq R[x_1, \ldots, x_n]$ is the map $H_I : \mathbb{N} \to \mathbb{N}$ given by $d \mapsto \binom{n+d}{d} - \dim I_{\leq d}$.

Note that, as usual in commutative algebra, the Hilbert function measures the codimension of $I_{\leq d}$ in its ambient space $R^{\mathrm{Mon}_{\leq \mathrm{d}}}$. A homogeneous variant of this Hilbert function applies only to homogeneous ideals and measures the codimension of I_d in $R^{\mathrm{Mon}_\mathrm{d}}$. The Hilbert function of a not necessarily homogeneous ideal I in $R[x_1, \ldots, x_n]$ equals the homogeneous Hilbert function of its homogenisation in $R[x_0, \ldots, x_n]$.

The following is a special case of [6, Corollary 3.6].

Theorem 2.5. *For a homogeneous tropical ideal $I \subseteq R[x_1, \ldots, x_n]$ and any $\mathbf{w} \in \mathbb{R}^n$, $\mathrm{in}_\mathbf{w} I \subseteq \mathbb{B}[x_1, \ldots, x_n]$ is a homogeneous tropical ideal, and $H_{\mathrm{in}_\mathbf{w} I} = H_I$.*

Theorem 2.5 allows one to pass to monomial initial ideals and show that the Hilbert function $H_I(d)$ of a homogeneous tropical ideal I becomes a polynomial in d for sufficiently large d, and also that homogeneous tropical ideals satisfy the ascending chain condition. Via homogenisation, one sees that both statements also hold for tropical ideals that are not homogeneous (but, as in the classical setting, the theorem does not apply directly, since for instance, when $n = 1$, $\mathrm{in}_{(1)}(0 \oplus x_1) = 0$ generates an ideal—the entire semiring—with a smaller Hilbert function than any tropical ideal containing $0 \oplus x_1$ but not 0).

Furthermore, Maclagan and Rincón prove that tropical ideals have tropical varieties that are finite polyhedral complexes [6, Theorem 5.11].

Theorem 2.6. *If $I \subseteq R[x_1, \ldots, x_n]$ is a tropical ideal then its (tropical)* **variety**

$$V(I) := \{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_\mathbf{w} I \text{ contains no monomial}\}$$

is the support of a finite polyhedral complex.

Indeed, if I is homogeneous, they show that the sets of $\mathbf{w}$ where $\mathrm{in}_\mathbf{w} I$ is constant form the relatively open polyhedra of a polyhedral complex with support $\mathbb{R}^n$ called the **Gröbner complex** of I, and that the cells where $\mathrm{in}_\mathbf{w} I$ contains no monomial form a subcomplex with support $V(I)$. By homogeneity, all cells then contain in their lineality space the linear span of the all-ones vector $\mathbf{1}$. In the case where $I \subseteq R[x_1, \ldots, x_n]$ is not necessarily homogeneous, let I^h be its homogenisation in $R[x_0, x_1, \ldots, x_n]$. Then $\mathbf{w} \mapsto (0, \mathbf{w})$ is a bijection between $V(I)$ and the intersection of $V(I^h)$ with the zeroth coordinate hyperplane, and we give $V(I)$ the corresponding polyhedral complex structure.

The variety of a tropical ideal comes equipped with positive integral weights on its maximal polyhedra; this is inspired by [7, Lemma 3.4.7] and studied more in depth in [5].

Definition 2.7. Let $I \subseteq R[x_1,\dots,x_n]$ be a tropical ideal, let σ be a maximal polyhedron of $V(I)$, and let $\mathbf{w}$ be in the relative interior of σ. The **multiplicity** of σ in $V(I)$ is defined as follows. First, let $I' \subseteq R[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be the (tropical) ideal in the Laurent polynomial ring generated by I. After an automorphism of the Laurent polynomial ring given by $\mathbf{x}^{\mathbf{u}} \mapsto \mathbf{x}^{A\mathbf{u}}$ with $A \in \mathrm{GL}_n(\mathbb{Z})$, we can assume that the affine span of σ is a translate of $\mathrm{span}(\mathbf{e}_1,\dots,\mathbf{e}_d)$ for some d. In this case, by [5], the tropical ideal $J := \mathrm{in}_{\mathbf{w}}(I') \cap \mathbb{B}[x_{d+1},\dots,x_n]$ is zero-dimensional, i.e., $H_J(e)$ is a constant for $e \gg 0$. The multiplicity of σ is defined to be equal to this constant, called the degree of J.

