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Powers of monomial ideals and the Ratliff-Rush operation

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Abstract. Powers of (monomial) ideals is a subject that still calls attraction in various ways. In this paper we present a nice presentation of high powers of ideals in a certain class in $\mathbb{C}[x_1, ..., x_n]$ and $\mathbb{C}[[x_1, ..., x_n]]$. As an interesting application it leads to an algorithm to compute Ratliff-Rush ideals for that class. The Ratliff-Rush operation itself has several applications, for instance, if I is a regular $\mathfrak{m}$-primary ideal in a local ring (R, m), then the Ratliff-Rush associated ideal $\tilde{I}$ is the unique largest ideal containing I with the same Hilbert polynomial as I.

Keywords: Ratliff-Rush operation, powers of monomial ideals, polynomial rings

1 Introduction

Let R be a commutative Noetherian ring and I a regular ideal in it, that is, an ideal containing a non-zerodivisor. The Ratliff-Rush ideal associated to I is defined as $\tilde{I} = \cup_{k\geq 0}(I^{k+1} : I^k)$. For simplicity we will call it the Ratliff-Rush operation on I, even though it does not preserve inclusion, as shown in [6]. In [5] it is proved that $\tilde{I}$ is the unique largest ideal that satisfies $I^l = \tilde{I}^l$ for all large l. An ideal I is called Ratliff-Rush if $I = \tilde{I}$. Properties of the Ratliff-Rush operation and its interaction with other algebraic operations have been studied by several authors, see [6, 5, 3]. In particular, we would like to mention the following two results. If I is an $\mathfrak{m}$-primary ideal in a local ring $(R, \mathfrak{m})$, then $\tilde{I}$ is the unique largest ideal containing I with the same Hilbert polynomial (the length of (R/I^l) for sufficiently large l) as I. It is also known that the associated graded ring $\oplus_{k\geq 0} I^k/I^{k+1}$ has positive depth if and only if all powers of I are Ratliff-Rush (see [3] for a proof). Recently there have been discovered connections to Castelnuovo-Mumford regularity (see [2]).

In this paper we describe an algorithm for computing the Ratliff-Rush ideal of $\mathfrak{m}$-primary monomial ideals of a certain class (we will call it a class of good ideals), which is a generalization of algorithms described in [4] and [1]: if we restrict to two variables, the ideals $I_{q_1,0}$ and I_{0,q_2}, defined in Section 5, are exactly I_T and I_S, defined in [4] and [1].

In Section 3 we introduce the notion of a good ideal. The idea is as follows: any $\mathfrak{m}$-primary monomial ideal has some $x_1^{d_1}, \ldots, x_n^{d_n}$ as minimal generators and thus defines a (non-disjoint) covering of $\mathbb{N}^n$ with rectangular "boxes" of sizes $d_1, \ldots, d_n$. Then I is called a good ideal if it satisfies the so-called box decomposition principle, namely, if for any positive integer l any minimal generator of I^l belongs to some box $B_{a_1,\ldots,a_n}$ with $a_1 + \ldots + a_n = l - 1$. We also discuss a necessary and a sufficient condition for being a good ideal. From this point we will work with good ideals, unless stated otherwise.

In Section 4 we associate an ideal to each box in the following way: if I is a good ideal and $B_{a_1,\ldots,a_n}$ is some box, then it contains some of the minimal generators of I^l, where $l = a_1 + \ldots + a_n + 1$. Since they are in $B_{a_1,\ldots,a_n}$, they are divisible by $(x_1^{d_1})^{a_1} \cdots (x_n^{d_n})^{a_n}$. Therefore, we can define

$$I_{a_1,\ldots,a_n} := \left\langle \frac{m}{(x_1^{d_1})^{a_1} \cdots (x_n^{d_n})^{a_n}} \mid m \in B_{a_1,\ldots,a_n} \cap G(I^l) \right\rangle.$$

We will conclude this section by showing that

$$I_{a_1,\ldots,a_n} = I^l : \langle (x_1^{d_1})^{a_1} \cdots (x_n^{d_n})^{a_n} \rangle,$$

which immediately implies the following property: if $(a_1, \ldots, a_n) \le (b_1, \ldots, b_n)$, then $I_{a_1,\ldots,a_n} \subseteq I_{b_1,\ldots,b_n}$. We also study the asymptotic behaviour of $I_{a_1,\ldots,a_n}$. Now that we know that $I_{a_1,\ldots,a_n}$ grows when $(a_1, \ldots, a_n)$ grows, and given that ideals can not grow forever, we are expecting some sort of stabilization in $I_{a_1,\ldots,a_n}$ when $(a_1, \ldots, a_n)$ is large enough. In other words, we are expecting some pattern on I^l for large l.

In Section 5 we prove the main theorem of this paper, namely, the following: if I is a good ideal, then $\tilde{I} = I_{q_1,0,\ldots,0} \cap I_{0,q_2,\ldots,0} \cap \ldots \cap I_{0,\ldots,0,q_n}$, where $I_{q_1,0,\ldots,0}$ is the stabilizing ideal of the chain $I_{0,0,\ldots,0} \subseteq I_{1,0,\ldots,0} \subseteq I_{2,0,\ldots,0} \subseteq \ldots$, $I_{0,q_2,\ldots,0}$ is the stabilizing ideal of the chain $I_{0,0,\ldots,0} \subseteq I_{0,1,\ldots,0} \subseteq I_{0,2,\ldots,0} \subseteq \ldots$ and so on. The pattern established in Section 4 plays an important role in the proof of the main theorem.

In Section 6 we show that computation of $I_{0,0,\ldots,q_i,0,\ldots,0}$ is much easier than it seems. In particular, we show that the corresponding chain stabilizes immediately as soon as we have two equal ideals.

Section 7 contains examples and explicit computations of $\tilde{I}$.

