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Unitary friezes and frieze vectors

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Abstract. We study friezes of type Q as homomorphisms from the cluster algebra to an arbitrary integral domain. In particular, we show that every positive integral frieze of affine Dynkin type $\mathbb{A}$ is unitary, which means it is obtained by specializing each cluster variable in one cluster to the constant 1. This completes the answer to the question of unitarity for all positive integral friezes of Dynkin and affine Dynkin types.

For an arbitrary quiver Q, we introduce a new class of integer vectors which we call frieze vectors. These frieze vectors are defined as solutions of certain Diophantine equations given by the cluster variables in the cluster algebra. We establish a bijection between the positive unitary frieze vectors and the clusters in the cluster algebra.

Résumé. Nous étudions les frieses de type Q comme étant des homomorphismes de l'algèbre amassée vers un anneau intègre. Nous montrons en particulier, que chaque friese entière positive de type $\mathbb{A}$-affine est unitaire, ce qui signifie que l'on peut la construire en spécialisant chaqu'une des variables d'un amas en la constante 1. Ainsi nous complétons la réponse à la question d'unitarité pour les frieses entières positives de type Dynkin ou affine.

Pour un carquois arbitraire Q, nous introduisons une nouvelle classe de vecteurs entiers que l'on appelle des vecteurs de friese. Ces vecteurs de friese sont définis comme des solutions de certaines équations diophantines données par les variables amassées de l'algèbre amassée. Nous montrons que les vecteurs de friese sont en bijection avec les amas de l'alg'ebre amassées.

Keywords: cluster algebra, frieze, frieze vector, unitary frieze, Auslander–Reiten quiver

1 Introduction

Friezes of type $\mathbb{A}_n$ were classified by Conway and Coxeter in 1973 [5]. More than 30 years later, Caldero and Chapoton discovered a connection between friezes and cluster algebras [3]. Since then friezes were studied by many authors; see [13] for an overview.

Classical friezes are defined as certain planar arrays of positive integers that satisfy a diamond relation. In this paper however, we take a different point of view and define a

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frieze to be a homomorphism from an arbitrary cluster algebra to an arbitrary integral domain. The usual planar array is obtained from the Auslander–Reiten quiver of the corresponding cluster category by replacing the indecomposable objects (i.e. the vertices of the Auslander–Reiten quiver) by the values of the homomorphism on the corresponding cluster algebra elements. Friezes as homomorphisms were also considered in [8, 9].

Let Q be a quiver without loops and 2-cycles and let $\mathcal{A}(Q)$ be the corresponding cluster algebra with trivial coefficients [7]. We define a *frieze of type* Q to be a ring homomorphism $\mathcal{F}\colon \mathcal{A}(Q) \to R$ from the cluster algebra to an integral domain R. The frieze $\mathcal{F}$ is called *non-zero* if every cluster variable is mapped to a non-zero element of R and $\mathcal{F}$ is said to be *unitary* if there exists a cluster $\mathbf{x}$ such that $\mathcal{F}(x)$ is a unit in R, for all $x \in \mathbf{x}$. Moreover $\mathcal{F}$ is called *integral* if $R = \mathbb{Z}$, and *positive* if $R = \mathbb{Z}$ and every cluster variable is mapped to a positive integer. Our definition of unitary friezes agrees with that of [12, 9] for positive integral friezes. Note however that if the integral frieze is not positive, we also allow specialization at -1.

Positive integral friezes of Dynkin type $\mathbb{A}_n$ are precisely the classical Conway–Coxeter friezes. The classical frieze is given by displaying the values of $\mathcal{F}$ on the cluster variables in the shape of the Auslander–Reiten quiver of the cluster category, see Section 2.1.

Every non-zero frieze is determined by its values $\mathcal{F}(\mathbf{x}) = (a_1, \dots, a_n)$ on an arbitrary cluster $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathcal{A}(Q)$. It is therefore natural to ask which values $(a_1, \dots, a_n)$ produce positive unitary integral friezes. We call such a vector $(a_1, \dots, a_n)$ a *unitary frieze vector relative to the cluster* $\mathbf{x}$. Our first main result is the following.

Theorem 1. *Let Q be a quiver without loops and 2-cycles and let $\mathbf{x} = (x_1, \dots, x_n)$ be an arbitrary cluster of $\mathcal{A}(Q)$. Then there is a bijection*

$$\begin{array}{rcl} \phi\colon \{\text{unordered clusters in } \mathcal{A}(Q)\} & \longrightarrow & \{\text{positive unitary frieze vectors relative to } \mathbf{x}\} \\ \mathbf{x}' = \{x'_1, \dots, x'_n\} & \longmapsto & \phi(\mathbf{x}') = (a_1, \dots, a_n). \end{array}$$

Thus every cluster $\mathbf{x}'$ defines a unique unitary frieze vector. One can thus think of the frieze vectors as another parametrization of the clusters in the cluster algebra. The frieze vectors are different from other known vectors appearing in cluster algebra theory like denominator vectors, c-vectors or g-vectors.