Remark 2.8. A slightly more coordinate-free version of Definition 2.7 is the following. Consider the linear span of σ, defined as

$$\mathrm{span}(\sigma) := \mathbb{R}_{\geq 0}\{\mathbf{v} - \mathbf{v}' \mid \mathbf{v}, \mathbf{v}' \in \sigma\}.$$

Let $S \subseteq \mathbb{B}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be the sub-semiring spanned by monomials $\mathbf{x}^{\mathbf{u}}$ of $\mathbf{w}$-weight $\mathbf{w}\cdot\mathbf{u}$ equal to zero for all $\mathbf{w} \in \mathrm{span}(\sigma)$. Then S itself is isomorphic to a Laurent polynomial semiring in $n-d$ variables. The multiplicity of σ is the degree of the zero-dimensional ideal $\mathrm{in}_{\mathbf{w}}(I') \cap S$.

We will need the following results.

Lemma 2.9. *Let I be a tropical ideal in $R[x_1,\dots,x_n]$. Denote by I' the ideal generated by I in $R[x_1^{\pm 1},\dots,x_n^{\pm n}]$, and set $I^{\mathrm{sat}} := I' \cap R[x_1,\dots,x_n]$. Then $I^{\mathrm{sat}} \supseteq I$ is a tropical ideal, and $V(I^{\mathrm{sat}}) = V(I)$ as weighted polyhedral complexes.*

We call I^{sat} the **saturation** of I with respect to $m := x_1 \cdots x_n$, and we call I **saturated** with respect to m if $I^{\mathrm{sat}} = I$.

Proof. That I^{sat} is a tropical ideal containing I is straightforward from the definition. Since $I^{\mathrm{sat}} \supseteq I$ we have $V(I^{\mathrm{sat}}) \subseteq V(I)$. Conversely, let $\mathbf{w} \in V(I)$ and $f \in I^{\mathrm{sat}}$. Then $\mathbf{x}^{\mathbf{u}} \odot f \in I$ for some $\mathbf{u} \in \mathbb{N}^n$, hence $\mathrm{in}_{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} \odot f)$ is not a monomial, and therefore neither is $\mathrm{in}_{\mathbf{w}} f$. This shows that $V(I) = V(I^{\mathrm{sat}})$. That the multiplicities are the same follows from the fact that the multiplicities in $V(I)$ are defined using I'. □

If Σ is a polyhedral complex in $\mathbb{R}^n$ and σ is a polyhedron in Σ, the **star** $\mathrm{star}_\sigma \Sigma$ of Σ at σ is a weighted polyhedral fan, whose cones are indexed by the cones τ of Σ containing σ. The cone indexed by such τ is

$$\overline{\tau} := \mathbb{R}_{\geq 0}\{\mathbf{v} - \mathbf{w} \mid \mathbf{v} \in \tau \text{ and } \mathbf{w} \in \sigma\},$$

with weight equal to the weight of τ in Σ.

The following is a result from [5].

Proposition 2.10. *Let I be a tropical ideal in $R[x_1,\dots,x_n]$, σ be a polyhedron in $V(I)$, and $\mathbf{w}$ be in the relative interior of σ. Then $\mathrm{in}_{\mathbf{w}} I \subseteq \mathbb{B}[x_1,\dots,x_n]$ is homogeneous with respect to every vector $\mathbf{v} \in \mathrm{span}(\sigma)$, and $V(\mathrm{in}_{\mathbf{w}} I) = \mathrm{star}_{\mathbf{w}} V(I)$ as weighted polyhedral complexes.*

3 The independence complex of a tropical ideal

Definition 3.1. Let $I \subseteq R[x_1, \ldots, x_n]$ be a tropical ideal. The **independence complex** of I is the simplicial complex

$$\mathcal{I}(I) := \{A \subseteq \{1, \ldots, n\} : I \cap R[x_i : i \in A] = \{\infty\}\}. \tag{3.1}$$

When $\mathcal{I}(I)$ is the collection of independent sets of a matroid M, we will say that I is a **matroidal tropical ideal**, and that M is its associated **algebraic matroid**.

The independence complex of a tropical ideal I can be recovered from its variety $V(I)$, at least if $R = \overline{\mathbb{R}}$.

Proposition 3.2. *If $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is a tropical ideal then*

$$\mathcal{I}(I) = \{A \subseteq \{1, \ldots, n\} : \pi_A(V(I)) = \mathbb{R}^A\}, \tag{3.2}$$

where $\pi_A \colon \mathbb{R}^n \to \mathbb{R}^A$ is the coordinate projection onto the coordinates indexed by A. In particular, the independence complex $\mathcal{I}(I)$ depends only on the variety $V(I)$.

