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The s-weak order and s-permutahedra

Cesar Ceballos*[1] and Viviane Pons†[2]

[1]*Faculty of Mathematics, University of Vienna, Vienna, Austria*
[2]*LRI, Univ. Paris-Sud - CNRS - Centrale Supelec - Univ. Paris-Saclay, Orsay, France*

Abstract. We introduce the s-weak order on decreasing trees, a lattice which generalizes the classical weak order on permutations. Restricting this lattice to certain trees gives rise to the s-Tamari lattice, a sublattice which generalizes the Tamari lattice. The s-weak order and the s-Tamari lattice have beautiful underlying geometric structures which we call the s-permutahedron and the s-associahedron. We provide geometric realizations of these objects in dimensions two and three, and conjecture that similar constructions exist in general. We also show that our construction is related to the ν-Tamari lattices of Préville-Ratelle and Viennot.

Résumé. Nous définissons le s-ordre faible sur les arbres décroissants : un treillis qui généralise l'ordre faible sur les permutations. En restreignant à certains arbres, nous obtenons le treillis de s-Tamari, un sous-treillis qui généralise le treillis de Tamari. Ces deux objets ont de jolies structures géomé-triques sous-jacentes que nous nommons le s-permutohèdre et s-associaèdre. Nous donnons les réalisations géométriques de ces objets en dimensions deux et trois et conjecturons que des constructions similaires existent en général. Par ailleurs, notre travaille est lié au treillis de ν-Tamari de Préville-Ratelle et Viennot.

Keywords: Weak order, Permutahedron, Tamari Lattice, Associahedron.

1 Introduction

The weak order is a partial order on the set of permutations of $[n]$, which is defined as the inclusion order of their corresponding inversion sets. This partial order turns out to be a lattice, whose Hasse diagram can be realized geometrically as the edge graph of a polytope called the permutahedron. Restricting the weak order to the set of 231-avoiding permutations gives rise to the Tamari lattice, a well-studied lattice whose Hasse diagram can be realized as the edge graph of another polytope called the associahedron [7]. A beautiful realization of the associahedron obtained by removing certain facets of the permutahedron is described by Hohlweg and Lange in [6].

*cesar.ceballos@univie.ac.at. Sup. by the Austrian Science Foundation FWF, grant F 5008-N15.
†viviane.pons@lri.fr
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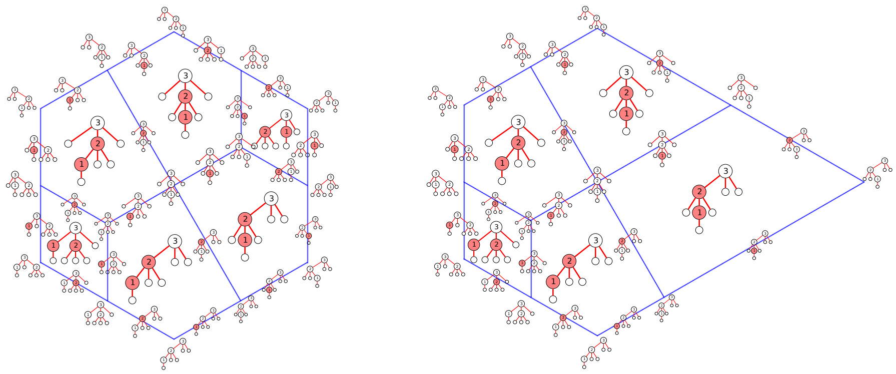


Figure 1: The s-permutahedron and the s-associahedron for $s = (0,2,2)$. Their edge graphs are the Hasse diagrams of the s-weak order and the s-Tamari lattice, respectively.

In this article, we introduce a wider generalization of these concepts indexed by a weak composition $s = (s(1), s(2), \ldots, s(n))$. Our generalizations recover the classical objects for $s = (1,1,\ldots,1)$. The *s-weak order* is a partial order on a set of certain trees which we call *s-decreasing trees*, and is defined as the inclusion order of their *s-tree inversions*. We show that the s-weak order has the structure of a lattice (Theorem 2.6), and give a complete characterization of its cover relations in terms of a simple combinatorial rule on the trees (Theorem 2.7). Restricting the s-weak order to certain trees, which play the role of 231-avoiding permutations, gives rise to a sublattice which we call the s-Tamari lattice (Theorem 3.2). As in the classical case, the s-weak order and the s-Tamari lattice possess beautiful underlying geometric structures which are illustrated in Figure 1. We described them in purely combinatorial terms as potential polyhedral complexes which we call the *s-permutahedron* and the *s-associahedron*. We conjecture that the s-permutahedron can be realized geometrically as a polytopal subdivision of a polytope (Conjecture 1), and that the s-associahedron can be obtained from it by removing certain facets (Conjecture 2)[1]. We show that these two conjectures hold in dimensions two and three ($n = 3$ and $n = 4$ resp.). Some 3-dimensional examples are illustrated in Figure 2.