Our second main result is about the unitarity of positive integral friezes. Since Conway and Coxeter's work in 1973, it is known that every positive integral frieze of Dynkin type $\mathbb{A}$ is unitary. For Dynkin types $\mathbb{D}$ and $\mathbb{E}$ there exist non-unitary positive integral friezes, see [9]. We extend these results to the affine Dynkin types as follows.

Theorem 2. *Let Q be a quiver of type $\widetilde{\mathbb{A}}_{p,q}$ and let $\mathcal{F}\colon \mathcal{A}(Q) \to \mathbb{Z}$ be a positive integral frieze. Then $\mathcal{F}$ is unitary.*

Our proof is constructive. We give an algorithm that starts from an arbitrary positive integral frieze $\mathcal{F}$ and produces the unique cluster $\mathbf{x}$ such that $\mathcal{F}(\mathbf{x}) = (1, \dots, 1)$. In the other affine types $\widetilde{\mathbb{D}}$ and $\widetilde{\mathbb{E}}$, there are non-unitary positive integral friezes.

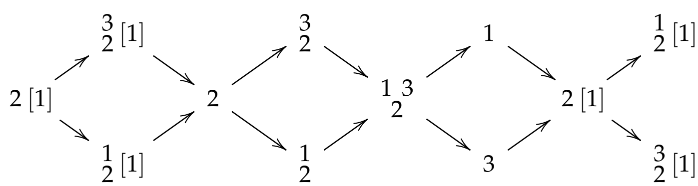
It is natural to ask if friezes of types $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ remain unitary if one replaces the ring of integers by other integral domains. However, already over the Gaussian integers we give an example of a non-unitary frieze of Dynkin type $\mathbb{A}_2$. The classification of friezes over the Gaussian integers or other integral domains besides $\mathbb{Z}$ is open even in type $\mathbb{A}$. For type $\mathbb{A}_1$ there are 12 non-zero friezes over the Gaussian integers, see [8].

This is an extended abstract of [10], and is organized as follows. In Section 2, we give several examples of friezes of type $\mathbb{A}_3$ over different rings and show how the friezes are a generalization of Conway–Coxeter friezes. Section 3 is devoted to the definition of frieze vectors and the proof of Theorem 1, and Theorem 2 is proved in Section 4.

2 Examples of friezes

2.1 The identity homomorphism

For example, let Q be the type $\mathbb{A}_3$ quiver $1 \to 2 \leftarrow 3$. We can visualize a frieze in the Auslander–Reiten quiver of the cluster category $\mathcal{C}_Q$ (see [4, 2]) as follows. First let us write down the Auslander–Reiten quiver.

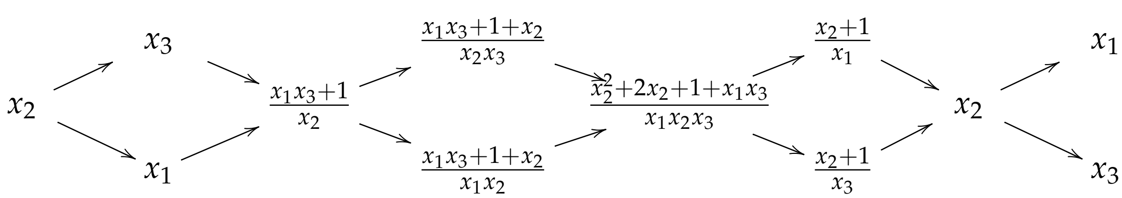


Here we use a standard notation for the representations of the quiver Q, see for example [14], and [1] denotes the shift. Vertices with the same label are identified, so the quiver lies on a Moebius strip. The Auslander–Reiten translation τ is the horizontal translation to the left. For example $\tau\, 3 = \begin{smallmatrix}1\\2\end{smallmatrix}$. The Auslander–Reiten triangles are given by the meshes in the Auslander–Reiten quiver, for example

$$\to \begin{smallmatrix}1\\2\end{smallmatrix}[1] \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \to \qquad \text{and} \qquad \to 2 \to \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \to \begin{smallmatrix}1\ 3\\2\end{smallmatrix} \to$$

are Auslander–Reiten triangles.

The identity homomorphism $\mathcal{A}(Q) \to \mathcal{A}(Q)$ is an example of a frieze of type $\mathbb{A}_3$:



Note that the Auslander–Reiten triangles give the usual diamond rules, for example

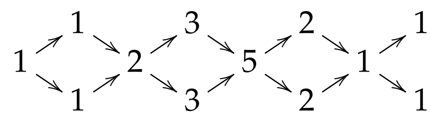
$$x_1 \, \frac{x_1x_3 + 1 + x_2}{x_1x_2} = \frac{x_1x_3 + 1}{x_2} + 1 \qquad \text{and}$$

$$\frac{x_1x_3+1}{x_2}\ \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} = \frac{x_1x_3+1+x_2}{x_1x_2}\ \frac{x_1x_3+1+x_2}{x_2x_3} + 1.$$

2.2 Specializations

We compute several specializations of the example above.