2 Preliminaries and notation

Let $R = \mathbb{C}[x_1, ..., x_n], n \ge 2$. We start by listing a few basic properties of monomial ideals in R that will be used later.

1. There is a natural bijection between monomials in $\mathbb{C}[x_1, \ldots, x_n]$ and points in $\mathbb{N}^n$ via $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \leftrightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We say that $(\beta_1, \beta_2, \ldots, \beta_n) \le (\alpha_1, \alpha_2, \ldots, \alpha_n)$ if $\beta_i \le \alpha_i$ for all $i \in \{1, 2, \ldots, n\}$. Clearly, $x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ divides $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ if and only if $(\beta_1, \beta_2, \ldots, \beta_n) \le (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Multiplication of monomials corresponds

to addition of points. We will say that a monomial belongs to a subset of $\mathbb{N}^n$, meaning that the corresponding point belongs to that subset. We will also say that a point belongs to an ideal I, meaning that the corresponding monomial belongs to I.

2. $I : (J_1 + J_2) = (I : J_1) \cap (I : J_2)$, $(I_1 + I_2) : \langle m \rangle = I_1 : \langle m \rangle + I_2 : \langle m \rangle$ and $\langle m_1 \rangle : \langle m_2 \rangle = \left\langle \frac{m_1}{\gcd(m_1, m_2)} \right\rangle$.

Let I be an $\mathfrak{m}$-primary monomial ideal of R, where $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$, that is, for some positive integers $d_1, \ldots, d_n$ we have $\{x_1^{d_1}, \ldots, x_n^{d_n}\} \subset G(I)$. Henceforth, by I we always mean an $\mathfrak{m}$-primary monomial ideal and denote $\mu_i := x_i^{d_i}$, $1 \leq i \leq n$. In this paper we do not consider any polynomials other than monomials since it will always be sufficient to prove statements for monomials only.

Definition 2.1. Let I be an ideal, let $a_1, \ldots, a_n$ be nonnegative integers and denote

$$B_{a_1,\ldots,a_n} := ([a_1 d_1, (a_1+1)d_1] \times \ldots \times [a_n d_n, (a_n+1)d_n]) \cap \mathbb{N}^n.$$

$B_{a_1,\ldots,a_n}$ will be called the **box** with coordinates $(a_1, \ldots, a_n)$, associated to I. Points of the type $(k_1 d_1, \ldots, k_n d_n)$ and the corresponding monomials, where all k_i are nonnegative integers, will be called **corners**.

Note that all minimal generators of I lie in $B_{0,\ldots,0}$.

3 Good and bad ideals

In this section we will introduce the notion of a good ideal, state a necessary and a sufficient condition for being a good ideal and give some examples.

Definition 3.1. We will say that an ideal I satisfies the **box decomposition principle** if the following holds: for every positive integer l, every minimal generator of I^l belongs to some box $B_{a_1,\ldots,a_n}$ such that $a_1 + \ldots + a_n = l - 1$. Ideals satisfying the box decomposition principle will be called **good**, otherwise they will be called **bad**.

Example 3.2. Consider the ideal $I = \langle x^3, y^3, z^3, xyz \rangle$ in $\mathbb{C}[x, y, z]$. Then $x^2 y^2 z^2$ is a minimal generator of I^2, but it only belongs to $B_{0,0,0}$ and $0 + 0 + 0 \neq 1$. Therefore, I is a bad ideal.

Example 3.3. Let $I = \langle x^3, y^3, z^3, x^2 y^2 z^2 \rangle$ in $\mathbb{C}[x, y, z]$. Then

$$G(I^2) = \{x^6, y^6, z^6, x^3 y^3, x^3 z^3, y^3 z^3, x^5 y^2 z^2, x^2 y^5 z^2, x^2 y^2 z^5\}.$$

$$G(I^2) \cap B_{1,0,0} = \{x^6, x^3 y^3, x^3 z^3, x^5 y^2 z^2\},$$

$$G(I^2) \cap B_{0,1,0} = \{y^6, x^3y^3, y^3z^3, x^2y^5z^2\},$$
$$G(I^2) \cap B_{0,0,1} = \{z^6, x^3z^3, y^3z^3, x^2y^2z^5\}.$$

Note that each minimal generator of I^2 belongs to at least one such box. Denote $S_{1,0,0} := G(I^2) \cap B_{1,0,0}$ and similarly $S_{0,1,0} := G(I^2) \cap B_{0,1,0}$ and $S_{0,0,1} := G(I^2) \cap B_{0,0,1}$. We see that $S_{1,0,0} = \mu_1 G(I)$, $S_{0,1,0} = \mu_2 G(I)$, $S_{0,0,1} = \mu_3 G(I)$, that is, $I^2 = \langle S_{1,0,0}, S_{0,1,0}, S_{0,0,1} \rangle = \mu_1 I + \mu_2 I + \mu_3 I$. Geometrically it means that I^2 is minimally generated by all appropriate shifts of I. Clearly, the pattern repeats in all powers of I:

$$I^l = \sum_{l_1+\ldots+l_n=l-1} \mu_1^{l_1} \ldots \mu_n^{l_n} I,$$

that is, I is a good ideal.

Now we are interested in necessary and sufficient conditions for an ideal to be good.

Theorem 3.4. *(A necessary condition) Let I be an ideal in $\mathbb{C}[x_1, ..., x_n]$. If I is a good ideal, then for any minimal generator $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ of I the following holds:*

$$\frac{\alpha_1}{d_1} + \cdots + \frac{\alpha_n}{d_n} \geq 1.$$

The idea of the proof is the following: assume that there is a minimal generator for which the above condition fails, that is, $m = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ with $\frac{\alpha_1}{d_1} + \cdots + \frac{\alpha_n}{d_n} = 1 - \epsilon$, $\epsilon > 0$. Then it is easy to show that the box decomposition principle fails for any $l > \frac{1}{\epsilon}$.