In [9], Préville-Ratelle and Viennot introduced another generalization of the Tamari lattice called the *ν-Tamari lattice*. Its Hasse diagram was recently proved to be realizable as the edge graph of the *ν-associahedron*, a polyhedral complex induced by an arrangement of tropical hyperplanes introduced in [3]. We prove that the ν-Tamari lattices and the ν-associahedra are isomorphic to the s-Tamari lattices and the s-associahedra, respectively, for specific choices of ν and s (Theorems 3.5 and 5.6). This gives a new perspective

[1] When s contains no zeros except possibly for $s(1)$.

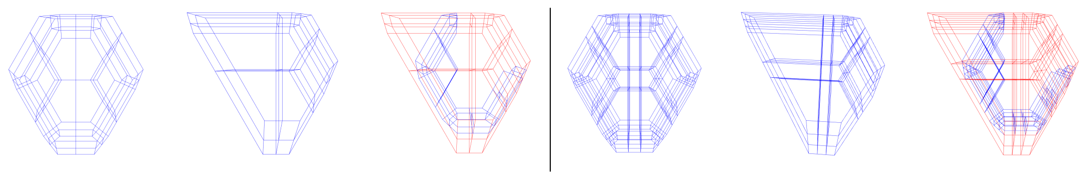


Figure 2: The s-permutahedron and the s-associahedron obtained from it by removing certain facets, for $s = (0,2,2,2)$ and $s = (0,3,3,3)$.

for the geometric realization problem of ν-associahedra.

The s-weak order has been considered in [8] for the case $s = (m, \dots, m)$ under the name "metasylvester lattice". The results in this paper are more general and we study the objects in consideration from a more geometrical point of view.

2 The s-weak order on decreasing trees

We denote by $\mathbb{P}$ the set of positive integers, $\mathbb{N}$ the set of nonnegative integers and for any $n \in \mathbb{P}$ let $[n] := \{1, 2, \dots, n\}$. A *weak composition* is a finite sequence $\mu = (\mu(1), \mu(2), \dots, \mu(n))$ of numbers $\mu(i) \in \mathbb{N}$. For a weak composition μ we define its *weight* $|\mu| := \sum_i \mu(i)$ and its *length* $\ell(\mu) := n$.

2.1 s-decreasing trees

Let $s = (s(1), s(2), \dots, s(n))$ be a weak composition (with possible zero entries). An *s-decreasing tree* is a plane rooted tree with n internal nodes labeled from 1 to n, such that node i has $s(i) + 1$ children (we write the corresponding subtrees $T^i_0, \dots, T^i_{s(i)}$) and all its descendants have smaller labels (see Figure 3). If $s(i) \neq 0$ for all i, the s-decreasing trees are in bijection with 212-avoiding permutations of the word $1^{s(1)}2^{s(2)} \dots n^{s(n)}$ (also known as Stirling s-permutations), see *e.g.* [2, Sec. 3.3.1] and the references therein.

Definition 2.1 (Tree-inversions). *Let T be an s-decreasing tree and $1 \leq x < y \leq n$. The* cardinality *$\#_T(y, x)$ of (y, x) in T is determined as follows: if x belongs T^y_0 or is left of y, then $\#_T(y, x) = 0$; if $x \in T^y_i$ with T^y_i a middle child of y (i.e. with $0 < i < s(y)$), then $\#_T(y, x) = i$; if x belongs to $T^y_{s(y)}$ or is right of y, then $\#_T(y, x) = s(y)$.*

This covers all positions of x relatively to y. If $\#_T(y, x) > 0$, we say that (y, x) is a tree-inversion *of T, and denote by $\mathrm{inv}(T)$ the multi-set of tree-inversions counted with their cardinalities. When the context is clear, we omit T and simply write $\#(y, x)$.*

An example is given in Figure 3. Note that if $s(y) = 0$ then $\#_T(y, x) = 0$ for every $x < y$. If $s = (m, m, \dots, m)$ with $m > 0$, this definition coincides with the one given in [8]. When $s = (1, 1, \ldots, 1)$, the s-decreasing trees are decreasing binary trees, which are in natural bijection with permutations, and tree-inversions are exactly inversions of permutations. We will provide a characterization of the sets of tree-inversions in general below.