(i) Specializing $x_1 = x_2 = x_3 = 1$, we obtain the unitary positive integral frieze

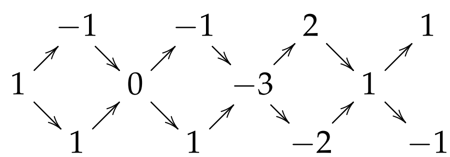


Here the previous examples of the diamond rules become simply

$$1 \cdot 3 = 2 + 1 \qquad \text{and} \qquad 2 \cdot 5 = 3 \cdot 3 + 1.$$

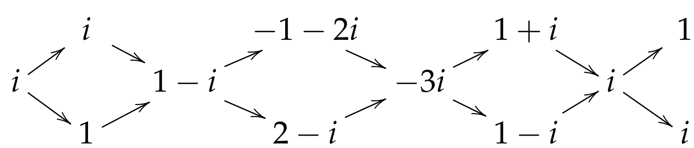
This is an example of a classical Conway–Coxeter frieze; let us point out that one can extend this frieze pattern by a row of 1's above and below the current pattern, which is how the Conway–Coxeter friezes are usually represented.

(ii) Specializing $x_1 = x_2 = 1$ and $x_3 = -1$, we obtain the following unitary integral frieze which is non-positive, not even non-zero.



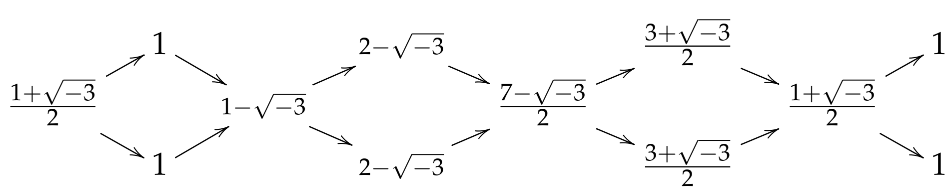
Here our example diamond relations become $1 \cdot 1 = 0 + 1$ and $0 \cdot (-3) = (-1) \cdot 1 + 1$.

(iii) Specializing $x_1 = 1$, $x_2 = i$, and $x_3 = i$, we obtain the following unitary non-zero frieze in the Gaussian integers $\mathbb{Z}[i]$.



Our example diamond relations become $1 \cdot (2-i) = (1-i) + 1$ and $(1-i) \cdot (-3i) = (-1-2i) \cdot (2-i) + 1$.

(iv) Specializing $x_1 = 1$, $x_2 = \frac{1+\sqrt{-3}}{2}$, $x_3 = 1$, we obtain the following unitary non-zero frieze in the quadratic integer ring $\mathbb{Z}[\sqrt{-3}]$. The units in this ring are $\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}$.



In this case, the examples of the diamond relations become $1 \cdot (2-\sqrt{-3}) = 1-\sqrt{-3}$ and $(1-\sqrt{-3})(\frac{7-\sqrt{-3}}{2}) = (2-\sqrt{-3})^2+1$.

3 Frieze vectors

In this section, we introduce a class of positive integer vectors and show that they are in bijection with the clusters of the cluster algebra.

3.1 Definition

We start with a general result on non-zero friezes.

Proposition 3.1. *Every non-zero frieze $\mathcal{F}\colon \mathcal{A}(Q) \to R$ is completely determined by its values on an arbitrary cluster in $\mathcal{A}(Q)$.*

Proof. Let $\mathbf{x} = (x_1, \cdots, x_n)$ be a cluster in $\mathcal{A}(Q)$ and let u be an arbitrary cluster variable in $\mathcal{A}(Q)$ that does not lie in $\mathbf{x}$. By the Laurent phenomenon [7], we can write u as a Laurent polynomial in $x_1, \ldots, x_n$, thus

$$u = \frac{f(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \qquad \text{with } f \in \mathbb{Z}[x_1, \ldots, x_n], d_i \geq 0.$$

Thus

$$\mathcal{F}(u) = \frac{f(\mathcal{F}(x_1), \ldots, \mathcal{F}(x_n))}{\mathcal{F}(x_1)^{d_1} \cdots \mathcal{F}(x_n)^{d_n}}$$

in the field of fractions of R. Note that this expression is well-defined since the frieze is non-zero. Therefore $\mathcal{F}(u)$ is determined by the values $\mathcal{F}(x_i)$. Since the cluster algebra is generated by its cluster variables, this completes the proof. □

Proposition 3.1 implies that given an arbitrary cluster $\mathbf{x} = (x_1, \ldots, x_n)$ we can obtain *every* non-zero frieze by specializing the cluster variables x_i of the cluster to certain ring elements $\mathcal{F}(x_i) = a_i \in R$. It is important to note that by far not every choice of elements $a_i \in R$ will produce a frieze with values in R, because in general the values will be in the field of fractions of R. This leads us to the following definition.

