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Explicit Combinatorial Formulas for Some Irreducible Characters of the $GL_k \times \mathbb{S}_n$-module of multivariate diagonal harmonics

Nancy Wallace*

LaCIM, Université du Québec à Montréal, Montréal (Québec), Canada

Abstract. We give an explicit combinatorial formula for some irreducible components of $GL_k \times \mathbb{S}_n$-modules of multivariate diagonal harmonics. To this end we introduce a new path combinatorial object $T_{n,s}$ allowing us to give the formula directly in terms of Schur functions.

Résumé. Dans ce long résumé, certaines composantes irréductibles des caractères de $GL_k \times \mathbb{S}_n$-modules harmoniques diagonaux multivariés sont donnés sous forme d'une formule combinatoire explicite. À cette fin, nous introduisons un nouvel objet de la combinatoire des chemins, $T_{n,s}$, à partir duquel nous associons une fonction de Schur à chaque chemin.

Keywords: Multivariate diagonal harmonics, path combinatorics, Macdonald eigenoperators

1 Introduction

The aim of his paper is to describe some features of the characters of "rectangular" $GL_k \times \mathbb{S}_n$-modules, $\mathcal{E}^{\langle k \rangle}_{m,n}$, introduced by F. Bergeron in [1].

When $k = 2$ and $m = n$, these modules are the modules of diagonal harmonics whose characters have been studied for many years. As shown in [9] and [12], the Frobenius transformation of its graded characters may be expressed as $\nabla(e_n)$ where ∇ is the Macdonald eigenoperator introduced in [4] and recalled in Section 3 and e_n is the n-th elementary symmetric function. A combinatorial interpretation that became known as the Shuffle Conjecture was introduced in [11] and proved recently by Carlson and Mellit [8, 15]. These characters also intervene in torus knot link homology and algebraic geometry; see, for instance [13, 14, 10, 16]. The case $k = 3$ was studied in [6].

In recent work [1] F. Bergeron made a breakthrough in the multivariate case (k arbitrary). He found interesting relations between various irreducible characters of the

*wallace.nancy@courrier.uqam.ca. Nancy Wallace is supported by a scholarship from the NSERC.

modules. This allowed the study of the character of $\mathcal{E}_{n,n}^{\langle k\rangle}$ developed in the elementary symmetric functions for $n \leq 4$ in [7] (in our notation this is the specialization $\sum c_{\lambda,\mu} s_\lambda(q,1,...,1,0,0,...)s_\mu(X)$, where there are $k-1$, 1's). These relations are also exploited here to obtain our main result.

To state our result we briefly fix the required notation. We encode the characters of the irreducible $GL_k \times \mathbb{S}_n$ -modules as products of Schur functions, $s_\lambda(\boldsymbol{Q})s_\mu(\boldsymbol{X})$ (see Section 3 for details). For easier reading we will also write $s_\lambda \otimes s_\mu$ for $s_\lambda(\boldsymbol{Q})s_\mu(\boldsymbol{X})$. As shown in [3] for the case $m = n$ there is a stability property that makes it possible to avoid mentioning k. Therefore the character of $\mathcal{E}_{m,n}^{\langle k\rangle}$ can be expressed in the form $\mathcal{E}_{m,n} = \sum_\mu \left(\sum_\lambda c_{\lambda,\mu} s_\lambda\right) \otimes s_\mu$. Our aim is to describe some features of $\mathcal{E}_{m,n}$, or equivalently, $\langle \mathcal{E}_{m,n}, s_\mu\rangle = \sum_\lambda c_{\lambda,\mu} s_\lambda$.

The main result of this paper is a combinatorial description of the multiplicity of $s_\lambda \otimes s_\mu$ in $\mathcal{E}_{n,n}$, when λ is hook shaped. The result is constructive, in that the hooks are determined combinatorially by a standard Young tableau and certain paths in a staircase shaped grid. More precisely, we give the following combinatorial description (the basic combinatorial notations used in Theorem 1 are recalled in Section 2).

Theorem 1. *If $r = 1$ and $\mu \in \{(n),(n-1,1),(n-2,1,1),(1^n)\}$ then:*

$$\langle \mathcal{E}_{rn,n}, s_\mu\rangle|_{\text{hooks}} = \sum_{\tau\in \mathrm{SYT}(\mu)} \sum_{\gamma \in T_{n,\mathrm{des}(\tau')}} s_{\mathrm{hook}(\gamma)} \tag{1.1}$$

where the first sum is over all standard Young tableaux of shape μ, the second sum is over paths γ in $T_{n,\mathrm{des}(\tau')}$ and $\mathrm{hook}(\gamma) = \left((r-1)\binom{n}{2} + \mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)}\right)$.