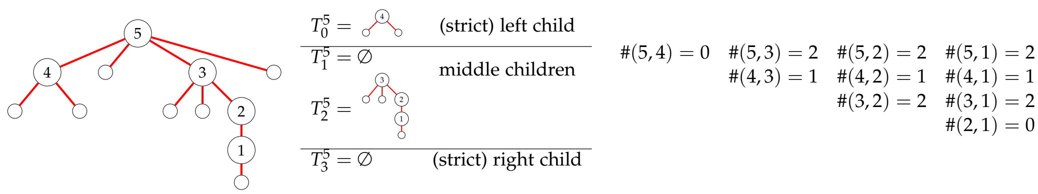


Figure 3: An s-decreasing tree and its tree-inversions for $s = (0, 0, 2, 1, 3)$.

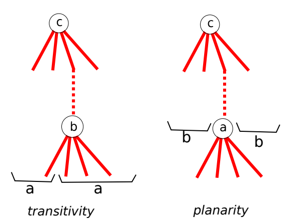
Definition 2.2. *A* multi inversion set *on* $1, \ldots, n$ *is a multi set* I *of inversions* (y, x) *with* $1 \leq x < y \leq n$. *We write* $\#_I(y, x)$ *for the number of occurrences of* (y, x) *in* I. *If there is no occurrence of* (y, x) *in* I *we write* $(y, x) \notin I$ *or equivalently* $\#_I(y, x) = 0$.

Given two multi inversion sets I *and* J, *we say that* I *is* included *in* J *(resp. strictly included) and write* $I \subseteq J$ *(resp.* $I \subset J$*) if* $\#_I(y, x) \leq \#_J(y, x)$ *(resp.* $\#_I(y, x) < \#_J(y, x)$*) for every* $1 \leq x < y \leq n$.

Given a weak composition s *with* $\ell(s) = n$, *we denote by* Σ_s *the* maximal s-inversion set *defined by* $\#_{\Sigma_s}(y, x) = s(y)$ *for all* $1 \leq x < y \leq n$.

Definition 2.3. *Let* s *be a weak composition with* $\ell(s) = n$ *and* $I \subseteq \Sigma_s$ *a multi inversion set. Then* I *is said to be an* s-tree-inversion set *if it satisfies the following two rules:*

- Transitivity: *if* $a < b < c$ *with* $\#(c, b) = i$, *then* $\#(b, a) = 0$ *or* $\#(c, a) \geq i$.
- Planarity: *if* $a < b < c$ *with* $\#(c, a) = i$, *then* $\#(b, a) = s(b)$ *or* $\#(c, b) \geq i$.



Proposition 2.4. *s-decreasing trees are in bijection with s-tree-inversions sets.*

2.2 Lattice definition and cover relations

Tree-inversions are an analogue of permutation inversions. This motivates the following definition of an analogue of the weak order on s-decreasing trees.

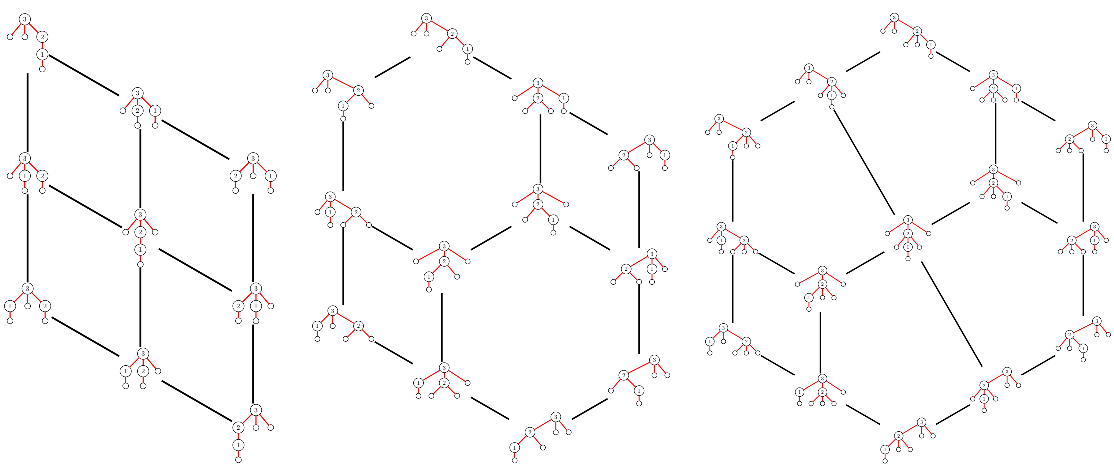
Definition 2.5. *Let* s *be a weak composition and* R *and* T *be two* s*-decreasing trees. We say that* $R \preccurlyeq T$ *if* $\mathrm{inv}(R) \subseteq \mathrm{inv}(T)$ *using the inclusion of multi inversion sets from* Definition 2.2. *We call the relation* $\preccurlyeq$ *the* s-weak order.