Definition 3.2. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a cluster of $\mathcal{A}(Q)$.

(1) A vector $(a_1, \ldots, a_n) \in R^n$ is called a *frieze vector relative to* $\mathbf{x}$ if the frieze $\mathcal{F}$ defined by $\mathcal{F}(x_i) = a_i$ has values in R. If the frieze $\mathcal{F}$ is unitary we say that the frieze vector $(a_1, \ldots, a_n)$ is *unitary*.

(2) A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ is called a *positive frieze vector relative to* $\mathbf{x}$ if the frieze $\mathcal{F}$ defined by $\mathcal{F}(x_i) = a_i$ is positive integral.

Proposition 3.3. (1) *Let* $(a_1,\ldots,a_n) \in R^n$ *such that every* a_i *is a unit in* R*. Then* $(a_1,\ldots,a_n)$ *is a (unitary) frieze vector relative to every cluster* $\mathbf{x} = (x_1,\ldots,x_n)$ *in* $\mathcal{A}(Q)$*.*

(2) *The vector* $(1,\ldots,1) \in \mathbb{Z}^n_{>0}$ *is a positive (unitary) frieze vector relative to every cluster* $\mathbf{x} = (x_1,\ldots,x_n)$ *in* $\mathcal{A}(Q)$*.*

Proof. (1) By the Laurent phenomenon, every cluster variable is a Laurent polynomial in $\mathbf{x}$. Since each x_i is specialized to a unit in R, the denominator of this Laurent polynomial also specializes to a unit in R. Therefore the image of every cluster variable lies in R, and hence $\mathcal{F}(\mathcal{A}(Q)) \subset R$.

(2) The frieze is integral by part (1) and positivity follows from the positivity theorem for cluster variables [11]. □

3.2 Acyclic type

In the case where the quiver Q is mutation equivalent to an acyclic quiver, we have the following characterization of frieze vectors.

Proposition 3.4. *Let* $(\mathbf{x} = (x_1,\ldots,x_n), Q)$ *be an acyclic seed. Then* $(a_1,\ldots,a_n) \in R^n$ *is a frieze vector relative to* $\mathbf{x}$ *if and only if* a_i is a divisor of $\prod_{i\to j} a_j + \prod_{i\leftarrow j} a_j$ *in* R*, for all* $i = 1,\ldots,n$*.*

Proof. Let x'_i denote the cluster variable obtained from $(\mathbf{x}, Q)$ by mutating in direction i. Then

$$x'_i = \frac{\prod_{i\to j} x_j + \prod_{i\leftarrow j} x_j}{x_i}.$$

By [1, Cor. 1.21], the cluster algebra is generated by the $2n$ variables $x_1,\ldots,x_n,x'_1,\ldots,x'_n$. Let $\mathcal{F}$ be the homomorphism defined by $\mathcal{F}(x_i) = a_i$. Then

$$\mathcal{F}(\mathcal{A}(Q)) \subset R \Leftrightarrow \mathcal{F}(x'_i) \in R \text{ for each } i \;\Leftrightarrow a_i \text{ divides } \prod_{i\to j} a_j + \prod_{i\leftarrow j} a_j \text{ in } R \text{ for all } i. \quad \square$$

3.3 Main result on frieze vectors

In this subsection we state and prove our first main result.

Proposition 3.5 ([10, Prop. 2.5])**.** *Let* $\mathcal{F}\colon \mathcal{A}(Q) \to \mathbb{Z}$ *be a positive unitary integral frieze and let* $\mathbf{x}$ *be a cluster such that* $\mathcal{F}(\mathbf{x}) = (1,\ldots,1)$*. Then for all cluster variables* $u \notin \mathbf{x}$ *we have* $\mathcal{F}(u) > 1$*. In particular* $\mathbf{x}$ *is the unique cluster such that* $\mathcal{F}(\mathbf{x}) = (1,\ldots,1)$*.*

Theorem 3.6. *Let* Q *be a quiver without loops and 2-cycles and let* $\mathbf{x} = (x_1,\ldots,x_n)$ *be an arbitrary cluster of* $\mathcal{A}(Q)$*. Then there is a bijection*

$$\begin{aligned} \phi\colon \{\text{unordered clusters in } \mathcal{A}(Q)\} &\longrightarrow \{\text{positive unitary frieze vectors relative to } \mathbf{x}\} \\ \mathbf{x}' = \{x'_1,\ldots,x'_n\} &\longmapsto \phi(\mathbf{x}') = (a_1,\ldots,a_n). \end{aligned}$$

Remark 3.7. (1) The theorem implies that every cluster $\mathbf{x}'$ defines a unique positive unitary frieze vector in $\mathbb{Z}^n_{>0}$.