Theorem 3.5. *(A sufficient condition) Let I be an ideal in $\mathbb{C}[x_1, ..., x_n]$. Assume that for any minimal generator $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ of I which is not a corner the following holds:*

$$\frac{\alpha_1}{d_1} + \cdots + \frac{\alpha_n}{d_n} \geq \frac{n}{2}.$$

Then I is a good ideal.

Proof. Let $m_1 = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $m_2 = x_1^{\beta_1} \cdots x_n^{\beta_n}$ with $\frac{\alpha_1}{d_1} + \cdots + \frac{\alpha_n}{d_n} \geq \frac{n}{2}$ and $\frac{\beta_1}{d_1} + \cdots + \frac{\beta_n}{d_n} \geq \frac{n}{2}$. It suffices to show that $m_1 m_2 = \mu_i x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ for some i and with $\frac{\gamma_1}{d_1} + \cdots + \frac{\gamma_n}{d_n} \geq \frac{n}{2}$. Note that $\frac{\alpha_1+\beta_1}{d_1} + \cdots + \frac{\alpha_n+\beta_n}{d_n} \geq n$, thus we must have $\frac{\alpha_i+\beta_i}{d_i} \geq 1$ for some i. We can assume $i = 1$, then $\frac{\alpha_1+\beta_1-d_1}{d_1} + \cdots + \frac{\alpha_n+\beta_n}{d_n} \geq n - 1 \geq \frac{n}{2}$. Setting $\gamma_1 = \alpha_1 + \beta_1 - d_1$ and $\gamma_i = \alpha_i + \beta_i$ for $2 \leq i \leq n$ finishes the proof. □

Remark 3.6. For $n = 2$ the necessary and sufficient conditions are equivalent.

Example 3.7. (A good ideal that does not satisfy the sufficient condition)

Let $I = \langle \mu_1, \mu_2, \mu_3, m \rangle = \langle x^5, y^5, z^5, xyz^4 \rangle \subset \mathbb{C}[x, y, z]$. The ideal satisfies the necessary condition, but not the sufficient one. For examining $G(I^l)$, we first of all notice that $m^5 = x^5y^5z^{20}$ is divisible by $\mu_1\mu_2\mu_3^3 \in I^5$, thus $m \notin G(I^5)$. Therefore, for any l, the minimal generators of I^l will be of the form $\mu_1^{k_1}\mu_2^{k_2}\mu_3^{k_3}m^k$, where $k_1 + k_2 + k_3 + k = l$ and $k \leq 4$. If $k = 0$, the monomial is just a corner and this case is trivial, so let $k \geq 1$. Clearly, such a monomial belongs to a box whose sum of coordinates is $l - 1$ if and only if m^k belongs to a box whose sum of coordinates is $k - 1$. So the only thing we need to check is whether m^k belongs to a box whose sum of coordinates is $k - 1$, $2 \leq k \leq 4$ (this is always true for $k = 1$). We see that $m^2 = x^2y^2z^8 \in B_{0,0,1}$, $m^3 = x^3y^3z^{12} \in B_{0,0,2}$, $m^4 = x^4y^4z^{16} \in B_{0,0,3}$. Therefore, I is a good ideal.

Example 3.8. (A bad ideal that satisfies the necessary condition)

Let $I = \langle x^5, y^5, z^5, x^2y^2z^2 \rangle \subset \mathbb{C}[x, y, z]$. The ideal satisfies the necessary condition, but not the sufficient one. We see that $x^4y^4z^4$ is a minimal generator of I^2 and it only belongs to $B_{0,0,0}$. Since $0 + 0 + 0 \neq 1$, I is a bad ideal.

We would also like to point out that for any given ideal there exists a way to determine whether it is good or bad, but we do not know of any characterisation.

4 Ideals inside boxes, their connection to each other and asymptotic behaviour

Definition 4.1. Let I be a good ideal and $a_1, \ldots, a_n$ nonnegative integers. We define

$$I_{a_1,\ldots,a_n} := \left\langle \frac{m}{\mu_1^{a_1} \cdots \mu_n^{a_n}} \mid m \in G(I^l) \cap B_{a_1,\ldots,a_n} \right\rangle,$$

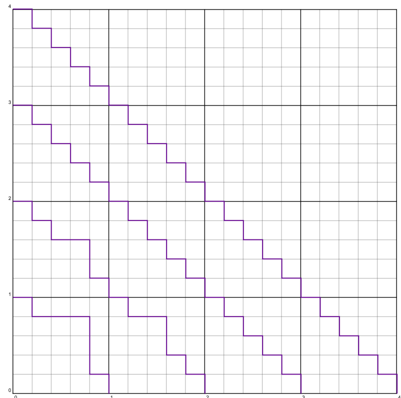
where $l = a_1 + \ldots + a_n + 1$. Note that this a minimal generating set of $I_{a_1,\ldots,a_n}$.

Example 4.2. Let $I = \langle x^5, y^5, xy^4, x^4y \rangle \subset \mathbb{C}[x, y]$. I is a good ideal by the sufficient condition. The picture below represents powers of I up to I^4.

Consider the box $B_{1,0}$. Then

$$G(I^2) \cap B_{0,1} = \{x^5y^5, x^6y^4, x^8y^2, x^9y, x^{10}\}.$$

Therefore, $I_{1,0} = \langle y^5, xy^4, x^3y^2, x^4y, x^5 \rangle$. Geometrically, this means viewing monomials in $B_{1,0}$ as if the smallest corner of $B_{1,0}$ was the origin. In this particular example we have $I_{0,0} = I$, $I_{1,0} = \langle y^5, xy^4, x^3y^2, x^4y, x^5 \rangle$, $I_{0,1} = \langle y^5, xy^4, x^2y^3, x^4y, x^5 \rangle$, $I_{a,b} = \langle y^5, xy^4, x^2y^3, x^3y^2, x^4y, x^5 \rangle$ for all other (a, b).