Furthermore, when $\mu = 1^n$, (1.1) holds for all positive integers r that satisfy the equality $\langle \mathcal{E}_{rn,n}, s_{k+1,1^{n-k-1}}\rangle = e_k^\perp \langle \mathcal{E}_{rn,n}, s_{1^n}\rangle$ for all k.

The combinatorial object $T_{n,s}$ ($T_{n,\mathrm{des}(\tau')}$ in (1.1)) represents a set of paths, these paths and their statistics (area and ht) will be defined in Section 4. This object was introduced by the author in order to eliminate alternating sums and obtain a Schur positive expression. When $s = 0$, they afford a generating function correlated to the q-Pochhammer symbol $(-qz;z)_n$. Symmetric functions s_λ, e_n, the operators ∇ and Δ' will be recollected in Section 3. Finally, Section 5 will be dedicated to (1.1), the definition of $|_{\text{hooks}}$, of the operator $e_k^\perp$ and a proposition restricting theorem 1 to the $GL_2 \times \mathbb{S}_n$-characters mentioned above, which gives formulas in terms of the major index for $\langle \nabla^r(e_n), s_\mu\rangle|_{\text{hooks}}$ and $\langle \Delta'_{e_k}(e_n), e_n\rangle|_{\text{hooks}}$.

2 Combinatorial Tools

A *partition* of n is a decreasing sequence of positive integers often represented as a Ferrer's diagram. For λ a *Ferrer's diagram* of shape $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ is a left justified pile of boxes having λ_i boxes in the i-th row. We will use the French notation so the second row lies on top of the first row (see Figure 1). We can see them as a subset of $\mathbb{N} \times \mathbb{N}$ if we put the bottom left corner of the diagram to the origin. In this setting, we can associate the bottom left corner of a box to the coordinate it lies on. We say a partition is *hook shaped* if it has the form $(a, 1, \ldots, 1) = (a, 1^k)$, where $a, k \in \mathbb{N}$. If to each box of a diagram we associate an entry it is called a *tableau*. A tableau is of shape λ if it is a filling by integers of a diagram of shape λ. For λ a partition of n, a tableau of shape λ with distinct entries from 1 to n strictly increasing in rows and columns is a *standard Young tableau*. The set of all standard Young tableaux of shape μ is denoted $\mathrm{SYT}(\mu)$. The *descent* set of a tableau is the set of entries i such that the entry $i+1$ lies in a row strictly above i. For a tableau τ, the descent set is denoted $\mathrm{Des}(\tau)$ and the cardinality is denoted $\mathrm{des}(\tau)$. The sum of the elements of $\mathrm{Des}(\tau)$ is called the *major index* and is denoted $\mathrm{maj}(\tau)$ (see Figure 3). The *conjugate* of a partition λ, (respectively a diagram, a tableau) is denoted λ', and is its reflection through the line $x = y$ (see Figure 2).

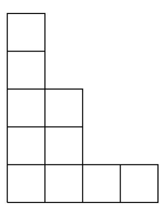


Figure 1: $\lambda = 42211$

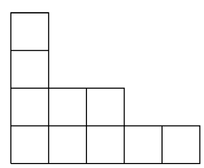


Figure 2: $\lambda' = 5311$

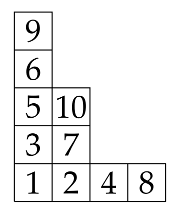


Figure 3: In $\mathrm{SYT}(42211)$; $\mathrm{Des}(\tau) = \{2, 4, 5, 8\}$, $\mathrm{des}(\tau) = 4$, $\mathrm{maj}(\tau) = 19$

We end this section by recalling the definition of the Gaussian polynomials, $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Let:

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}\ ,\ \ [n]!_q = \prod_{i=1}^{n} [i]_q \text{ and } \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[n-k]!_q [k]!_q}.$$

So when $q = 1$ this gives the usual binomial coefficient. It is well known that the Gaussian polynomials are related to the north-east paths of a $k \times (n-k)$ grid: if $\mathcal{C}_k^n$ denotes the set of such paths, and the *area* of a path γ, denoted $\mathrm{area}(\gamma)$, is defined as the number of boxes beneath the path, then $\sum_{\gamma \in \mathcal{C}_k^n} q^{area(\gamma)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

3 The space and characters

Before introducing our new combinatorial objects we provide more details about the modules $\mathcal{E}_{n,n}^{\langle k \rangle}$ and why they are interesting. (This section is not necessary for understanding our main result.)