(2) We stress that, while the order of the cluster variables $x'_1, \ldots, x'_n$ is irrelevant, the order of the entries of the frieze vector $\phi(\mathbf{x}') = (a_1, \ldots, a_n)$ is important. In other words, if σ is a permutation then $\phi(\sigma\mathbf{x}') = \phi(\mathbf{x}')$, but $\sigma\phi(\mathbf{x}') \neq \phi(\mathbf{x}')$ in general.

Proof. Each cluster variable $x_1, \ldots, x_n$ in the fixed cluster $\mathbf{x}$ can be expressed as a Laurent polynomial in the cluster $\mathbf{x}'$, say $x_i = \mathcal{L}_i(x'_1, \ldots, x'_n)$. We define the map ϕ by $\phi(\mathbf{x}') = (a_1, \ldots, a_n)$, with $a_i = \mathcal{L}_i(1, \ldots, 1)$. In other words, $\phi(\mathbf{x}')$ is equal to the vector $\mathcal{F}(\mathbf{x}) = (a_1, \ldots, a_n)$, where $\mathcal{F}$ is the frieze defined by specializing the cluster variables in $\mathbf{x}'$ to 1. By Proposition 3.3, the frieze $\mathcal{F}$ is unitary, integral and positive. Thus $(a_1, \ldots, a_n)$ is a positive unitary frieze vector relative to $\mathbf{x}$. Furthermore, since every variable in $\mathbf{x}'$ is specialized to 1, we clearly have $\phi(\sigma\mathbf{x}') = \phi(\mathbf{x}')$, for every permutation σ. Thus the map ϕ is well-defined.

To show that ϕ is surjective, let $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ be any positive unitary frieze vector relative to $\mathbf{x}$. By definition, the corresponding frieze defined by $\mathcal{F}(x_i) = a_i$ is positive and unitary, so there exists a cluster $\mathbf{x}' = (x'_1, \ldots, x'_n)$ such that $\mathcal{F}(x'_i) = 1$, for $i = 1, \ldots, n$. By construction of ϕ, we have $\phi(\mathbf{x}') = (a_1, \ldots, a_n)$, so ϕ is surjective.

To show injectivity, let $\mathbf{x}', \mathbf{x}''$ be clusters in $\mathcal{A}(Q)$ with $\phi(\mathbf{x}') = \phi(\mathbf{x}'')$. Let $\mathcal{F}'$ and $\mathcal{F}''$ be the unitary friezes defined by $\mathcal{F}'(x'_i) = 1$ and $\mathcal{F}''(x''_i) = 1$, respectively. Since $\phi(\mathbf{x}') = \phi(\mathbf{x}'')$, both friezes have the same values on $\mathbf{x}$, thus $\mathcal{F}'(\mathbf{x}) = \mathcal{F}''(\mathbf{x}) = (a_1, \ldots, a_n)$. Now Proposition 3.1 implies that $\mathcal{F}' = \mathcal{F}''$, and Proposition 3.5 yields $\mathbf{x}' = \mathbf{x}''$. □

Remark 3.8. The inverse of the bijection ϕ is given as follows. Given a positive unitary frieze vector $(a_1, \ldots, a_n)$, we compute the corresponding unitary frieze $\mathcal{F}$ by specializing $(x_1, \ldots, x_n) = (a_1, \ldots, a_n)$. By Proposition 3.5, this frieze has a unique cluster $\mathbf{x}'$ such that $\mathcal{F}(\mathbf{x}') = (1, \ldots, 1)$. Then $\phi^{-1}(a_1, \ldots, a_n) = \mathbf{x}'$.

3.4 Example

Thanks to Proposition 3.4, the positive integral frieze vectors (a_1, a_2, a_3) relative to the seed $(x_1, x_2, x_3), 1 \to 2 \leftarrow 3$ are characterized by the condition that the three expressions $\frac{a_2+1}{a_1}, \frac{a_1a_3+1}{a_2}, \frac{a_2+1}{a_3}$ are integers. The 14 frieze vectors (a_1, a_2, a_3) are the following.

$(1,1,1)$ $(1,1,2)$ $(1,2,1)$ $(1,2,3)$ $(1,3,2)$ $(2,1,1)$ $(2,1,2)$ $(2,3,1)$ $(2,3,4)$
$(2,5,2)$ $(3,2,1)$ $(3,2,3)$ $(3,5,3)$ $(4,3,2)$

Equivalently, we can think of the conditions as Diophantine equations in two sets of integers as follows.

$$a_1b_1 = a_2 + 1, \; a_2b_2 = a_1a_3 + 1, \; a_3b_3 = a_2 + 1.$$

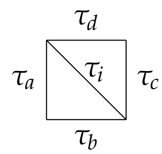


Figure 1: Quadrilateral in the triangulation T.