It is immediate to see that this defines a partial order, see Figure 4 for some examples. Note that when $s = (1, 1, \dots, 1)$, this is the classical weak order on permutations which is known to be a lattice. In the case where $s = (m, m, \dots, m)$, this is the *metasylvester lattice* defined in [8]. For general s (possibly including some zero entries), the proof of [8] does not apply. We use the following two operations.

- the *union* $I \cup J$ of two multi inversion sets I and J is the smallest multi inversion set, in terms of inclusion, containing both I and J.
- the *transitive closure* I^{tc} of a multi inversion set I is the smallest transitive multi inversion set, in terms of inclusion, containing I.

Theorem 2.6. *For any weak composition s, the s-weak order on s-decreasing trees is a lattice. The join of two s-decreasing trees T and R is determined by*

$$\mathrm{inv}(T \vee R) = (\mathrm{inv}(T) \cup \mathrm{inv}(R))^{\mathrm{tc}}. \tag{2.1}$$



$s = (0, 0, 2)$ $s = (0, 1, 2)$ $s = (0, 2, 2)$

Figure 4: Examples of s-weak lattices.

Indeed, planarity is stable through union and transitive closure. The cover relations of the s-weak-order can be described in terms of certain rotations on trees which we call *s-tree rotations*. Such rotations can be performed along *tree-ascents*, which generalize the classical notion of ascents of a permutation. A *tree-ascent* of an s-decreasing tree T is a pair (a, c) such that:

(i) $a \in T_i^c$ for some $0 \le i < s(c)$ (a is a non-right descendant of c);

(ii) if $a \in T_i^b$ and $c > b > a$, then $i = s(b)$ (a is a right descendant of b);

(iii) if $s(a) > 0$, then $T^a_{s(a)}$ is empty (strict right descendant of a is empty).

Having a tree-ascent (a, c) means that we can move the tree rooted in a along the node c. The tree rooted in a needs to have no strict right child. It will move, taking along its middle children but not its left child. This can be expressed formally by the *s-tree rotation*: we increase the cardinality $\#(c, a)$ by one and take the transitive closure of the obtained multi inversion set. We write the new multi inversion set $(\mathrm{inv}(T) + (c, a))^{\mathrm{tc}}$ and prove that it is indeed a tree-inversion set. The corresponding tree is called the *s-tree rotation* of T along (a, c).

Theorem 2.7. *The cover relations of the s-weak order are in correspondence with s-tree rotations.*

3 The s-Tamari lattice and the ν-Tamari lattice

3.1 The s-Tamari lattice

Definition 3.1. *An s-decreasing tree T is called an* s-Tamari tree *if for any $a < b < c$*

$$\#_T(c, a) \leq \#_T(c, b).$$

In other words, the labels in T^c_i are smaller than all the labels in T^c_j for $i < j$[2]*. The multi set of inversions of an s-Tamari tree is called an s-Tamari inversion set.*

Theorem 3.2. *The collection of s-Tamari trees forms a sublattice of the s-week order, which we call the* s-Tamari lattice*.*

An example is given in Figure 5. Similarly as in the s-weak order, the cover relations of the s-Tamari lattice can be described in terms of certain tree rotations. We say that (a, c) is an *s-Tamari-ascent* of T is a is a non-right (direct) child of c. As before, we can compute the multi inversion set $(\mathrm{inv}(T) + (c, a))^{\mathrm{tc}}$. We prove that this corresponds to an s-Tamari tree which we call the *s-Tamari rotation* of T along (a, c). Basically, a is still moving along c but now it takes along its middle children and strict right child.