It is easy to show that if I is a good ideal, then any corner $\mu_1^{k_1}\cdots\mu_n^{k_n}$ is a minimal generator of $I^{k_1+\ldots+k_n}$, therefore, $\left\{\mu_j\prod_{i=1}^n\mu_i^{a_i}\mid 1\le j\le n\right\}\subseteq I^l\cap B_{a_1,\ldots,a_n}$, where $l=a_1+\ldots+a_n+1$ and therefore $\{\mu_1,\ldots,\mu_n\}\subseteq G(I_{a_1,\ldots,a_n})$ for all $a_1,\ldots,a_n$.

Proposition 4.3. *Let I be a good ideal and $a_1,\ldots,a_n$ nonnegative integers. Then*

$$I_{a_1,\ldots,a_n}=I^l:\langle\mu_1^{a_1}\cdots\mu_n^{a_n}\rangle,$$

where $l=a_1+\ldots+a_n+1$.

Proof. It is clear from the definition that $I_{a_1,\ldots,a_n}\subseteq I^l:\langle\mu_1^{a_1}\cdots\mu_n^{a_n}\rangle$. For the other inclusion, let $m\in I^l:\langle\mu_1^{a_1}\cdots\mu_n^{a_n}\rangle$. Then $m\mu_1^{a_1}\cdots\mu_n^{a_n}\in I^l$, that is, $m\mu_1^{a_1}\cdots\mu_n^{a_n}$ is a multiple of some $g\in G(I^l)$, say, $m\mu_1^{a_1}\cdots\mu_n^{a_n}=gg_1$. Being a minimal generator of I^l, g belongs to some box, say, $B_{b_1,\ldots,b_n}$ with $b_1+\ldots+b_n=l-1=a_1+\ldots+a_n$. If $(a_1,\ldots,a_n)=(b_1,\ldots,b_n)$, then m is a multiple of $\frac{g}{\mu_1^{a_1}\cdots\mu_n^{a_n}}$, which is a generator of $I_{a_1,\ldots,a_n}$ and thus we are done. If $(a_1,\ldots,a_n)\neq(b_1,\ldots,b_n)$, then there is some $a_i<b_i$. Without loss of generality, we assume that $a_1<b_1$. Then the right hand side of $m\mu_1^{a_1}\cdots\mu_n^{a_n}=gg_1$ is divisible by $\mu_1^{b_1}$, thus m is divisible by μ_1, and μ_1 is a minimal generator of $I_{a_1,\ldots,a_n}$ by the discussion before this proposition. Therefore, $m\in I_{a_1,\ldots,a_n}$. □

Corollary 4.4. *Let I be a good ideal and let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be nonnegative integers such that $(a_1,\ldots,a_n)\le(b_1,\ldots,b_n)$. Then $I_{a_1,\ldots,a_n}\subseteq I_{b_1,\ldots,b_n}$.*

Now we know that $I_{a_1,\ldots,a_n}$ grows as $(a_1,\ldots,a_n)$ grows. Since $I_{a_1,\ldots,a_n}$ can not increase forever, one expects some pattern on high powers of I, which is indeed the case.

Definition 4.5. Let $a_1,\ldots,a_n$ be nonnegative integers. We will use the following notation:

$C_{\underline{a_1},\underline{a_2},\ldots,\underline{a_k},a_{k+1},a_{k+2},\ldots,a_n}:=\{(b_1,\ldots,b_n)\in\mathbb{N}^n\mid b_1=a_1,\ldots,b_k=a_k,b_{k+1}\ge a_{k+1},\ldots,b_n\ge a_n\}.$

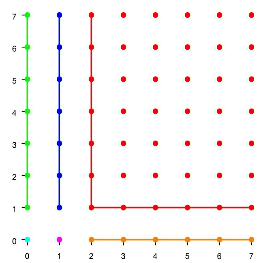
We will use a similar notation for any configuration of fixed and non-fixed coordinates. Sets of this type will be called **cones**, for any cone the number of non-fixed coordinates will be called its **dimension** and $(a_1,\ldots,a_n)$ will be called its **vertex**. Note that $\mathbb{N}^n=C_{0,0,\ldots,0}$.

Definition 4.6. Let $a_1, \dots, a_n$ be nonnegative integers. By $A_{a_1,\dots,a_n}$ we denote the set of all cones that satisfy the following conditions:

1. if $(b_1, \dots, b_n)$ is the vertex of the cone, then $b_i \leq a_i$ for all $1 \leq i \leq n$;
2. for all $1 \leq i \leq n$ the following holds: if $b_i = a_i$, then b_i is not underlined and if $b_i < a_i$, then b_i is underlined.

Note that the unique cone of dimension n in $A_{a_1,\dots,a_n}$ is $C_{a_1,\dots,a_n}$

Example 4.7. Let $n = 2$, $a_1 = 2$, $a_2 = 1$. Then $A_{2,1} = \{C_{\underline{0},\underline{0}}, C_{\underline{0},1}, C_{\underline{1},\underline{0}}, C_{\underline{1},1}, C_{2,\underline{0}}, C_{2,1}\}$



The picture above represents the six cones from $A_{2,1}$. The boundary lines are only drawn for better visibility. Clearly, the number of boundary lines equals the dimension of the cone.