The vectors (b_1, b_2, b_3), in the same order as the frieze vectors above, are the following.

$(2,2,2)$ $(2,3,1)$ $(3,1,3)$ $(3,2,1)$ $(4,1,2)$ $(1,3,2)$ $(1,5,1)$ $(2,1,4)$ $(2,3,1)$
$(3,1,3)$ $(1,2,3)$ $(1,5,1)$ $(2,2,2)$ $(1,3,2)$

4 Friezes of type $\widetilde{\mathbb{A}}$

In this section, we study the special case of integral friezes of affine Dynkin type $\mathbb{A}$. We show that every positive integral frieze of this type is unitary.

Let Q be a quiver that is mutation equivalent to a quiver Q' of type $\widetilde{\mathbb{A}}_{p,q}$. The cluster algebra $\mathcal{A}(Q)$ is of surface type and the corresponding surface is an annulus with p marked points on one boundary component and q marked points on the other boundary component, see [6]. The cluster variables x_γ in $\mathcal{A}(Q)$ are in bijection with the arcs γ in the annulus. We call a cluster variable x_γ *transjective* if its arc γ has its two endpoints on two different boundary components (bridging arc) and we call the cluster variable x_γ *regular* if the arc γ has both endpoints on the same boundary component (peripheral arc). The terminology transjective versus regular comes from the cluster category $\mathcal{C}_Q$.

Lemma 4.1. *Let $\mathcal{F}\colon \mathcal{A}(Q) \to \mathbb{Z}$ be a positive integral frieze of type $\widetilde{\mathbb{A}}_{p,q}$ and let $\mathbf{x} = (x_1, \dots, x_n)$ be a cluster such that $\mathcal{F}(x) = 1$ for each regular cluster variable $x \in \mathbf{x}$ if any. Let i be such that $\mathcal{F}(x_i) \geq \mathcal{F}(x_j)$ for all j, and suppose that $\mathcal{F}(x_i) > 1$. Let x_i' be the cluster variable obtained from $\mathbf{x}$ by mutation at i. Then $\mathcal{F}(x_i') < \mathcal{F}(x_i)$ and if x_i' is a regular cluster variable then $\mathcal{F}(x_i') = 1$.*

Proof. Let τ_j be the arc corresponding to the cluster variable x_j, so that $T = (\tau_1, \dots, \tau_n)$ is the triangulation corresponding to the cluster $\mathbf{x}$. The mutation in direction i is given by flipping the arc τ_i in T, and the exchange relation in the cluster algebra is of the form

$$x_i x_i' = x_a x_c + x_b x_d \tag{4.1}$$

where τ_i is the diagonal in the quadrilateral in T with sides $\tau_a, \tau_b, \tau_c, \tau_d$ as in Figure 1 some of which may be boundary edges.

Since $\mathcal{F}(x_i) > 1$ but $\mathcal{F}(x) = 1$ for every regular cluster variable $x \in \mathbf{x}$, we have that x_i is transjective. Hence τ_i is a bridging arc, so its endpoints lie on different boundary

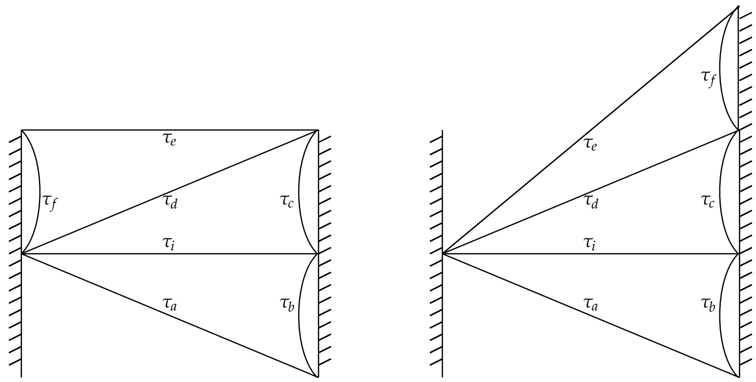


Figure 2: Two possible configurations in T if τ_c is a peripheral arc or a boundary edge.

components. Therefore one of the arcs τ_a, τ_b is bridging and the other is peripheral (or a boundary edge), and one of τ_c, τ_d is bridging and the other is peripheral (or a boundary edge). Assume without loss of generality that τ_a is bridging and consider two cases.