Proof. Let $A \subseteq \{1, \ldots, n\}$. If $A \notin \mathcal{I}(I)$ then there exists $f \in I \cap \overline{\mathbb{R}}[x_i : i \in A]$ such that $f \neq \infty$, and $V(I) \subseteq V(f)$. We then have $\pi_A(V(I)) \subseteq \pi_A(V(f)) \subsetneq \mathbb{R}^A$, as claimed. For the reverse inclusion, suppose that $\pi_A(V(I)) \subsetneq \mathbb{R}^A$, and let $\mathbf{w} \in \mathbb{R}^A \setminus \pi_A(V(I))$. For any polynomial $f \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$, denote by $f|_{\mathbf{w}}$ the polynomial in $\overline{\mathbb{R}}[x_i : i \notin A]$ obtained by specializing each variable x_i with $i \in A$ to $w_i \in \mathbb{R}$. Consider the ideal $I|_{\mathbf{w}} \subseteq \overline{\mathbb{R}}[x_i : i \notin A]$ defined as $I|_{\mathbf{w}} := \{f|_{\mathbf{w}} : f \in I\}$. By [5], the ideal $I|_{\mathbf{w}}$ is a tropical ideal. Moreover, we must have $V(I|_{\mathbf{w}}) = \emptyset$, as any point $\mathbf{v} \in V(I|_{\mathbf{w}})$ would lift to the point $(\mathbf{v}, \mathbf{w}) \in V(I)$, contradicting that $\mathbf{w} \notin \pi_A(V(I))$. By the weak Nullstellensatz [6, Corollary 5.17], the tropical ideal $I|_{\mathbf{w}}$ must contain the constant polynomial 0. But then $0 = f|_{\mathbf{w}}$ for some $f \in I$, which in particular implies that $f \in I \cap \overline{\mathbb{R}}[x_i : i \in A]$ and $f \neq \infty$. □

Recall that the Hilbert function $H_I(e)$ of a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ eventually agrees with a polynomial in e, called the **Hilbert polynomial** of I [6, Proposition 3.8]. The **dimension** $\dim(I)$ of I is defined as the degree of its Hilbert polynomial.

Corollary 3.3. *For any tropical ideal I we have*

$$\dim \mathcal{I}(I) + 1 = \dim V(I) = \dim I.$$

Proof. From (3.2) it is clear that $\dim V(I) \geq \dim \mathcal{I}(I) + 1$. Now, if $V(I)$ contains a polyhedron σ of dimension d then there is some coordinate projection $\pi_A(\sigma)$ that is d-dimensional, and thus from (3.1) we see that $A \in \mathcal{I}(I)$ and thus $\dim \mathcal{I}(I) + 1 \geq d$. This shows that $\dim \mathcal{I}(I) + 1 = \dim V(I)$. The equality $\dim V(I) = \dim I$ is proved in [5]. □

In the classical setting, primality of an ideal implies matroidality. We have no idea about a similarly appealing sufficient condition for matroidality of general tropical ideals.

Example 3.4. If $J \subseteq K[x_1, \dots, x_n]$ is a prime ideal, where K is a field with a non-Archimedean valuation, then $\text{trop}(J)$ is a matroidal tropical ideal. Its associated algebraic matroid is the matroid that captures algebraic independence among the coordinate functions $x_1, \dots, x_n$ in the field of fractions of $K[x_1, \dots, x_n]/J$. $\diamondsuit$

Problem 3.5. Find algebraic conditions on a tropical ideal that imply matroidality.

As shown in Example 3.4, any (classically) algebraic matroid is the algebraic matroid of a tropical ideal. However, in principle, it is possible that the class of matroids that are "tropically algebraic" is strictly larger than the usual class of algebraic matroids.

Question 3.6. Which matroids arise as the algebraic matroid of a tropical ideal?

4 Not every Bergman fan is the variety of a tropical ideal

We now prove that not every balanced polyhedral complex can be obtained as the variety of a tropical ideal. Our counterexample will in fact be the Bergman fan of a matroid; see [1] for details.