Let $X = (x_{i,j})_{i,j}$ where $1 \leq i \leq k$, $1 \leq j \leq n$ and let $R_n^{\langle k \rangle} = \mathbb{Q}[X]$ denote the polynomial ring in the variables X. For (τ, σ) in $GL_k \times \mathbb{S}_n$ the group $GL_k \times \mathbb{S}_n$ acts on $R_n^{\langle k \rangle}$ as follows:

$$(\tau, \sigma).F(X) = F(\tau \cdot X \cdot \sigma)$$

With this action we can define $\mathcal{E}_{n,n}^{\langle k \rangle}$ as the smallest submodule of $R_n^{\langle k \rangle}$ that contains the Vandermonde determinant, is closed under all higher polarization operators $\sum_{j=0}^{n} x_{r,j} \partial_{x_{s,j}}$ and is closed under all partial derivatives $\partial_{x_{s,j}}$.

When we separate this modules into a direct sum of the smallest possible modules that preserve this action, we say that it is a decomposition into irreducible modules. A module can be encoded by its character. There is a one to one correspondence between irreducible modules and irreducible characters. It is known that the irreducible characters for GL_k and the Frobenius transform of irreducible characters of $\mathbb{S}_n$ are the set of Schur functions, denoted $\{s_\lambda\}$, indexed over all partitions.

Meanwhile, the *ring of symmetric polynomials* is a set of polynomials which are invariant by permutation of the variables $Y = \{y_1, y_2, \ldots, y_n\}$. So Λ is embedded in $\mathbb{Q}[Y]$. In other words for all $\sigma \in \mathbb{S}_n$ we have:

$$f(y_1, y_2, \ldots, y_n) = f(y_{\sigma^{-1}(1)}, y_{\sigma^{-1}(2)}, \ldots, y_{\sigma^{-1}(n)}).$$

The *ring of symmetric functions*, denoted Λ, can be thought of as the ring of symmetric polynomials in an infinite set of variables. It is a graded ring and has the Schur functions as a basis. Therefore the ring of symmetric functions is the right setting for our study. *Elementary symmetric functions* are defined by $e_n(X) = \sum_{i_1 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$ and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$. They also form a basis for the ring of symmetric functions Λ.

If we start with the ring $\mathbb{Q}(q,t)[Y]$ we also get a ring with Schur functions as a basis. Additionally, the combinatorial Macdonald polynomials, denoted H_μ, are a basis for $\mathbb{Q}(q,t)[Y]$. They appear as eigenvectors of special operators (see [2] for more on this), ∇ and the Δ'_{e_m}, introduced in [4], [5]. These Macdonald operators are defined as follows:

$$\nabla(H_\mu) = \prod_{(i,j) \in \mu} q^i t^j H_\mu \quad \text{and} \quad \Delta'_{e_m}(H_\mu) = \left(\sum_{\substack{S \subset \mu/(0,0) \\ |S| = m}} \prod_{(i,j) \in S} q^i t^j \right) H_\mu.$$

By definition we have $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$, which gives the character decomposition of the $GL_2 \times \mathbb{S}_n$ case stated in the introduction. It was proven that the coefficients are symmetric polynomials in the q and t variables, therefore, one could write the coefficients in the form $\sum_{\lambda,\mu} c_{\lambda,\mu} s_\lambda(q,t,0,\ldots) s_\mu(X)$. More generally, the GL_k characters can be obtained by setting $q_{k+1} = q_{k+2} = \cdots = 0$. We can therefore use a more general notation $\mathcal{E}_{n,n}$.

We will also discuss characters of the form $\mathcal{E}_{rn,n}$ which are related to $\nabla^r(e_n)$ when we restrict to GL_2. They are constructed by adding a set of inert variables considered to be of degree zero. For more details see [1].

4 Defining $T_{n,s}$

Our formula will be formulated in terms of new combinatorial objects that we denote by $T_{n,s}$. It is the set of north-east paths in an $n-2$ staircase shaped grid lying in $\mathbb{N}^2$, starting at $(0,s)$ and ending at a point in the set $\{(x,y) \mid x+y=n-2, x \geq 0 \text{ and } y \geq s\}$. For an example see Figure 4. The relevant paths can be represented as words of length $n-s-2$ in the alphabet $\{N,E\}$.

The *area*, denoted area, of a path is the number of boxes south-east of the path (see Figure 5). The *height* of a path, ht, is the y coordinate of its end point (see Figure 6).

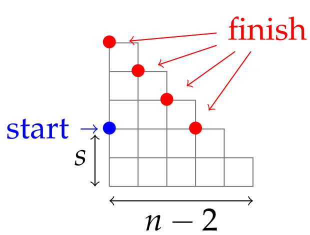


Figure 4: $T_{7,2}$

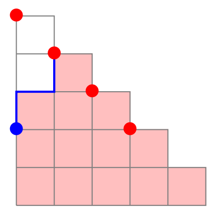


Figure 5: $\mathrm{area}(NEN) = 13$

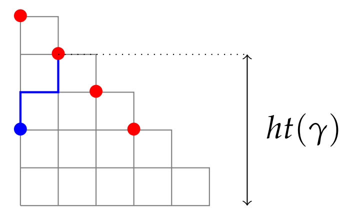


Figure 6: $\mathrm{ht}(NEN) = 4$

When $s > n-1$ we set $T_{n,s} = \{N^{n-2}\}$ for reasons stated later on. Since

$$\sum_{\gamma \in C_k^n} q^{\mathrm{area}(\gamma)} = \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

it can be observed that summing over the paths ending at height $j+s$ will contribute $q^{\binom{s+j+1}{2}+s(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q$ to the generating function. We thus get the following result.