Theorem 3.3. *The cover relations of the s-Tamari lattice are in correspondence with s-Tamari rotations.*

3.2 Isomorphism with the ν-Tamari lattice

We will show that the s-Tamari lattice is isomorphic to the ν-Tamari lattice introduced by Préville-Ratelle and Viennot in [9]. We recall the definition of this lattice in terms of certain combinatorial objects called ν-trees following [4].

[2] This is a natural generalization of 231-avoiding permutations for s-decreasing trees.

Let ν be a lattice path in the plane which starts at the origin and consists of a finite number of north and east unit steps. We denote by A_ν the set of lattice points weakly above ν inside the smallest rectangle containing ν. We say that two points $p, q \in A_\nu$ are *ν-incompatible* if and only if p is southwest or northeast of q and the south-east corner of the smallest rectangle containing p and q is weakly above ν. Otherwise, we say that p and q are *ν-compatible*.

Definition 3.4. *A* ν-tree *is a maximal set of pairwise ν-compatible elements in A_ν.*

Each ν-tree can be regarded as a rooted binary tree: each vertex is connected to the next vertex to its right if any, and to the next vertex below it if any. This gives an identification between ν-trees and rooted binary trees [4, Lem. 2.4].

Let p, q, r be three vertices of a ν-tree $\mathcal{T}$, such that q lies below p and to the left of r. Then q may be exchanged by the vertex q' located to the right of p and above r, creating a new ν-Tamari tree $\mathcal{T}'$. In such a case, q is called a *ν-ascent* of T, and exchanging q by q' is called a *ν-Tamari rotation*. The *ν-Tamari lattice* is the lattice of ν-trees whose cover relations are ν-Tamari rotations. An example is shown in Figure 5.

Let $s = (s(1), \ldots, s(n))$ be a weak composition and denote by $\nu(s)$ the lattice path $\nu(s) := NE^{s(n)} \ldots NE^{s(1)}$. From now on we fix $\nu = \nu(s)$. By [4, Lem. 2.11], all ν-trees have exactly $1 + s(1) + \cdots + s(n)$ vertices, which is the number of vertices of every s-tree.

Let T be an s-Tamari tree. We say that its vertices (internal nodes and leaves) are traversed from *right-to-left* if we visit the root r first and then traverse from right-to-left each of the trees T_i^r such that T_i^r is visited before T_j^r for $i > j$. We label the vertices of T with the numbers $0, 1, \ldots, n$, such that a vertex u has label i if the number of internal nodes traversed strictly before u is equal to i. We denote by $\varphi(T)$ the unique ν-tree containing as many vertices with y-coordinate equal to i as there are vertices in T with label i. Such a ν-tree can be uniquely constructed using the right flushing algorithm described in [4, Section 3.2]. See Figure 5 for an illustration.

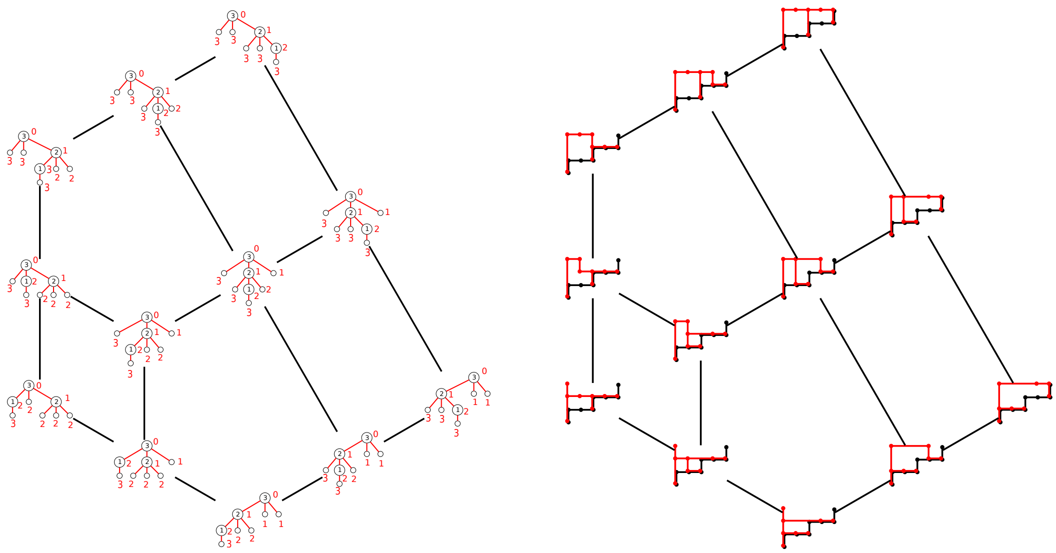
Theorem 3.5. *The map φ is isomorphism between the s-Tamari lattice and the $\nu(s)$-Tamari lattice.*