Definition 4.1. Let M be a loopless matroid of rank d on the ground set $\{1, \dots, n\}$. The **Bergman fan** $\mathcal{B}(M)$ of M is the pure d-dimensional polyhedral fan in $\mathbb{R}^n$ consisting of the cones of the form

$$\sigma_{\mathcal{F}} := \text{cone}(\mathbf{e}_{F_1}, \mathbf{e}_{F_2}, \dots, \mathbf{e}_{F_k}) + \mathbb{R} \cdot \mathbf{e}_{\{1,\dots,n\}}$$

where $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq \{1, \dots, n\}\}$ is a chain of flats in the lattice of flats $\mathcal{L}(M)$ of M, and where $\mathbf{e}_S$ stands for the sum of the standard basis vectors $\mathbf{e}_i$ with i running through S. The Bergman fan of any matroid is given the structure of a balanced polyhedral complex by defining the multiplicity of each maximal cone to be equal to 1.

Bergman fans of matroids are the tropical linear spaces (more specifically, their part inside the torus $\mathbb{R}^n$) that correspond to valuated matroids where the basis valuations all take values in $\mathbb{B}$.

Let $U_{2,3}$ be the uniform matroid of rank 2 on the ground set $\{1, 2, 3\}$, and let V_8 be the Vámos matroid (of rank 4 on 8 elements). The following is our main result.

Theorem 4.2. *There is no tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_3, y_1, \dots, y_8]$ such that $V(I)$ is equal to $\mathcal{B}(U_{2,3} \oplus V_8)$ as weighted polyhedral complexes, even up to common refinement.*

Note that we do not require the polyhedral structure on $V(I)$ coming from the Gröbner complex of the homogenisation of I to be equal to the fan structure on the Bergman fan described above.

To prove the theorem, in addition to the fundamental results from Section 2, we will need results relating $V(I)$ to H_I for any tropical ideal I.

Lemma 4.3. *Let $L, L' \subseteq R^N$ be tropical linear spaces. If $\dim L + \dim L' > |N|$, then $L \cap L' \neq \{(\infty, \dots, \infty)\}$.*

Proof. The notion of stable intersection for tropical linear spaces was studied by Speyer in [11] when the underlying matroids of both tropical linear spaces were uniform matroids, and later generalized by Mundinger [8] for arbitrary tropical linear spaces in R^N. The stable intersection $L \cap_{\mathrm{st}} L'$ is a tropical linear space contained in both L and L', and it has dimension a least $\dim L + \dim L' - |N| > 0$, which implies the desired result. □

Proposition 4.4. *Let $I \subseteq R[x_1, \dots, x_n]$ be a tropical ideal. If the independence complex $\mathcal{I}(I)$ contains a subset A of size r, then $H_I(d) \geq \binom{r+d}{d}$ for all $d \in \mathbb{N}$.*

Proof. The space $R[x_i : i \in A]_{\leq d}$ is a tropical linear space in $R^{\mathrm{Mon}_{\leq d}}$ of dimension $\binom{r+d}{d}$ and, by assumption, it does not intersect $I_{\leq d}$. Hence by Lemma 4.3, $\dim I_{\leq d} \leq \binom{n+d}{d} - \binom{r+d}{d}$, and therefore $H_I(d) \geq \binom{r+d}{d}$. □

Proposition 4.5. *Let $I \subsetneq R[x_1, \dots, x_n]$ be a tropical ideal, and set $r := H_I(1) - 1$. Then $H_I(d) \leq \binom{r+d}{d}$ for all $d \in \mathbb{N}$.*

Proof. Let $I^h \subseteq R[x_0, \dots, x_n]$ be the homogenisation of I. Then $\dim(I^h)_d = \dim I_{\leq d}$ for all $d \in \mathbb{N}$, and in particular $\dim(I^h)_1 = \dim I_{\leq 1} = n + 1 - H_I(1) = n - r$. Moreover, by applying Theorem 2.5 with a sufficiently general weight vector $\mathbf{w}$, the Hilbert function of I^h is also that of some monomial ideal J. We find that J contains precisely $n - r$ of the $n + 1$ variables $x_0, \dots, x_n$, and therefore all their multiples. This implies that $\dim J_d \geq \binom{n+d}{d} - \binom{r+d}{d}$, where the last term counts monomials in the remaining $r + 1$ variables of degree d. We then have

$$H_I(d) = \binom{n+d}{d} - \dim I_{\leq d} = \binom{n+d}{d} - \dim J_d \leq \binom{n+d}{d} - \binom{n+d}{d} + \binom{r+d}{d},$$

as desired. □

The following proposition shows that the algebraic matroid of a Bergman fan $\mathcal{B}(M)$ (as in Proposition 3.2) is equal to the matroid M.