Suppose first that τ_c is a bridging arc. Then the relation (4.1) implies

$$\mathcal{F}(x_i') = (\mathcal{F}(x_a)\mathcal{F}(x_c) + 1)/\mathcal{F}(x_i) \tag{4.2}$$

because the frieze has value 1 on the two regular variables (or boundary edge weights) x_b and x_d. Note that in this case the flipped arc τ_i' is bridging. Recall that $\mathcal{F}(x_a) \leq \mathcal{F}(x_i)$ and $\mathcal{F}(x_c) \leq \mathcal{F}(x_i)$. If $\mathcal{F}(x_a) = \mathcal{F}(x_i)$ then the right hand side of (4.2) would be equal to $\mathcal{F}(x_c) + 1/\mathcal{F}(x_i)$ which is not an integer. Thus $\mathcal{F}(x_a) < \mathcal{F}(x_i)$ and similarly $\mathcal{F}(x_c) < \mathcal{F}(x_i)$. Therefore the right hand side of (4.2) is at most $((\mathcal{F}(x_i) - 1)^2 + 1)/\mathcal{F}(x_i) = \mathcal{F}(x_i) - 2 + (2/\mathcal{F}(x_i))$ which is strictly smaller than $\mathcal{F}(x_i)$, and we are done.

Suppose now that τ_c is peripheral. Then τ_d is bridging and the relation (4.1) implies

$$\mathcal{F}(x_i') = (\mathcal{F}(x_a) + \mathcal{F}(x_d))/\mathcal{F}(x_i) \tag{4.3}$$

Note that in this case the arc τ_i' is peripheral and forms a triangle with the two peripheral arcs τ_b and τ_c. We will show that $\mathcal{F}(x_i') = 1$. Since $\mathcal{F}(x_i)$ is the maximal frieze value in $\mathbf{x}$, equation (4.3) yields $\mathcal{F}(x_i') \leq 2\mathcal{F}(x_i)/\mathcal{F}(x_i) = 2$. If $\mathcal{F}(x_i') = 1$ we are done. Assume therefore that $\mathcal{F}(x_i') = 2$. Then equation (4.3) implies

$$\mathcal{F}(x_a) = \mathcal{F}(x_d) = \mathcal{F}(x_i) \geq 2. \tag{4.4}$$

Consider the quadrilateral in T in which τ_d is the diagonal and denote its sides $\tau_i, \tau_c, \tau_e, \tau_f$ where τ_i, τ_e are bridging arcs and τ_c, τ_f are peripheral, see Figure 2.

Let x_d' be the cluster variable obtained by mutating $\mathbf{x}$ in direction d. Then in the situation of the left picture in Figure 2 we have

$$\mathcal{F}(x_d') = (\mathcal{F}(x_i)\mathcal{F}(x_e) + 1)/\mathcal{F}(x_d) = \mathcal{F}(x_e) + 1/\mathcal{F}(x_i),$$

where the last equality holds by (4.4). But since $\mathcal{F}(x_i) \geq 2$, this expression is not an integer, so we have a contradiction.

Therefore we must be in the situation of the right picture in Figure 2, and we have

$$\mathcal{F}(x'_d) = (\mathcal{F}(x_i) + \mathcal{F}(x_e))/\mathcal{F}(x_d) = 1 + \mathcal{F}(x_e)/\mathcal{F}(x_i),$$

where the last identity holds by (4.4). Since $\mathcal{F}(x_i) \geq \mathcal{F}(x_e)$ and $\mathcal{F}$ is a positive integral frieze, we must have $\mathcal{F}(x_i) = \mathcal{F}(x_e)$ and $\mathcal{F}(x'_d) = 2$.

We have thus shown that if $\mathcal{F}(x'_i) = 2$ then the triangulation T contains a fan of bridging arcs τ_i, τ_d, τ_e and $\mathcal{F}(x'_d) = 2, \mathcal{F}(x_e) = \mathcal{F}(x_i)$. We can now repeat this argument by considering the cluster variable x'_e obtained by mutating $\mathbf{x}$ in direction e, and recursively with every new bridging arc in the fan and we obtain a fan of bridging arcs in T and each arc in this fan has the same frieze value $\mathcal{F}(x_i) \geq 2$. Since T is a triangulation of the annulus, this fan is finite, and the two arcs bounding it correspond to a sink and a source in the quiver Q_T. Mutating at one of those arcs again gives a contradiction as in Figure 2 (left). We have shown that $\mathcal{F}(x'_i)$ cannot be equal to 2, and thus $\mathcal{F}(x'_i) = 1$. □

We are now ready for the main theorem of this section.

Theorem 4.2. *Let Q be a quiver of type $\widetilde{\mathbb{A}}_{p,q}$ and let $\mathcal{F}\colon \mathcal{A}(Q) \to \mathbb{Z}$ be a positive integral frieze. Then $\mathcal{F}$ is unitary.*