4 The s-Permutahedron

Let T be an s-decreasing tree and A be a subset of tree-ascents of T. We will denote by $T + A$ the s-decreasing tree with inversion set $(\mathrm{inv}(T) + A)^{\mathrm{tc}}$.

Definition 4.1 (The s-permutahedron)**.** *The* s-permutahedron $\mathrm{Perm}(s)$ *is the complex*[3] *whose faces are pairs (T, A) where T is an s-decreasing tree and A is a subset of tree-ascents of T. The dimension of (T, A) is equal to $|A|$. In particular,*

[3] We have strong indications that it is a polyhedral complex, which we should prove in a longer version of this abstract.



right-to-left traversal $\qquad$ $\nu(s) = NE^2NE^2NE^0$

Figure 5: The s-Tamari lattice and the $\nu(s)$-Tamari lattice for $s = (0, 2, 2)$.

1. *the vertices of* $\mathrm{Perm}(s)$ *are s-decreasing trees T, and*

2. *two trees T and T′ are connected by an edge if they are related by an s-tree rotation.*

The face (T, A) *is contained in* (T', A') *if and only if* $[T, T + A] \subseteq [T', T' + A']$ *as intervals in the s-weak order. See an example in Figure 1.*

Proposition 4.2. *The f-vector of* $\mathrm{Perm}(s)$ *is the coefficient vector of*

$$\sum_T (1 + t)^{\mathrm{asc}(T)},$$

where the sum ranges over all s-decreasing trees T and $\mathrm{asc}(T)$ *denotes the number of s-tree ascents of T.*

The polynomial $E(s) = \sum_T t^{\mathrm{asc}(T)}$ may be regarded as an s-generalization of the Eulerian polynomials, since they coincide for $s = (1, \ldots, 1)$.

The Hasse diagram of the s-permutahedron seems to be realizable as the edge graph of a polytopal subdivision of a polytope. This polytope should be combinatorially isomorphic to the zonotope

$$Z(s) = \sum_{1 \le i < j \le n} s(j)\Delta_{ij}, \tag{4.1}$$

where $\Delta_{ij} = \mathrm{conv}\{e_i, e_j\} \subset \mathbb{R}^n$. In particular, if s has no zeros (except possibly for $s(1)$) then $Z(s)$ is combinatorially an $(n-1)$-dimensional permutahedron.

Conjecture 1. *The s-permutahedron can be realized as a polyhedral subdivision of a polytope which is combinatorially isomorphic to* $Z(s)$.

5 The s-associahedron and the ν-associahedron

5.1 The s-associahedron

Let T be an s-Tamari tree and A be a subset of Tamari-ascents of T. For simplicity, we will denote by $T + A$ the s-Tamari tree with inversion set $(\mathrm{inv}(T) + A)^{\mathsf{tc}}$.

Definition 5.1 (The s-associahedron)**.** *The s-associahedron* $\mathrm{Asso}(s)$ *is the polyhedral complex*[4] *whose faces are pairs* (T, A) *where T is an s-Tamari tree and A is a subset of Tamari-ascents of T. The dimension of* (T, A) *is equal to* $|A|$*. In particular,*

1. *the vertices of* $\mathrm{Asso}(s)$ *are s-Tamari trees T, and*
2. *two s-Tamari trees are connected by an edge if they are related by an s-Tamari rotation.*

The face (T, A) *is contained in* (T', A') *if and only if* $[T, T + A] \subseteq [T', T' + A']$ *as intervals in the s-Tamari lattice. See an example in Figure 1.*

Proposition 5.2. *The f-vector of* $\mathrm{Asso}(s)$ *is the coefficient vector of*

$$\sum_T (1+t)^{\mathrm{tasc}(T)},$$

where the sum ranges over all s-Tamari trees T and $\mathrm{tasc}(T)$ *denotes the number of s-Tamari ascents of T.*

The entries of $\sum_T t^{\mathrm{tasc}(T)}$ may be regarded as s-generalizations of the Narayana numbers. The s-Narayana numbers have already been considered in [3, 2] in this general set up, and in [1] for the special case of rational Catalan combinatorics.