Proposition 2. *Let $T_{n,s}(q,z) = \sum_{\gamma \in T_{n,s}} q^{\mathrm{area}(\gamma)} z^{\mathrm{ht}(\gamma)}$. Then for $r = n-s-2$ we have:*

$$T_{n,s}(q,z) = \sum_{j=0}^{r} q^{\binom{s+j+1}{2}+s(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q z^{j+s} = T_{r,0}(q,z) z^s q^{rs+\binom{s+1}{2}}$$

In particular, if $s = 0$, we have:

$$T_{n,0}(q,z) = \sum_{j=0}^{n-2} q^{\binom{j+1}{2}} \begin{bmatrix} n-2 \\ j \end{bmatrix}_q z^j = (-qz;q)_{n-2}$$

The following result shows how this object is used to eliminate an alternating sum and make the relevant formula positive.

Proposition 3. *Let $g_j : \mathbb{N}^* \to \mathbb{Z}$ be such that $g_j(k) - g_j(k-1) = k + j$ for all $k \geq 1$ and $g_j(0) - \binom{j+1}{2} = g_i(0) - \binom{i+1}{2}$ for all i, j. Then*

$$\sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)} z^{j-1} = T_{n,0}(q,z) q^{g_j(0)-\binom{j+1}{2}}.$$

In particular we have:

$$\sum_{k=0}^{n-j-1} (-1)^k \begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)} = \begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_q q^{g_j(0)}. \tag{4.1}$$

We will only give a sketch of proof for $g_j(k) = \binom{j+k+1}{2}$. Notice that by the previous proposition we only need to prove the second part. For some fixed k and j we can represent the term $\begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)}$ by the set of paths in $\mathcal{C}_{j+k}^{n-1}$ to which we add $g_j(k)$ boxes and subtract k boxes (see Figure 7). Then there is a bijection between paths ending with a north step in $\mathcal{C}_{j+k+1}^{n-1}$ (blue in Figure 8) and paths ending with an east step in $\mathcal{C}_{j+k}^{n-1}$ (red in Figure 8). We only need to change the last step. They both account for the same number of boxes (see Figure 8 as an example). Since the terms have coefficient $(-1)^k$ they cancel out pairwise in the sum. So the only steps left are the ones when $k = 0$ and the path in $\mathcal{C}_j^{n-1}$ ends with a north step. Eliminating the last north step doesn't affect the area if there are no east steps afterwards. Therefore we can consider the paths in $\mathcal{C}_{j-1}^{n-2}$ with the same statistic, which is what we needed.

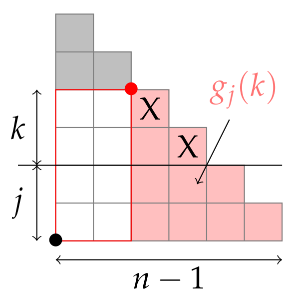


Figure 7: Representation of the term $\begin{bmatrix} n-1 \\ j+k \end{bmatrix}_q q^{-k+g_j(k)}$ in (4.1)

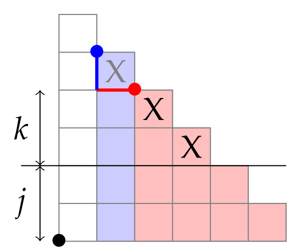


Figure 8: Comparing a path ending with a North step in $\mathcal{C}_{j+k+1}^{n-1}$ and a path with an east step in $\mathcal{C}_{j+k}^{n-1}$

5 The formula

First let us recall that the dual Pieri rule describes the multiplication of a Schur function by e_k. The adjoint of the dual Pieri rule for the Hall scalar product, denoted $e_k^{\perp}$, is defined on the Schur basis and extended linearly. More precisely, $e_k^\perp s_\lambda$ is the sum $\sum_\mu s_\mu$ over all partitions μ obtained by deleting k boxes each lying in a different row (see Figure 9).