Proof. We need to show that there exists a cluster $\mathbf{x}'$ such that $\mathcal{F}(\mathbf{x}') = (1, \ldots, 1)$. Let $\mathbf{x}_0$ be a cluster consisting entirely of transjective cluster variables. Its triangulation T_0 consists entirely of bridging arcs. Then $\mathbf{x}_0 = (x_1, \ldots, x_n)$ is a cluster that satisfies the condition of Lemma 4.1. If $\mathcal{F}(\mathbf{x}_0) = (1, \ldots, 1)$ we are done. Otherwise Lemma 4.1 implies that mutating at a cluster variable x_i with maximal frieze value will produce a cluster $\mathbf{x}_1 = (\mathbf{x}_0 \setminus \{x_i\}) \cup \{x'_i\}$ such that $\mathcal{F}(x'_i) < \mathcal{F}(x_i)$ and if x'_i is regular then $\mathcal{F}(x'_i) = 1$. Therefore, if $\mathcal{F}(\mathbf{x}_1) \neq (1, \ldots, 1)$ then the cluster $\mathbf{x}_1$ also satisfies the hypothesis of Lemma 4.1, and we can repeat this procedure to produce a sequence of clusters $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_s, \ldots$ such that $\mathbf{x}_s = (\mathbf{x}_{s-1} \setminus \{x\}) \cup \{x'\}$ with $\mathcal{F}(\mathbf{x}_s) \neq (1, \ldots, 1)$ and $\mathcal{F}(x') < \mathcal{F}(x)$. Since the frieze is positive integral this process must stop. Thus there is a cluster $\mathbf{x}_t$ such that $\mathcal{F}(\mathbf{x}_t) = (1, \ldots, 1)$. □

4.1 Friezes of type $\widetilde{\mathbb{A}}_{2,1}$

There are precisely two positive integral friezes of type $\widetilde{\mathbb{A}}_{2,1}$ up to symmetry, and by Theorem 4.2 both are unitary. In Figure 3, the cluster $\mathbf{x}$ with $\mathcal{F}(\mathbf{x}) = (1, 1, 1)$ is transjective and in Figure 4 one of the cluster variables in $\mathbf{x}$ is regular. In the figures, we show the values of the friezes on the transjective component of the Auslander–Reiten quiver.

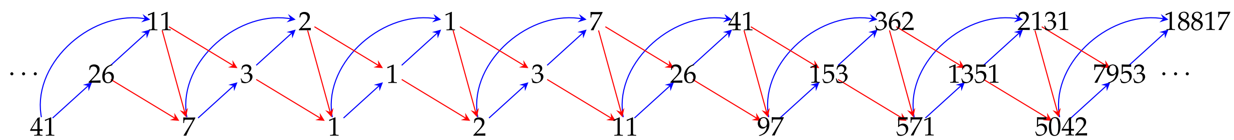


Figure 3: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The two peripheral arcs have frieze values 2 and 3.

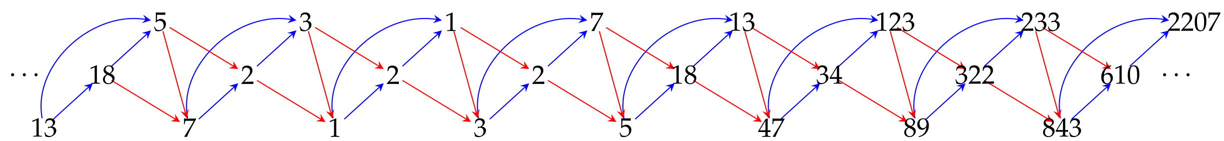
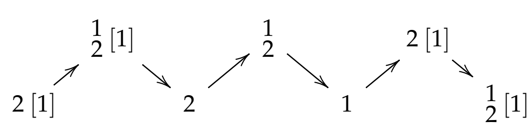


Figure 4: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The two peripheral arcs have frieze values 1 and 5.

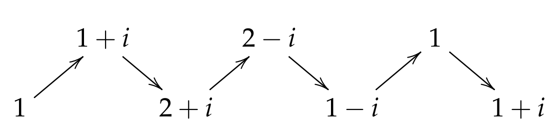
4.2 Further unitarity questions

It was shown in [5] that every positive integral frieze of Dynkin type $\mathbb{A}_n$ is unitary, and by Theorem 4.2, the same is true for affine type $\widetilde{\mathbb{A}}_{p,q}$. It is natural to ask if these results can be extended to friezes with values in other integral domains, for example in quadratic integer rings. However the following example shows that the result already fails over the Gaussian integers.

Example 4.3. Let Q be the quiver $1 \to 2$ with the Auslander–Reiten quiver of $\mathcal{C}_Q$:



Define a non-unitary frieze $\mathcal{F}\colon \mathcal{A}(Q) \to \mathbb{Z}[i]$ by $\mathcal{F}(x_1) = 1$, $\mathcal{F}(x_2) = 1+i$. We can visualize $\mathcal{F}$ in the Auslander–Reiten quiver as we do in Section 2:



4.2.1 Other Dynkin or affine types

For Dynkin types $\mathbb{D}$ and $\mathbb{E}$ there are non-unitary positive integral friezes [9], and these examples give rise to non-unitary positive integral friezes in the affine types $\widetilde{\mathbb{D}}$ and $\widetilde{\mathbb{E}}$.

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