Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #89, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

The Hopf monoid of orbit polytopes

Mariel Supina*

Department of Mathematics, University of California, Berkeley, CA, USA

Abstract. Many families of combinatorial objects have a Hopf monoid structure. Aguiar and Ardila introduced the Hopf monoid of generalized permutahedra and showed that it contains various other notable combinatorial families as Hopf submonoids, including graphs, posets, and matroids. We introduce the Hopf monoid of orbit polytopes, which is generated by the generalized permutahedra that are invariant under the action of the symmetric group. We show that modulo normal equivalence, these polytopes are in bijection with integer compositions. We interpret the Hopf structure through this lens, and we show that applying the first Fock functor to this Hopf monoid gives a Hopf algebra of compositions. We describe the character group of the Hopf monoid of orbit polytopes in terms of noncommutative symmetric functions, and we give a combinatorial interpretation of the antipode.

Keywords: Hopf monoids, generalized permutahedra, noncommutative symmetric functions, weight polytopes, orbit polytopes, integer compositions

1 Introduction

In [1], Aguiar and Ardila introduced the Hopf monoid of generalized permutahedra and proved that much of its algebraic structure can be interpreted combinatorially. Many other combinatorial families form Hopf submonoids of generalized permutahedra, so this theory produced new proofs of known results about graphs, matroids, posets, and other objects. It also led to some new and surprising theorems. Associated to a Hopf monoid is a group of multiplicative functions called the character group. Aguiar and Ardila showed that the character groups of the Hopf monoids of permutahedra and associahedra are isomorphic to the groups of formal power series under multiplication and composition, respectively. Using their formula for the antipode of generalized permutahedra, they found that permutahedra have information about multiplicative inverses of power series encoded in their face structure, and associahedra have analogous information about compositional inverses of power series.

A subject of ongoing study is to examine other Hopf submonoids of generalized permutahedra and compute the character groups of these. In this extended abstract,

*supina@math.berkeley.edu. Supported by the Chancellor's Fellowship of the University of California Berkeley and the Graduate Fellowships for STEM Diversity.

we consider the Hopf monoid generated by orbit polytopes. These are the generalized permutahedra that are invariant under the action of the symmetric group, so this Hopf monoid contains permutahedra as a Hopf submonoid.

This paper presents two main results. First, Theorem 4.15 describes a Hopf algebra of compositions which results from applying the first Fock functor to the Hopf monoid of orbit polytopes. Second, Theorem 5.5 shows that the character group of the Hopf monoid of orbit polytopes is isomorphic to a subgroup of the group of invertible elements in the completion of the Hopf algebra of noncommutative symmetric functions ($NSym$). In Section 2, we introduce some necessary background. Section 3 formally defines orbit polytopes and shows how, up to normal equivalence, they can be viewed as compositions. In Section 4, we interpret the product and coproduct of generalized permutahedra in the case of orbit polytopes, and we show that these operations can be neatly described in terms of compositions. Section 5 describes the character group, and Section 6 contains a formula for the antipode of orbit polytopes.

2 Preliminaries

Let $\mathbb{R}I$ be the linearization of the finite set I, so $\mathbb{R}I$ is a real vector space with basis $\{e_i : i \in I\}$. Let $\mathbb{R}^I$ be its dual, the set of linear functionals $y : \mathbb{R}I \to \mathbb{R}$. We begin by introducing an important equivalence relation on polytopes which will be very useful in our investigation of orbit polytopes. Let $P \subset \mathbb{R}I$ be a polytope and F be a face of P (we write $F \leq P$).

Definition 2.1. The *normal cone* of F is the cone of linear functionals

$$\mathcal{N}_P(F) := \{y \in \mathbb{R}^I : y \text{ attains its maximum value on } P \text{ at every point in } F\} \subseteq \mathbb{R}^I,$$

i.e. $\mathcal{N}_P(F)$ is the cone of linear functionals that define a face of P containing F.

Definition 2.2. The *normal fan* $\mathcal{N}_P$ of P is the fan in $\mathbb{R}^I$ consisting of the normal cones of each face of P;

$$\mathcal{N}_P := \{\mathcal{N}_P(F) : F \leq P\}.$$

Definition 2.3. The polytopes P and Q are *normally equivalent* if $\mathcal{N}_P = \mathcal{N}_Q$.