Proposition 4.6 ([13, Lemma 3])**.** *The independence complex of the Bergman fan $\mathcal{B}(M)$ of a loopless matroid M is the same as the independence complex of M.*

We now present a key step towards proving our main result.

Proposition 4.7. *Let M be a loopless matroid on the ground set $\{1,\dots,n\}$. Suppose $J \subseteq \mathbb{B}[x_1,\dots,x_n]$ is a homogeneous tropical ideal, saturated with respect to $x_1\cdots x_n$, whose variety $V(J)$ has a common refinement, as weighted polyhedral complexes, with the Bergman fan $\mathcal{B}(M)$ (with weight 1 in all its maximal cones). Then the matroid $M(J_1)$ is equal to M, under the identification $x_i \leftrightarrow i$ of ground sets.*

Proof. Let $B = \{b_1,\dots,b_d\}$ be a basis of M. For $0 \le i \le d$, consider the flat F_i of M obtained as the closure of the set $\{b_1,\dots,b_i\}$, and let σ be the maximal cone of $\mathcal{B}(M)$ corresponding to the chain of flats $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d-1} \subsetneq F_d = \{1,\dots,n\}$. Let $\tau \subseteq \sigma$ be a maximal cone in a common refinement of both $V(J)$ and $\mathcal{B}(M)$. The linear span $\operatorname{span}(\tau) = \operatorname{span}(\sigma)$ consists of all vectors $\mathbf{w} \in \mathbb{R}^n$ for which $w_i = w_j$ whenever $\{i,j\} \subseteq F_k \setminus F_{k-1}$ for some $k = 1,\dots,d$. A monomial $\mathbf{x}^{\mathbf{u}}$ in $\mathbb{B}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ has $\mathbf{w}$-weight equal to zero for all such $\mathbf{w}$ if and only if for every k we have $\sum_{i\in F_k\setminus F_{k-1}} u_i = 0$. As in Remark 2.8, let S be the subsemiring of $\mathbb{B}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ consisting of all polynomials involving only such monomials, and let J' be the (tropical) ideal in $\mathbb{B}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ generated by J.

Take $\mathbf{v}$ to be a vector in the relative interior of τ. Since τ has multiplicity 1 in $V(J)$, $\operatorname{in}_{\mathbf{v}}(J') \cap S$ is zero-dimensional of degree 1, and contains no monomials. Hence for any pair of distinct monomials $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{u}'}$ in S, $\operatorname{in}_{\mathbf{v}}(J') \cap S$ contains the binomial $\mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{u}'}$. In particular, if $\{i \neq j\} \subseteq F_k \setminus F_{k-1}$ for some k then $0 \oplus x_i^{-1}x_j \in \operatorname{in}_{\mathbf{v}}(J') \cap S$, and thus $x_i \oplus x_j \in \operatorname{in}_{\mathbf{v}}(J')$. As J is homogeneous and saturated with respect to $x_1\cdots x_n$, this implies that there is a polynomial of the form $x_i \oplus x_j \oplus f$ in J_1 where f is a sum of variables all contained in F_{k-1}. It follows that x_i is in the closure of $F_{k-1} \cup \{x_j\}$ in the matroid $M(J_1)$. We conclude that $\{b_1,\dots,b_d\}$ is a generating set in the matroid $M(J_1)$, and thus $\operatorname{rank}(M(J_1)) \le \operatorname{rank}(M)$. Now, the tropical prevariety cut out by the linear polynomials in J is equal to $\mathcal{B}(M(J_1))$, so we have $\mathcal{B}(M(J_1)) \supseteq V(J) = \mathcal{B}(M)$. It follows from [10, Lemma 7.4] that $\mathcal{B}(M(J_1)) = \mathcal{B}(M)$, and thus $M(J_1) = M$, completing the proof. □

We conclude with the proof of the main theorem.