Lemma 4.8. *Let $a_1, \dots, a_n$ be nonnegative integers. Then cones in $A_{a_1,\dots,a_n}$ form a disjoint covering of $\mathbb{N}^n$.*

The previous lemma can be restated in a more general context:

Theorem 4.9. *Given any cone C in $\mathbb{N}^n$ of dimension k and a point $\boldsymbol{a} \in C$, we can decompose C into a disjoint union of finitely many cones, where exactly one cone has dimension k and vertex $\boldsymbol{a}$, and all other cones have strictly lower dimensions.*

Example 4.10. Let $n = 5$ and consider $C_{\underline{5},7,\underline{4},2,\underline{3}}$. Consider $(a_1, \dots, a_5) = (5, 9, 4, 3, 3) \in C_{\underline{5},7,\underline{4},2,\underline{3}}$. The first, the third and the fifth coordinates are fixed once and forever, that is, all cones will have the form $C_{\underline{5},?,\underline{4},?,\underline{3}}$. We are left with the second and the fourth coordinate, that is, $(7, 2)$ for the cone and $(9, 3)$ for the point. Shifting in the negative direction by $(7, 2)$, we will get $(0, 0)$ and $(2, 1)$ respectively. Thus it is enough to find the decomposition of $\mathbb{N}^2$ with respect to $(2, 1)$, which has been done in Example 4.7. We obtained $A_{2,1} = \{C_{\underline{0},\underline{0}}, C_{\underline{0},1}, C_{\underline{1},\underline{0}}, C_{\underline{1},1}, C_{2,\underline{0}}, C_{2,1}\}$. Shifting in the positive direction by $(7, 2)$ gives us $\{C_{\underline{7},\underline{2}}, C_{\underline{7},3}, C_{\underline{8},\underline{2}}, C_{\underline{8},3}, C_{9,\underline{2}}, C_{9,3}\}$ and inserting back the first, the third and the fifth coordinates gives us $\{C_{\underline{5},\underline{7},\underline{4},\underline{2},\underline{3}}, C_{\underline{5},\underline{7},\underline{4},3,\underline{3}}, C_{\underline{5},\underline{8},\underline{4},\underline{2},\underline{3}}, C_{\underline{5},\underline{8},\underline{4},3,\underline{3}}, C_{\underline{5},9,\underline{4},\underline{2},\underline{3}}, C_{\underline{5},9,\underline{4},3,\underline{3}}\}$. Therefore, $C_{\underline{5},7,\underline{4},2,\underline{3}}$ is a disjoint union of these six cones.

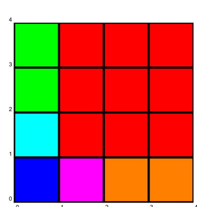
Now we will use these results on monomial ideals. Let I be a good ideal. Then for any vector of nonnegative integers $(a_1, \ldots, a_n)$ we have defined a box $B_{a_1,\ldots,a_n}$ and the corresponding ideal $I_{a_1,\ldots,a_n}$. There is a bijection between points in $\mathbb{N}^n$ and boxes/ideals; recall that if $(a_1, \ldots, a_n) \le (b_1, \ldots, b_n)$, then $I_{a_1,\ldots,a_n} \subseteq I_{b_1,\ldots,b_n}$ by Corollary 4.4.

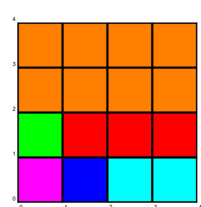
Theorem 4.11. *For any good ideal I there exists a finite coloring of $\mathbb{N}^n$ such that if $(a_1, \ldots, a_n)$ has the same color as $(b_1, \ldots, b_n)$, then $I_{a_1,\ldots,a_n} = I_{b_1,\ldots,b_n}$ and for each color the set of points with this color forms a cone.*

Proof. We use induction on the highest dimension of uncolored cones. We are starting with an n-dimensional cone $\mathbb{N}^n$. We will show how to obtain finitely many cones of strictly lower dimensions, each of which will then be treated similarly in a recursive way. First of all, note that it is possible to find a point $(a_1, \ldots, a_n)$ such that the following holds: if $(b_1, \ldots, b_n) \ge (a_1, \ldots, a_n)$, then $I_{a_1,\ldots,a_n} = I_{b_1,\ldots,b_n}$. Indeed, if we assume the converse, then for every point of $\mathbb{N}^n$ there exists a strictly larger point that corresponds to a strictly larger ideal, therefore, we can build an infinite chain of strictly increasing ideals, which is impossible by Noetherianity of the polynomial ring. So existence of such a point $(a_1, \ldots, a_n)$ is justified. Then from the Theorem 4.9, $\mathbb{N}^n$ can be covered with a disjoint union of (finitely many) cones in $A_{a_1,\ldots,a_n}$. The unique n-dimensional cone in $A_{a_1,\ldots,a_n}$ is $C_{a_1,\ldots,a_n}$ and, as we have just figured out, we may paint all points in this cone with the same color. Now we are left with a disjoint union of cones of dimension at most $n-1$ which need to be painted and we apply induction on each of them, lowering the maximal dimension by 1 again. Since it is a finite process, in the end we will obtain a finite coloring of $\mathbb{N}^n$. □

We remark that the coloring described above is not unique since it depends on the choice of $(a_1, \ldots, a_n)$ and its lower dimensional analogues.

Example 4.12. Let I be the ideal in Example 4.2. We can choose $(a_1, a_2) = (1, 1)$ since $I_{b_1,b_2} = I_{1,1}$ for all $b_1 \ge 1$ and $b_2 \ge 1$. Then $\mathbb{N}^2$ is a disjoint union of $C_{1,1}$, $C_{\underline{0},1}$, $C_{1,\underline{0}}$ and $C_{\underline{0},\underline{0}}$. Now consider $C_{\underline{0},1}$. We see that $I_{0,b} = I_{0,2}$ for all $b \ge 2$. Therefore, we consider the decomposition of $C_{\underline{0},1}$ with respect to $(0, 2)$: $C_{\underline{0},1}$ is a disjoin union of $C_{\underline{0},2}$ and $C_{\underline{0},\underline{1}}$. Similarly, $C_{1,\underline{0}}$ is a disjoint union of $C_{2,\underline{0}}$ and $C_{\underline{1},\underline{0}}$. The left picture below describes the coloring we have just discussed. The picture on the right describes another possible coloring if, for instance, we choose $(a_1, a_2) = (0, 2)$.