Definition 2.4. The *braid arrangement* $\mathcal{B}_I$ is the hyperplane arrangement in $\mathbb{R}^I$ consisting of the hyperplanes $x_i = x_j$ for $i, j \in I$ with $i \neq j$. It divides $\mathbb{R}I$ into closed full-dimensional cones, or *chambers*. The *braid fan* is the fan formed by taking the set of chambers of the braid arrangement and all of their faces.

The cones of the braid fan $\mathcal{B}_I$ are in natural bijection with ordered partitions of the set I. Fix a chamber of $\mathcal{B}_I$ and call this the *fundamental chamber*. (For example, we could choose the fundamental chamber of $\mathcal{B}_{[n]}$ to be the set of points in $\mathbb{R}^{[n]}$ with coordinates in decreasing order.) One can show that the faces of the fundamental chamber are in natural bijection with compositions of the integer $|I|$.

Definition 2.5. A *generalized permutahedron* is a polytope whose normal fan is a coarsening of the braid fan.

Generalized permutahedra are polytopes with very nice combinatorial and algebraic properties. They are equivalent to polymatroids up to translation. They can be obtained by moving the vertices of a standard permutahedron in such a way that the edge directions are preserved [5]. Proposition 2.8 gives an equivalent definition of generalized permutahedra using submodular functions.

Definition 2.6. A *submodular function* is a function $z : I \to \mathbb{R}$ where I is a finite set and z satisfies $z(\varnothing) = 0$ and $z(S \cap T) + z(S \cup T) \leq z(S) + z(T)$ for all $S, T \subseteq I$.

Definition 2.7. Let $z : I \to \mathbb{R}$ be a submodular function. The *base polytope* of z is

$$\mathcal{P}(z) = \left\{ x \in \mathbb{R}I : \sum_{i \in I} x_i = z(I) \text{ and } \forall\, \varnothing \subsetneq A \subsetneq I,\ \sum_{a \in A} x_a \leq z(A) \right\}.$$

Proposition 2.8 ([3])**.** *A polytope is a generalized permutahedron if and only if it is the base polytope of a submodular function.*

3 Orbit Polytopes

In this section, we introduce orbit polytopes, the main combinatorial objects studied in this paper.

3.1 Definition of an Orbit Polytope

The symmetric group S_I acts on $\mathbb{R}I$ by permuting coordinates. If σ is a permutation in S_I and $p = (p_i : i \in I) \in \mathbb{R}I$, then this action is given by $\sigma(p)_i = p_{\sigma^{-1}(i)}$.

Definition 3.1. Let $p \in \mathbb{R}I$. The *orbit polytope* of p is the polytope

$$\mathcal{O}(p) := \operatorname{conv}\{\sigma(p) : \sigma \in S_I\}.$$

Orbit polytopes are also called permutahedra [5], but we avoid this terminology in order to distinguish the Hopf monoid of orbit polytopes from the Hopf monoid of (standard) permutahedra discussed in [1].

Orbit polytopes are closely related to *weight polytopes*, a general construction arising in representation theory and the theory of finite reflection groups. The vertices of weight

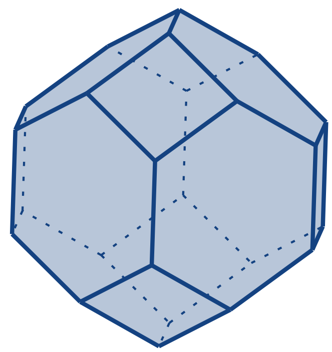

Figure 1: $\mathcal{O}(4,3,2,1)$, the standard 4-permutahedron

polytopes are given by the orbit of a special point, called a weight, under a relevant action. The weights arising in the representation theory of the general linear group are all integer points; thus orbit polytopes with integer vertices are the same as weight polytopes for GL_n. These are also the same as weight polytopes arising from reflection groups of type A.

Example 3.2. The orbit polytope $\mathcal{O}(1,0,0) := \text{conv}\{(0,0,1),(0,1,0),(1,0,0)\}$ is the standard 3-simplex.