Proof of Theorem 4.2. Suppose that such an I exists, and denote $M := U_{2,3} \oplus V_8$. Let σ be a polyhedron in $V(I)$ whose affine span is $\mathbb{R}\cdot\mathbf{1}$ (which is contained in the lineality space of $\mathcal{B}(M)$), and let $\mathbf{w}$ be in the relative interior of σ. Set $J' := \operatorname{in}_{\mathbf{w}} I \subseteq \mathbb{B}[x_1,\dots,x_3,y_1,\dots,y_8]$. By Proposition 2.10, the tropical ideal J' is homogeneous (with respect to $\mathbf{1}$) and has variety $V(J') = \operatorname{star}_{\mathbf{w}} V(I)$, which is equal to $\mathcal{B}(M)$ up to common refinement. Consider the homogeneous ideal $J := (J')^{\mathrm{sat}}$. By Lemma 2.9, we have that $V(J)$ is also equal to $\mathcal{B}(M)$ up to common refinement, and so by Proposition 4.7, $M(J_1)$ is equal to M. Since $\operatorname{rk} M = 6$, we find that $H_J(1) = 1 + 6 = 7$ and thus, by Proposition 4.5, $H_J(d) \le \binom{6+d}{d}$ for all d. On the other hand, since $V(J) = \mathcal{B}(M)$, by Propositions 3.2 and 4.6 the tropical ideal J is matroidal, with associated algebraic matroid $M = U_{2,3} \oplus V_8$. Hence, by Proposition 4.4 we have $H_J(d) \ge \binom{6+d}{d}$. We conclude that $H_J(d) = \binom{6+d}{d}$.

Denote $Q := M(J_2)$. The matroid Q has rank $H_J(2) - H_J(1) = 21$ on the ground set $S_1 \sqcup S_2 \sqcup S_3$ where $S_1 := \{x_i x_j \mid 1 \leq i \leq j \leq 3\}$, $S_2 := \{y_i y_j \mid 1 \leq i \leq j \leq 8\}$, and $S_3 := \{x_i y_j \mid 1 \leq i \leq 3, 1 \leq j \leq 8\}$. The restriction $Q|S_1$ is spanned by all products of two elements in a basis of $M(J_1)|\{x_1, x_2, x_3\}$, hence has rank at most $\binom{2+1}{2} = 3$. Similarly, the restriction $Q|S_2$ has rank at most $\binom{4+1}{2} = 10$. Hence $Q|S_3$ has rank at least $21 - 3 - 10 = 8$.

Since J is saturated, for each $1 \leq i \leq 3$, multiplication by x_i yields an isomorphism between the matroid $M(J_1)|\{y_1, \ldots, y_8\} \cong V_8$ and the restriction of Q to $x_i \cdot \{y_1, \ldots, y_8\} \subseteq S_3$. Similarly, for each $1 \leq j \leq 8$, the restriction of Q to $y_j \cdot \{x_1, \ldots, x_3\}$ is isomorphic to $U_{2,3}$. Hence $Q|S_3$ is a quasi-product of $U_{2,3}$ and V_8 in the sense of [4]. But the main result of [4] shows that such a quasi-product has rank at most 7, a contradiction. □

References

[1] F. Ardila and C. J. Klivans. "The Bergman complex of a matroid and phylogenetic trees". *J. Comb. Theory Ser. B* **96**.1 (2006), pp. 38–49. Link.

[2] A. L. Birkmeyer, A. Gathmann, and K. Schmitz. "The realizability of curves in a tropical plane". *Discrete Comput. Geom.* **57**.1 (2017), pp. 12–55. Link.

[3] E. Brugallé and K. Shaw. "Obstructions to approximating tropical curves in surfaces via intersection theory". *Can. J. Math.* **67**.3 (2015), pp. 527–572. Link.

[4] M. Las Vergnas. "On products of matroids". *Discrete Math.* **36** (1981), pp. 49–55. Link.

[5] D. Maclagan and F. Rincón. "The balancing property for tropical ideals". In preparation.

[6] D. Maclagan and F. Rincón. "Tropical ideals". *Compos. Math.* **154**.3 (2018), pp. 640–670. Link.

[7] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics 161. American Mathematical Society, Providence, RI, 2015.

[8] J. Mundinger. "The image of a tropical linear space". 2018. arXiv:1808.02150.

[9] K. Murota and A. Tamura. "On circuit valuation of matroids". *Adv. in Appl. Math.* **26**.3 (2001), pp. 192–225. Link.

[10] F. Rincón. "Isotropical linear spaces and valuated Delta-matroids". *J. Comb. Theory Ser. A* **119**.1 (2012), pp. 14–32. Link.

[11] D. E. Speyer. "Tropical linear spaces". *SIAM J. Discrete Math.* **22**.4 (2008), pp. 1527–1558. Link.

[12] D. E. Speyer. "Parameterizing tropical curves I: Curves of genus zero and one". *Algebra Number Theory* **8**.4 (2014), pp. 963–998. Link.

[13] J. Yu. "Algebraic matroids and set-theoretic realizability of tropical varieties". *J. Comb. Theory Ser. A* **147** (2017), pp. 41–45. Link.