Given a good ideal I, any coloring as in Theorem 4.11 represents a finite disjoint union of cones. Each cone has a vertex. Let L denote the maximum of sums of coordinates of these vertices. This number depends on I and on the coloring we choose, but we will not put any additional indices: as soon as we found some coloring (which exists according to Theorem 4.11), we simply work with it henceforth. For example, for both colorings in the picture above we have $L = 2$. The geometric meaning of this number is the following: starting from I^{L+1}, we know exactly how powers of I look like, given that we know the coloring. For instance, for the left coloring in the picture above we know that every power of I starting from I^3 consists of a green box, an orange box and several red boxes and we exactly know where each of them is. This means, there is a pattern on high powers of I, and this is a key point for finding the Ratliff-Rush closure of I.

5 The main result

Now we are ready to prove our main theorem, but first we need a preliminary lemma.

Lemma 5.1. *Let I be a good ideal and let Q be any nonnegative integer. Then there exists a number $L(Q)$ such that for any $l \geq L(Q)$ the following holds: for every minimal generator m of I^l there is an i such that $m = m'\mu_i^Q$ and m' is a minimal generator of I^{l-Q}.*

Proof. If $Q = 0$, the claim is trivial. Let $Q > 0$ and let L be the number defined in the end of Section 4. Take $L(Q) = L + nQ - n + 2$ and let $l \geq L(Q)$. Let m be a minimal generator of I^l, then it belongs to some box $B_{b_1,\ldots,b_n}$ with $b_1 + \ldots + b_n = l - 1 \geq L + nQ - n + 1$. We also know that $(b_1, \ldots, b_n)$ belongs to one of the cones from our coloring; assume that the vertex of this cone is $(a_1, \ldots, a_n)$ (some coordinates are underlined, some are not underlined). Now we want to find a coordinate b_i such that $(b_1, \ldots, b_{i-1}, b_i - Q, b_{i+1}, \ldots, b_n)$ belongs to the same cone. Assume that it is not possible. Then it follows that $b_1 - Q \leq a_1 - 1, \ldots, b_n - Q \leq a_n - 1$. These inequalities yield a contradiction $L < b_1 + \ldots + b_n - nQ + n \leq a_1 + \ldots + a_n \leq L$, where the last inequality follows from the definition of L. So we can find an index i such that $b_i - Q \geq a_i$ (in particular, this implies that a_i is not underlined). Without loss of generality we assume that $i = 1$. That means, $(b_1, \ldots, b_n)$ and $(b_1 - Q, b_2, \ldots, b_n)$ are both in the same cone. This implies that their colors are equal, which means $I_{b_1,\ldots,b_n} = I_{b_1-Q,b_2,\ldots,b_n}$. In other words, the set of monomials in $B_{b_1,\ldots,b_n} \cap G(I^l)$ coincides with the set of monomials in $B_{b_1-Q,b_2,\ldots,b_n} \cap G(I^{l-Q})$ up to a shift by μ_1^Q. Therefore, if $m \in B_{b_1,\ldots,b_n}$ is a minimal generator of I^l, then $\frac{m}{\mu_1^Q} \in B_{b_1-Q,b_2,\ldots,b_n}$ is a minimal generator of I^{l-Q}, as desired. □

Now let us consider the following line of boxes which is in bijection with nonnegative integer points on the x-axis: $B_{0,0,\ldots,0}$, $B_{1,0,\ldots,0}$, $B_{2,0,\ldots,0}$ etc. Let $B_{q_1,0\ldots,0}$ be the stabilizing box of this sequence in a sense that if $t \geq q_1$, then $I_{t,0,\ldots,0} = I_{q_1,0,\ldots,0}$. Similarly, considering lines of boxes going along the other coordinate axes, we will get $q_2, q_3, \ldots, q_n$. Denote $q := \max\{q_1, \ldots, q_n\}$.

Theorem 5.2. *Let I be a good ideal, let L and q_i be as above. Then $\tilde{I} = I_{q_1,0,\ldots,0} \cap I_{0,q_2,\ldots,0} \cap \ldots \cap I_{0,\ldots,0,q_n}$.*

Proof. $\subseteq$ Let $l \geq q$. We will show that $I^{l+1} : I^l \subseteq I_{q_1,0,\ldots,0} \cap I_{0,q_2,\ldots,0} \cap \ldots \cap I_{0,\ldots,0,q_n}$. In fact, we will show that $I^{l+1} : I^l \subseteq I_{q_1,0,\ldots,0}$, other inclusions are analogous. Since $I^{l+1} : I^l \subseteq I^{l+1} : \langle \mu_1^l \rangle$, it is sufficient to show that $I^{l+1} : \langle \mu_1^l \rangle \subseteq I_{q_1,0,\ldots,0}$. By Proposition 4.3, $I^{l+1} : \langle \mu_1^l \rangle = I_{l,0,\ldots,0}$ which equals $I_{q_1,0,\ldots,0}$, given the way $I_{q_1,0,\ldots,0}$ was defined and given that $l \geq q \geq q_1$. Therefore, everything follows.