Example 3.3. For any $\lambda \in \mathbb{R}$, $\mathcal{O}(\lambda,\ldots,\lambda)$ is a single point in $\mathbb{R}I$.

Example 3.4. Let $I = [n]$ and $p = (n, n-1, \ldots, 1) \in \mathbb{R}^n$. Then $\mathcal{O}(p)$ is the standard n-permutahedron (see Figure 1).

Proposition 3.5. *Orbit polytopes are generalized permutahedra.*

3.2 Orbit Polytopes as Base Polytopes of Submodular Functions

It is clear from Definition 3.1 that orbit polytopes are invariant under the action of S_I. However, one might wonder whether there exist other generalized permutahedra that are invariant under this action. Submodular functions are a useful tool for answering this question. A helpful discussion of submodular functions and their relationship to generalized permutahedra can be found in [1].

Definition 3.6. A submodular function z is *S_I-invariant* if $z(S) = z(T)$ when $|S| = |T|$.

Proposition 3.7. *A polytope in $\mathbb{R}I$ is an orbit polytope if and only if it is the base polytope of an S_I-invariant submodular function.*

Corollary 3.8. *Orbit polytopes are exactly the generalized permutahedra which are invariant under the S_I action on $\mathbb{R}I$.*

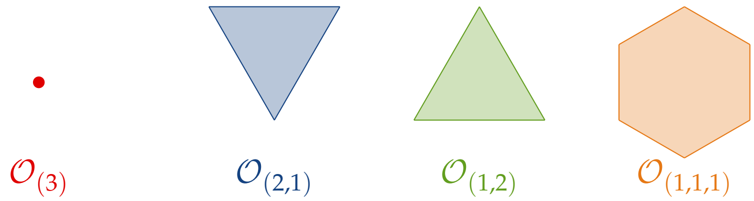

Figure 2: All normal equivalence classes of orbit polytopes in $\mathbb{R}I$ when $|I| = 3$

3.3 Orbit Polytopes Modulo Normal Equivalence

The goal of this section is to show that normal equivalence classes of orbit polytopes $\mathcal{O}(p)$ for $p \in \mathbb{R}I$ are in bijection with compositions of the integer $n := |I|$. In the following definition, we think of the braid arrangement as living in $\mathbb{R}I$ instead of $\mathbb{R}^I$ (as in Definition 2.4) by identifying the two isomorphic vector spaces.

Definition 3.9. Let $p \in \mathbb{R}I$. The *composition of* p is the integer composition of n corresponding to the unique face of the fundamental chamber of the braid arrangement $\mathcal{B}_I$ that contains some point in the S_I-orbit of p.

Definition 3.10. The *composition of an orbit polytope* is the composition of any of its vertices.

Proposition 3.11. *Let $\mathcal{O}$ and $\mathcal{O}'$ be orbit polytopes. Then $\mathcal{O}$ and $\mathcal{O}'$ are normally equivalent if and only if they have the same composition. In other words, normal equivalence classes of orbit polytopes $\mathcal{O}(p)$ for $p \in \mathbb{R}I$ are in bijection with compositions of n.*

Notation 3.12. Let α be a composition of n. We write $\mathcal{O}_{\alpha,I}$ for the normal equivalence class of orbit polytopes in $\mathbb{R}I$ with composition α. If the set I is clear from context, we may simply write $\mathcal{O}_\alpha$.

Example 3.13 (Normal equivalence classes for $n = 3$)**.** Let $I = \{1, 2, 3\}$. There are four compositions of the integer $3 = |I|$, so there are four normal equivalence classes of orbit polytopes in $\mathbb{R}I$ (see Figure 2).

Example 3.14 (Notable families of orbit polytopes)**.** The following compositions of n correspond to normal equivalence classes of well-studied families of polytopes in $\mathbb{R}I$ with $|I| = n$:

- (n): single point
- $(1, \ldots, 1)$: standard n-permutahedron
- $(1, n-1)$: standard n-simplex
- $(k, n-k)$: uniform matroid polytope $U_{k,n}$; these are also known as hypersimplices

4 Algebraic Structures on Orbit Polytopes

In [1], the authors introduce a product and coproduct which give generalized permutahedra the structure of a Hopf monoid. When restricted to orbit polytopes, these operations have a neat interpretation in terms of compositions.