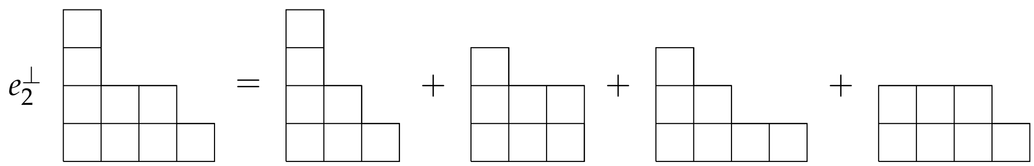


Figure 9: Example: $e_2^\perp s_{4311} = s_{3211} + s_{331} + s_{421} + s_{43}$

For the following lemma we first consider the application $\psi : \Lambda \to \mathbb{Q}[q,t]$ which is defined on the Schur basis by $\psi(s_\lambda) = q^{\lambda_1} t^{\ell(\lambda)-1}$ and then extended linearly.

The following lemma allows us to lift $\mathcal{E}_{m,n}^{\langle 2\rangle}$ to $\mathcal{E}_{m,n}$. Note that we can not obtain all the coefficients of $\mathcal{E}_{m,n}$ this way, but we know which are left out. If we consider the Schur decomposition of $\langle \mathcal{E}_{m,n}, e_n\rangle$, the following lemma shows how to obtain all the coefficients of the Schur functions indexed by partitions of shape $(a, b+1, 1^j)$, with a and j arbitrary and b fixed. If $\sum_\lambda c_\lambda s_\lambda$ is a linear combination of Schur functions, and $\mathcal{V}$ is a set of shapes, then $\sum_\lambda c_\lambda s_\lambda|_{\mathcal{V}} = \sum_{\lambda\in\mathcal{V}} c_\lambda s_\lambda$. On a symmetric function in the Schur basis, we set the restriction $|_{\text{1part}}$ (respectively $|_{\text{hooks}}$) to be the partial sum over the Schur functions indexed by partitions having only one part (respectively that are hook shaped).

Lemma 4. *Let b be a constant in $\mathbb{N}$ and $G \in \Lambda_Q$ be a symmetric functions in the variables $Q = \{q_1, q_2, \ldots\}$. Let $\mathcal{V}_b$ be the set of partitions of shape $(a, b, 1^k)$ with k and a arbitrary. If $f_0(q) = G|_{\mathcal{V}_b}$ and*

$$f_i(q) = \psi((e_i^\perp G|_{\mathcal{V}_{b+1}})^{\langle 2\rangle}|_{\mathcal{V}_b})t^{-1},$$

then:

$$\psi(G|_{\mathcal{V}_{b+1}})(q,t) = \sum_{j\geq 1}\sum_{k=0}^{j}(-1)^k f_{j-k}(q) q^{-k} t^j$$

Note that when $b \geq 2$, no formula is known for $\langle \nabla(e_n), s_{k,1^{n-k}}\rangle|_{\mathcal{V}_b}$ at this moment. Given this formula, the lemma gives a way to find the formulas for $\langle \mathcal{E}_n, s_{1^n}\rangle|_{\mathcal{V}_{b+1}}$.

We should also notice that $\psi(G|_{\text{hooks}})(q,t)$ is the sum of the restriction to $\mathcal{V}_1$ and the restriction to one parts ($|_{\mathcal{V}_0}$). This is the reason why, in Section 4, we used the convention that $T_{n,s} = \{N^{n-2}\}$ if $s > n-2$. Since the path N^{n-2} relates to the restriction to one part.

We can now prove the main theorem piece by piece. Using the combinatorics of m-Schröder paths, we found that for $\psi(\langle \nabla^r(s_{1^n}), s_{j+1,1^{n-j}}\rangle|_{\text{1 Part}})$ we have:

$$f_j^{(r)}(q) = q^{r\binom{n}{2}-\binom{j}{2}} \begin{bmatrix} n-1 \\ j \end{bmatrix}_{q^{-1}}$$

We can prove the next proposition with this, the previous lemma and a theorem in [1] stating that $\langle \mathcal{E}_{n,n}, s_{j+1,1^{n-j}}\rangle = e_j^\perp \langle \mathcal{E}_{n,n}, s_{1^n}\rangle$.

Proposition 5. *For $r = 1$ we have:*

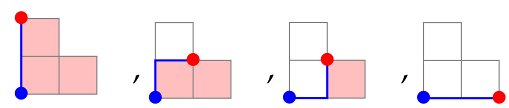
$$\langle \mathcal{E}_{rn,n}, e_n \rangle|_{\text{hooks}} = \sum_{\gamma \in T_n} s_{\text{hook}\,1(\gamma)} \tag{5.1}$$

where hook 1(γ) *is the partition* $\left((r-1)\binom{n}{2} + \text{area}(\gamma) + ht(\gamma) + 1, 1^{n-2-ht(\gamma)}\right)$.