$\supseteq$ Let $m \in I_{q_1,\ldots,0} \cap I_{0,q_2,\ldots,0} \cap \ldots \cap I_{0,\ldots,0,q_n}$, let $l \geq L(q) = L + nq - n + 2$ (as in Lemma 5.1). We will show that for every $m_l \in I^l$ we have $mm_l \in I^{l+1}$. It is enough to consider m_l to be minimal generators of I^l. First of all, from Lemma 5.1 we know that we can factor out some μ_i^q from m_l and get a minimal generator of I^{l-q}, that is, $m_l = \mu_i^q m_{l-q}$ for some index i and m_{l-q} a minimal generator in I^{l-q}. Also, since m belongs (in particular) to $I_{0,\ldots,0,q_i,0\ldots,0} = I^{q_i+1} : \langle \mu_i^{q_i} \rangle$, it means, $m\mu_i^{q_i} \in I^{q_i+1}$. Therefore, $mm_l = m\mu_i^{q_i}\mu_i^{q-q_i}m_{l-q} \in I^{l+1}$ since $m\mu_i^{q_i} \in I^{q_i+1}$, $\mu_i^{q-q_i} \in I^{q-q_i}$, $m_{l-q} \in I^{l-q}$. □

6 Explicit computation of $I_{0,\ldots,0,q_i,0,\ldots,0}$

We have seen that, given a good ideal I, its Ratliff-Rush closure is computed as $\tilde{I} = I_{q_1,0,\ldots,0} \cap I_{0,q_2,\ldots,0} \cap \ldots \cap I_{0,\ldots,0,q_n}$. Therefore, we would like to know more about $I_{0,\ldots,0,q_i,0\ldots,0}$. Let $i = 1$, other cases are analogous. So far we only know that $I_{t,0,\ldots,0\ldots,0} = I^{t+1} : \langle \mu_1^t \rangle$. Computation of I^t might take much time if t is large enough. In addition, we do not know yet at which moment the line has stabilized. So far the process seems more complicated than it is. We will state a few remarks to make this computation easier.

Remark 6.1. If I is a good ideal, then $I_{t+1,0,\ldots,0} = (I_{t,0\ldots,0} \cdot I) : \langle \mu_1 \rangle$ for all $t \geq 0$.

According to Remark 6.1, we have

$$I_{t+1,0,\ldots,0} = \left\langle \frac{fm}{\gcd(fm, \mu_1)} \mid f \in G(I_{t,0,\ldots,0}), m \in G(I) \right\rangle.$$

Let $f \in G(I_{t,0,\ldots,0})$ and $m \in G(I)$. Write $fm = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. If $\alpha_i \geq d_i$ for any $2 \leq i \leq n$, then $\frac{fm}{\gcd(fm,\mu_1)}$ is a multiple of $\mu_i \in G(I_{t+1,0,\ldots,0})$. Therefore, in the above formula we may force that $deg_{x_i}(fm) < d_i$ for $2 \leq i \leq n$. Moreover, consider $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mu_1^t = f\mu_1^t m \in I^{t+2}$ since $f\mu_1^t \in I^{t+1}$ according to Proposition 4.3. Since $\alpha_i < d_i$ for all $2 \leq i \leq n$, we must have $\alpha_1 \geq d_1$, otherwise $f\mu_1^t m$ would belong to a box with the sum of coordinates at most t. Thus $\gcd(fm, \mu_1) = \mu_1$. Therefore, we conclude that

$$I_{t+1,0,\ldots,0} = \left\langle \frac{fm}{\mu_1} \mid f \in G(I_{t,0,\ldots,0}), m \in G(I), deg_{x_i}(fm) < d_i \text{ for } 2 \leq i \leq n \right\rangle.$$

Remark 6.2. If $I_{t,0,\ldots,0} = I_{t+1,0,\ldots,0}$, then the line has stabilized, that is, $I_{k,0,\ldots,0} = I_{t,0,\ldots,0}$ for all $k \geq t$. This is a direct corollary of Remark 6.1.

Remark 6.3. Assume that we have computed $I_{t,0,\ldots,0}$ and $I_{t+1,0,\ldots,0}$ and let $E_t := G(I_{t,0,\ldots,0})$ and $F_{t+1} := \{$minimal generators of $I_{t+1,\ldots,0}$ which are not in $I_{t,\ldots,0}\}$. Clearly, E_{t+1} is the reduced union of E_t and F_{t+1}. From Remark 6.1 we remember that $I_{t+2,0,\ldots,0} = (I_{t+1,0,\ldots,0} \cdot I) : \langle \mu_1 \rangle = (\langle E_t \cup F_{t+1} \rangle \cdot I) : \langle \mu_1 \rangle = ((I_{t,\ldots,0} + \langle F_{t+1} \rangle) \cdot I) : \langle \mu_1 \rangle = (I_{t,\ldots,0} \cdot I + \langle F_{t+1} \rangle \cdot I) : \langle \mu_1 \rangle = (I_{t,\ldots,0} \cdot I) : \langle \mu_1 \rangle + (\langle F_{t+1} \rangle \cdot I) : \langle \mu_1 \rangle = I_{t+1,0,\ldots,0} + (\langle F_{t+1} \rangle \cdot I) : \langle \mu_1 \rangle$. Therefore, we conclude that minimal generators of $I_{t+2,0,\ldots,0}$ which are not in $I_{t+1,0,\ldots,0}$ (our future F_{t+2}) could only be among $(\langle F_{t+1} \rangle \cdot I) : \langle \mu_1 \rangle$, that is, only new monomials from the previous iteration can give rise to new monomials in the next iteration. Therefore, in order to compute F_{t+2} we need to compute $(\langle F_{t+1} \rangle \cdot I) : \langle \mu_1 \rangle$, reduce this set and throw away monomials that are already in E_{t+1}. We can start with $E_{-1} = \emptyset$, $F_0 = G(I)$.