Conjecture 2. *If s has no zeros (except for $s(1)$), there exists a geometric realization of* $\mathrm{Perm}(s)$ *such that* $\mathrm{Asso}(s)$ *can be obtained from it by removing certain facets.*[5]

5.2 Isomorphism with the ν-associahedron

It was shown in [3] that the Hasse diagram of the ν-Tamari lattice can be realized as the edge graph of a polyhedral complex, called the ν-associahedron. We recall its definition following the conventions in [4].

Let ν be a lattice path as before. A *ν-face* is a pairwise ν-compatible subset of A_ν. A ν-face $\mathcal{F}$ is said to be *covering* if it contains the top left corner in A_ν and at least one point in each row and each column.

[4] This follows from Theorem 5.6 and [3, Thm. 5.2].

[5] If s contains zeros other than $s(1)$ then $\mathrm{Asso}(s)$ is not convex (Theorem 5.6 and [3, Cor. 5.13]).

Definition 5.3. *The* ν-associahedron $\mathrm{Asso}(\nu)$ *is the polyhedral complex of covering* ν*-faces ordered by reverse inclusion. The dimension of a covering* ν*-face* $\mathcal{F}$ *is* $\ell(\nu)+1-|\mathcal{F}|$*, where* $\ell(\nu)$ *is the length of* ν*. In particular,*

1. *the vertices of* $\mathrm{Asso}(\nu)$ *are* ν*-trees, and*

2. *two* ν*-trees are connected by an edge if they are related by a* ν*-Tamari rotation.*

Before matching the definitions of s-associahedra and ν-associahedra we need the following lemmas.

Lemma 5.4. *Covering* ν*-faces* $\mathcal{F}$ *are in bijection with pairs* $(\mathcal{T},\mathcal{A})$ *such that* $\mathcal{T}$ *is a* ν*-tree and* $\mathcal{A}$ *is a subset of* ν*-ascents of* $\mathcal{T}$*. The bijection is determined by* $\mathcal{F}=\mathcal{T}\setminus\mathcal{A}$ *and* $\dim(\mathcal{F})=|\mathcal{A}|$*.*

For $a\in\mathcal{A}$ we denote by $\mathcal{T}_a$ the ν-tree obtained from $\mathcal{T}$ by applying a ν-Tamari rotation at the ν-ascent a, and by $\mathcal{T}+\mathcal{A}$ the join of the set $\{\mathcal{T}_a : a\in\mathcal{A}\}$.

Lemma 5.5. *Let* $(\mathcal{T},\mathcal{A})$ *and* $(\mathcal{T}',\mathcal{A}')$ *be two pairs of a* ν*-tree with a subset of* ν*-ascents, and let* $\mathcal{F}=\mathcal{T}\setminus\mathcal{A}$ *and* $\mathcal{F}'=\mathcal{T}'\setminus\mathcal{A}'$ *be their corresponding covering* ν*-faces. Then* $\mathcal{F}\supseteq\mathcal{F}'$ *if and only if* $[\mathcal{T},\mathcal{T}+\mathcal{A}]\subseteq[\mathcal{T}',\mathcal{T}'+\mathcal{A}']$ *as intervals in the* ν*-Tamari lattice.*

The bijection φ between s-Tamari trees and $\nu(s)$-trees described in Section 3.2 extends naturally to a bijection $\overline{\varphi}$ between the faces of the s-associahedron and the faces of the $\nu(s)$-associahedron. For each pair (T,A) of an s-Tamari tree T and a subset A of Tamari-ascents of T, we can associate a pair $(\mathcal{T},\mathcal{A})$ of a ν-tree $\mathcal{T}=\varphi(T)$ and a subsets $\mathcal{A}$ of ν-assents of $\mathcal{T}$ corresponding to A. We denote by $\overline{\varphi}$ the map that sends the pair (T,A) to $(\mathcal{T},\mathcal{A})$. Lemmas 5.4 and 5.5 imply the following result.