Moreover, for all $r \in \mathbb{N}^*$ such that the equality $\langle \mathcal{E}_{rn,n}, s_{j+1,1^{n-j}} \rangle = e_j^{\perp} \langle \mathcal{E}_{rn,n}, s_{1^n} \rangle$ is true, then (5.1) holds.

Figure 10 gives an example for $r = 1$ and $n = 4$. One might want to take note that for $n = 4$ there are only hooks so $\langle \mathcal{E}_{4,4}, e_4 \rangle|_{hooks} = \langle \mathcal{E}_{4,4}, e_4 \rangle$. This is why we can state in the introduction that (1.1) specializes to the equation of theorem 3.2.5 in [7] using a different basis.

The elements of $T_{2,0}$:



area(γ) :	3	2	1	0
$ht(\gamma)$:	2	1	1	0
$hook(\gamma)$:	$(6, 1^0)$	$(4, 1^1)$	$(3, 1^1)$	$(1, 1^2)$
$s_{hook(\gamma)}$:	s_6	s_{41}	s_{31}	s_{111}

$$\langle \mathcal{E}_{4,4}, e_4 \rangle|_{hooks} = s_6 + s_{41} + s_{31} + s_{111}$$

Figure 10: Example of $\langle \mathcal{E}_{4,4}, e_4 \rangle|_{hooks}$

For those who are used to seeing $\langle \mathcal{E}_{n,n}^{\langle 2 \rangle}, s_\mu \rangle = \langle \nabla(e_n), s_\mu \rangle$ in terms of Dyck paths and Schröder paths note that each path in $T_{n,k}$ is associated to a subset of Schröder paths. The next proposition is obtained by restricting the formula to $\mathcal{E}_{n,n}^{\langle 2 \rangle}$ and by using a bijection between a subset of Schröder paths with d diagonal steps and the set SYT$(d + 1, 1^{n-d-1}) \times \{1, 2, \ldots, n-d-1\}$. The subset is such that the paths end with a north step and the area statistic of that path is equal to 1.

Proposition 6. *If $r = 1$ and $\mu \in \{(k, 1^{n-k}) \mid 1 \leq k \leq n\}$ then we have:*

$$\langle \nabla^r(e_n), s_\mu \rangle|_{\text{hooks}} = \sum_{\tau \in \text{SYT}(\mu)} s_{r\binom{n}{2} - \text{maj}(\tau')}(q,t) + \sum_{i=2}^{\text{des}(\tau)} s_{r\binom{n}{2} - \text{maj}(\tau') - i, 1}(q,t). \tag{5.2}$$

Additionally, for all k we have:

$$\langle \Delta'_{e_k}(e_n), e_n \rangle|_{\text{hooks}} = \sum_{\tau \in \text{SYT}((n-k, 1^k))} s_{\text{maj}(\tau)}(q,t) + \sum_{i=2}^{k} s_{\text{maj}(\tau) - i, 1}(q,t). \tag{5.3}$$

Furthermore for all $\mu \in \{(n), (n-1,1), (n-2,1,1), (1^n)\}$, $0 \le k < n-1$ we have:

$$e^{\perp}_{n-k-1}\left(\langle \mathcal{E}_{n,n}, s_\mu\rangle|_{\text{hooks}}\right)^{\langle 2\rangle} = \sum_{\tau\in \mathrm{SYT}(\mu)} \sum s_{(k-1+\text{area}(\gamma)-\text{maj}(\tau'),1)} + \sum s_{(k+\text{area}(\gamma)-\text{maj}(\tau'))}. \tag{5.4}$$

Where the second sum is over paths in $T_{n,\text{des}(\tau')}$ of height $k-2$ and $k-1$ and the third sum is over paths in $T_{n,\text{des}(\tau')}$ of height $k-1$ and k.

Finally, if $t=0$ (5.4) holds for all μ and (5.2) holds for all r whenever μ is hook shaped. Equation (5.3) already holds for all q and t.

This proposition proves that the main theorem holds in the case where μ is of shapes $(n-2,1,1)$, $(n-1,1)$ or (n). This is because the height of the hooks is bounded by 2 in these cases. Note that, when $\mu \neq 1^n$ and $r>1$, computer experiments indicate that some hook shaped terms are not accounted for by (1.1) of Theorem 1.

It is shown in [1] that $\langle \mathcal{E}_{n,n}, s_{(k+1,1^{n-k-1})}\rangle = e_k^{\perp}\langle \mathcal{E}_{n,n}, s_{1^n}\rangle$. By Proposition 8 this can be done directly in terms of paths. It also supports the assumption that (1.1) is true for all hooked shape μ when $r=1$.