Remark 6.4. We can exclude all μ_i from all sets and it will not affect the algorithm. In other words, we can replace $G(I)$ by $P(I) := G(I) \backslash \{\mu_1, \ldots, \mu_n\}$ everywhere and then include the corners in the very end of the algorithm. We have already discussed why we can exclude μ_i for $2 \leq i \leq n$ (we want $deg_{x_i}(fm) < d_i$ for $2 \leq i \leq n$). We can also exclude μ_1 since multiplying and then dividing by μ_1 does not give us any new monomials.

7 Examples

Example 7.1. Let $I = \langle \mu_1, \mu_2, \mu_3, m_1, m_2, m_3 \rangle = \langle x^{29}, y^{29}, z^{29}, x^{28}y^8z^8, x^8y^{28}z^8, x^8y^8z^{28} \rangle \subset \mathbb{C}[x, y, z]$. Since I satisfies the sufficient condition, it is a good ideal. Computations in Singular show that

$$I^2 : I = I + \langle x^{27}y^{27}z^{27} \rangle,$$
$$I^3 : I^2 = I^4 : I^3 = I + \langle x^{26}y^{27}z^{27}, x^{27}y^{26}z^{27}, x^{27}y^{27}z^{26} \rangle,$$
$$I^5 : I^4 = I^6 : I^5 = \cdots = I^{10} : I^9 = I + \langle x^{26}y^{26}z^{26} \rangle.$$

It is natural to conjecture that $\tilde{I} = I + \langle x^{26}y^{26}z^{26} \rangle$. Now let us see what we get if we apply the algorithm above. We start with $E_{-1} = \emptyset$, $F_0 = P(I) = \{m_1, m_2, m_3\}$. Then we obtain E_0 by reducing $E_{-1} \cup F_0$, that is, $E_0 = P(I)$. In order to compute F_1, we take all products of $F_0 = P(I)$ with $P(I)$, keeping in mind that $y-$ and $z-$ coordinates of each product need to be less than 29, and divide each such product by μ_1. The only such monomial is $\frac{m_1^2}{\mu_1} = \frac{x^{56}y^{16}z^{16}}{x^{29}} = x^{27}y^{16}z^{16}$. This monomial is not in $\langle E_0 \rangle$, therefore, we add it to our set F_1 (and this set is already reduced). Thus $E_0 = P(I)$, $F_1 = \{x^{27}y^{16}z^{16}\}$. Now $E_1 = E_0 \cup F_1 = P(I) \cup \{x^{27}y^{16}z^{16}\}$ (this union is already reduced), and in order to compute F_2 we need to multiply $x^{27}y^{16}z^{16}$ with monomials from $P(I)$ (keeping in mind the condition on $y-$ and $z-$ coordinates) and divide the products by μ_1. The only possible monomial is $\frac{x^{27}y^{16}z^{16} \cdot m_1}{\mu_1} = x^{26}y^{24}z^{24}$. This monomial is not in $\langle E_1 \rangle$, therefore, $F_2 = \{x^{26}y^{24}z^{24}\}$.

$E_2 = E_1 \cup F_2 = P(I) \cup \{x^{27}y^{16}z^{16}, x^{26}y^{24}z^{24}\}$ (this set is already reduced) and if we try to compute F_3, we see that we can not get any new monomials. Therefore, $F_3 = \emptyset$ and the stabilizing point is $I_{2,0,0} = \langle E_2 \cup \{\mu_1, \dots, \mu_n\}\rangle = I + \langle x^{27}y^{16}z^{16}, x^{26}y^{24}z^{24}\rangle$. By symmetry, $I_{0,2,0} = I + \langle x^{16}y^{27}z^{16}, x^{24}y^{26}z^{24}\rangle$ and $I_{0,0,2} = I + \langle x^{16}y^{16}z^{27}, x^{24}y^{24}z^{26}\rangle$. According to the theorem, $\tilde{I} = I_{2,0,0} \cap I_{0,2,0} \cap I_{0,0,2} = I + \langle x^{26}y^{26}z^{26}\rangle$, just as expected.

Example 7.2. Let $I = \langle x^{41}, y^{41}, z^{41}, x^{40}y^5z^5, x^5y^{40}z^5, x^5y^5z^{40}\rangle \subset \mathbb{C}[x, y, z]$. It can be shown that I is a good ideal. All the new monomials can only be obtained from powers of non-corners: $I_{1,0,0} = I + x^{39}y^{10}z^{10}, I_{2,0,0} = I_{1,0,0} + x^{38}y^{15}z^{15}, \dots, I_{6,0,0} = I_{5,0,0} + x^{34}y^{35}z^{35}$. Here the line stabilizes. We similarly get $I_{0,6,0}$ and $I_{0,0,6}$. Intersecting them we will get

$$\tilde{I} = I_{6,0,0} \cap I_{0,6,0} \cap I_{0,0,6} = I + \langle x^{34}y^{35}z^{35}, x^{35}y^{34}z^{35}, x^{35}y^{35}z^{34}\rangle.$$

Computing successive quotients via computer algebra gives is the following: $I^2 : I^1$ has 7 minimal generators, that is, $|G(I^2 : I^1)| = 7$; $|G(I^3 : I^2)| = 9$; $|G(I^4 : I^3)| = 12$; $|G(I^5 : I^4)| = 16$; $|G(I^6 : I^5)| = 21$; $|G(I^7 : I^6)| = 27$; $|G(I^8 : I^7)| = 31$; $|G(I^9 : I^8)| = 33$; $|G(I^{10} : I^9)| = 33$; $|G(I^{11} : I^{10})| = 31$; $|G(I^{12} : I^{11})| = 24$; $|G(I^{13} : I^{12})| = 18$; $|G(I^{14} : I^{13})| = 13$; $|G(I^{15} : I^{14})| = 9$ and it finally coincides with the ideal obtained above. It takes much time to perform these computations using computer algebra, whereas the computation of $I_{6,0,0}$, $I_{0,6,0}$ and $I_{0,0,6}$ and their intersection is much easier.

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