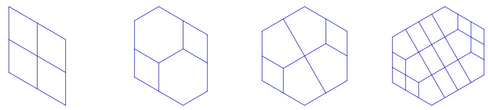
Theorem 5.6. *The map* $\overline{\varphi}$ *is an isomorphism between* $\mathrm{Asso}(s)$ *and* $\mathrm{Asso}(\nu(s))$*.*

6 Geometric realizations in dimensions 2 and 3

There is a natural way of assigning coordinates to each s-decreasing tree. Let $e_{ij}:=e_i-e_j$ for $i<j$, where $e_1,\dots,e_n\in\mathbb{R}^n$ are the standard basis vectors in $\mathbb{R}^n$. Let $s=(s(1),s(2),\dots,s(n))$ be a weak composition, T be an s-decreasing tree and A be a subset of tree-ascents of T. We define

$$v_T=\sum_{i<j}\#_T(j,i)e_{ij}\quad\text{and}\quad F_{(T,A)}=\mathrm{conv}\{v_{T'}:T'\in[T,T+A]\}.$$

For $n=3$ and $s(3)\neq 0$, this gives a 2-dimensional realization of the s-permutahedron in the subspace $\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1+x_2+x_3=0\}\subset\mathbb{R}^3$, see Figure 6. This realization "cuts" a polygon (a hexagon if $s_2\neq 0$ or a quadrilateral if $s(2)=0$) into smaller polygons. Each polygon is convex and corresponds to one facet of the s-permutahedron.



$s = (0,0,2)$ $s = (0,1,2)$ $s = (0,2,2)$ $s = (0,4,3)$

Figure 6: Some geometric realizations of s-permutahedra in dimension 2.

One would hope that this construction directly extends to higher dimensions, but it is not the case. For $n \geq 4$, the convex hull of all v_T's is the zonotope $Z(s)$ from (4.1), which is still cut into identifiable pieces; however, those pieces do not form convex polytopes. We were able to fix this realization in dimension 3 ($n = 4$ and $s(4) \neq 0$), using a procedure illustrated in Figure 7. The first image shows the direct realization obtained by v_T: we notice some bent edges and can identify what we call a "broken pattern" in trees related to those edges. The solution is to push the selected trees into a given direction by a parameter given by a broken pattern itself. In certain compositions s, this push leads to a "collision" (see middle of Figure 7) which again forces us to push certain trees further away. The process can be explicitly described for dimension 3. The new coordinates are given by $\overline{v}_T = \sum_{i<j}(3\,\#_T(j,i) + f_T(j,i))e_{ij}$ where $f_T(j,i) = 0$ for $j \neq 3$ and

$$f_T(3,i) = \begin{cases} 0 & \text{if } \#_T(3,i) = 0, \\ s_3 + (\#_T(4,1) - \#_T(4,3)) + (\#_T(4,2) - \#_T(4,3)) & \text{if } 0 < \#_T(3,i) < s_3, \\ 2s_3 & \text{if } \#_T(3,i) = s_3. \end{cases}$$

See Figure 2 for examples. You can find 3-dimensional animations of these polyhedral subdivisions and more in this webpage[6]. We conjecture that such a construction also exists for higher dimensions.

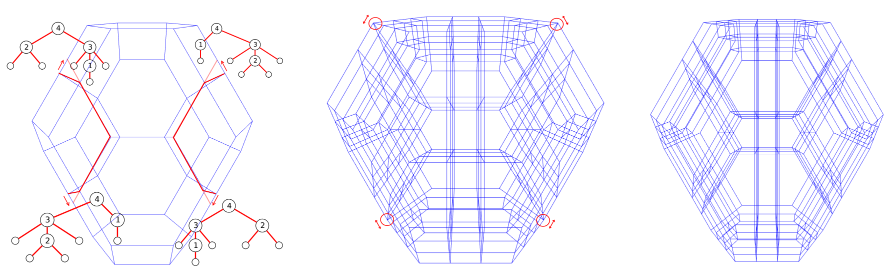


Figure 7: Construction of a 3 dimensional realization.

Once we have a geometric realization of the s-permutahedron, we are able to identify what we call *Tamari-valid* faces and construct a realization of the s-associahedron by

[6] https://www.lri.fr/~pons/static/spermutahedron/

removing faces of the s-permutahedron. This works for both our $2D$ and $3D$ realizations, and is an analogue of a construction in [6] giving rise to Loday's realization of the associahedron. The process is illustrated in Figure 8 in $2D$, see Figure 2 for $3D$ examples.

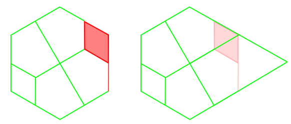


Figure 8: Realization of s-Associahedron for $s = (0, 2, 2)$.

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