Lets consider the three following sets of paths:

$$\bigcup_{\tau\in \mathrm{SYT}((k+1,1^{n-k-1}))} \{\gamma = N^j E\gamma' \in T_{n-2,\text{des}(\tau')} \mid j \ge n-k-\min(\text{Des}(\tau'))\} = T^+_{n,k}$$

$$\bigcup_{\tau\in \mathrm{SYT}((k+1,1^{n-k-1}))} \{\gamma = N^j E\gamma' \in T_{n-2,\text{des}(\tau')} \mid j < n-k-\min(\text{Des}(\tau'))\} = T^-_{n,k}$$

and

$$T_n^k = \{\gamma \in T_{n,0} \mid n-2-\text{ht}(\gamma) \ge k\}.$$

For k between 1 and $n-1$, we will now define two families $\{\underline{e^{\perp}_{k-}}\}$ and $\{\underline{e^{\perp}_{k+}}\}$ of maps:

$$\underline{e^{\perp}_{k-}} : T_n^{k-1}\backslash\{E^{n-2}\} \to T^-_{n,k} \quad\text{and}\quad \underline{e^{\perp}_{k+}} : T_n^{k} \to T^+_{n,k}$$

For $\gamma \in T_{n,0}$, lets consider the prefix of γ ending with the k-th east step. The prefix exists by definition of T_n^k. Let $p_1,\dots,p_k$ denote the integers such that p_i is the number of north steps before the i-th east step. To this we associate τ', the standard Young tableau such that $\text{Des}(\tau') = \{n-2-i+1-p_i-1 \mid 1\le i\le k\}$. Then $\underline{e^{\perp}_{k+}}(\gamma)$ is the path in $T_{n,\text{des}(\tau')}$ given by discarding all the k first east steps of γ. (See Figure 11)

The map $\underline{e^{\perp}_{k-}}(\gamma)$ is defined in a similar way. We denote by $p_1,\dots,p_{k-1}$ the integers such that p_i is the number of north steps before the i-th east step. Let h be the number of east steps before the first north step. We choose τ' to be the standard Young tableau such that $\text{Des}(\tau') = \{n-2-i+1-p_i+1 \mid 1\le i\le k-1\} \cup \{\max(1, h-k+2)\}$. Then $\underline{e^{\perp}_{k-}}(\gamma)$ is the path in $T_{n,\text{des}(\tau')}$ given by discarding all the $k-1$ first east steps of and the first north step γ. (See Figure 12)

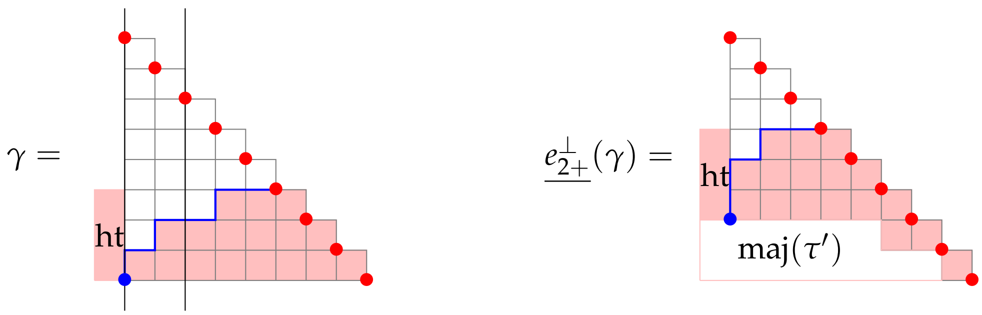


Figure 11: For $NENEENEE \in T_{9,0}$, $\underline{e_{2+}^{\perp}}(NENEENEE) = NNENEE \in T_{9,\mathrm{des}(\tau')}$, where $\mathrm{Des}(\tau') = \{6,8\}$

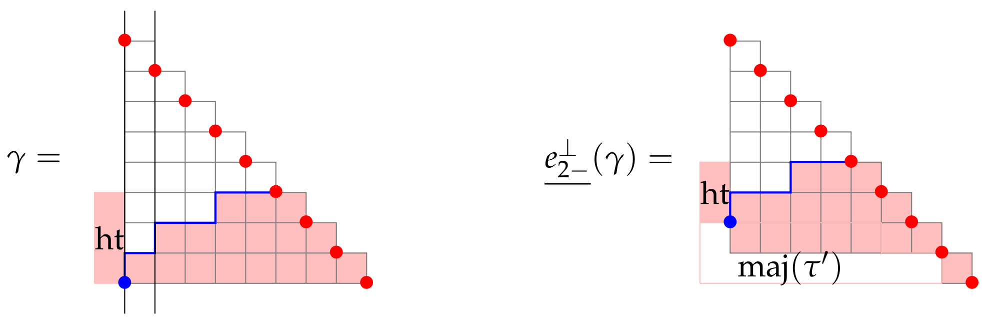


Figure 12: For $NENEENEE \in T_{9,0}$, $\underline{e_{2-}^{\perp}}(NENEENEE) = NNEENEE \in T_{9,\mathrm{des}(\tau')}$, where $\mathrm{Des}(\tau') = \{1,8\}$

We will recall that for $r = 1$ the hooks in Theorem 1 are given by:

$$\mathrm{hook}(\gamma) = \left(\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) - \mathrm{maj}(\tau') + 1, 1^{n-2-\mathrm{ht}(\gamma)}\right)$$

Lemma 7. *For all k the maps $\underline{e_{k+}^{\perp}}$ is a well-defined bijection such that*

$$\mathrm{hook}(\underline{e_{k+}^{\perp}}(\gamma)) = \left(\mathrm{area}(\gamma) + \mathrm{ht}(\gamma) + 1, 1^{n-2-\mathrm{ht}(\gamma)-k}\right).$$

For all k the map $\underline{e_{k-}^{\perp}}$ is a well defined injective map such that

$$\mathrm{hook}(\underline{e_{k-}^{\perp}}(\gamma)) = \left(\mathrm{area}(\gamma) + \mathrm{ht}(\gamma), 1^{n-1-\mathrm{ht}(\gamma)-k}\right).$$

Furthermore,

$$e_k^{\perp}(s_{\mathrm{hook}(\gamma)}) = s_{\mathrm{hook}(\underline{e_{k+}^{\perp}}(\gamma))} + s_{\mathrm{hook}(\underline{e_{k-}^{\perp}}(\gamma))}$$

For the next proposition we extend our maps so that $\underline{e_{k+}^{\perp}}(\gamma) = \emptyset$, if $\gamma \in T_{n,0} \backslash T_n^k$, $\underline{e_{k-}^{\perp}}(\gamma) = \emptyset$ if $\gamma \in T_{n,0} \backslash T_n^{k-1}$ and $s_{\mathrm{hook}(\emptyset)} = 0$.

Proposition 8. *For all k, we have:*

$$e_k^{\perp}\left(\langle \mathcal{E}_{n,n}, e_n\rangle|_{\text{hooks}}\right) = \sum_{\gamma\in T_{n,0}} s_{\text{hook}(\underline{e_{k+}^{\perp}}(\gamma))} + s_{\text{hook}(\underline{e_{k-}^{\perp}}(\gamma))},$$

where hook *is defined as in Theorem 1.*

In addition, $\sum_{\tau\in\text{SYT}(k+1,1^{n-k-1})}\sum_{\gamma\in T_{n,\text{des}(\tau')}} s_{\text{hook}(\gamma)} - e_k^{\perp}\left(\langle \mathcal{E}_{n,n}, s_{1^n}\rangle|_{\text{hooks}}\right)$ *has a Schur positive expansion.*

This last proposition gives reason to believe that (1.1) holds for all μ, a hook, since the missing terms should be obtained by the restriction to shapes having two columns. The following proposition starts the second column.

Proposition 9. *For $\mathcal{W} = \{(a,2,1^k) \mid k\in\mathbb{N}, a\in\mathbb{N}_{\geq 2}\}$, we have:*

$$\langle \mathcal{E}_n, e_n\rangle|_{\mathcal{W}} = \sum_{i=2}^{n-3}\sum s_{\text{Shape}\,U},$$

Where the second sum is over paths $\gamma\in T_{n,0}$ such that $\gamma\neq N\tilde{\gamma}N^{i-1}$ and $i\leq \text{ht}(\gamma)\leq n-3$. Additionally $\text{Shape}\,U$ *is the partition given by* $\left(\text{area}(\gamma)+\text{ht}(\gamma)+1-i, 2, 1^{n-3-\text{ht}(\gamma)}\right)$.

We haven't proven that the equation $e_1^{\perp}(\langle \mathcal{E}_n, s_{1^n}\rangle|_{\mathcal{W}})|_{\text{hooks}}$ is equivalent to the remainder of the remainder of $\sum_{\tau\in\text{SYT}(2,1^{n-2})}\sum_{\gamma\in T_{n,\text{des}(\tau')}} s_{\text{hook}(\gamma)} - e_1^{\perp}\left(\langle \mathcal{E}_{n,n}, s_{1^n}\rangle|_{\text{hooks}}\right)$. Such an equivalence would prove the case $\mu = (2,1^{n-2})$ of (1.1).

6 Conclusion and further questions

It would be interesting to show that the formula hold for all μ when $r=1$ and all r when $\mu = 1^n$. Has we mentioned previously, this could be obtained, for μ a hook, by having a formula for the restriction to Schur's having two columns. Writing the q,t statistics of the m-Schröder paths in term of Schur functions would give rise to a more general formula. Finally, the author is already working on showing how existing formulas for special cases are equivalent to the proposed formula.

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