4.1 Toward a Product

Definition 4.1. Let $P \subset \mathbb{R}S$ and $Q \subset \mathbb{R}T$ be any polytopes. Then the *product of P and Q* is the polytope

$$P \cdot Q := \{(p,q) \in \mathbb{R}(S \sqcup T) : p \in P, q \in Q\} \subset \mathbb{R}(S \sqcup T).$$

The identity of this product is the unique empty orbit polytope, which lives in $\mathbb{R}\varnothing$.

The vertices of the product of two polytopes are exactly the products of vertices of the two polytopes. This means that the vertex set of a product of nonempty orbit polytopes will *not* contain the entire orbit of any point under the action of the symmetric group, unless we are multiplying points with all coordinates the same. Therefore, the product of two nonempty orbit polytopes will never be normally equivalent to an orbit polytope unless both polytopes are a single point.

Proposition 4.2. *Let $P \subset \mathbb{R}I$ be a product of finitely many orbit polytopes. Then up to commutativity of $\cdot$, P has a unique expression of the form*

$$P = \mathcal{O}(p_1) \cdot \ldots \cdot \mathcal{O}(p_k)$$

where $p_j \in \mathbb{R}S_j$ for $1 \leq j \leq k$, and $S_1 \sqcup \cdots \sqcup S_k$ is a partition of I into nonempty sets, and $\mathcal{O}(p_j)$ is not a single point unless $|S_j| = 1$.

4.2 Toward a Coproduct

Proposition 4.3. *Let $p \in \mathbb{R}I$ where $|I| = n$. Let $\mathcal{O} = \mathcal{O}(p) \subset \mathbb{R}I$ be the orbit polytope of p and let $I = S \sqcup T$. Suppose that $F \leq \mathcal{O}$ is the face of $\mathcal{O}$ maximizing the indicator functional $\mathbf{1}_S$ of S, where $\mathbf{1}_S(x) := \sum_{s \in S} x_s$. Then there exist unique orbit polytopes $\mathcal{O}|_S \subset \mathbb{R}S$ and $\mathcal{O}/_S \subset \mathbb{R}T$ such that*

$$F = \mathcal{O}|_S \cdot \mathcal{O}/_S.$$

We call $\mathcal{O}|_S$ "$\mathcal{O}$ restricted to S," and we call $\mathcal{O}/_S$ "$\mathcal{O}$ contracted by S." It is straightforward to show that if $\mathcal{O}$ is a product of finitely many orbit polytopes, a version of Proposition 4.3 still holds. That is, the $\mathbf{1}_S$-maximal face of $\mathcal{O}$ decomposes as $\mathcal{O}|_S \cdot \mathcal{O}/_S$, where $\mathcal{O}|_S$ is a finite product of orbit polytopes that lives in $\mathbb{R}S$ and $\mathcal{O}/_S$ is a finite product of orbit polytopes that lives in $\mathbb{R}T$.

4.3 Species

Let Set be the category of sets with arbitrary morphisms, and let $\mathrm{Set}^{\times}$ be the category of finite sets with bijections.

Definition 4.4. A *set species* is a functor $\mathrm{F} : \mathrm{Set}^{\times} \to \mathrm{Set}$. If I is a finite set, then F maps I to a set $\mathrm{F}[I]$ which can be considered to contain "structures of type F labeled by I." If $\sigma : I \to J$ is a bijection of finite sets, then F maps σ to a morphism $\mathrm{F}[\sigma] : \mathrm{F}[I] \to \mathrm{F}[J]$ which can be thought of as the map "relabeling the elements of $\mathrm{F}[I]$ according to σ."

Definition 4.5. The *set species of orbit polytopes*, denoted OP, maps a finite set I to the set $\mathrm{OP}[I]$ of finite products of orbit polytopes living in $\mathbb{R}I$. For a bijection of finite sets $\sigma : I \to J$, we get the map $\mathrm{OP}[\sigma] : \mathrm{OP}[I] \to \mathrm{OP}[J]$ induced from the isomorphism from $\mathbb{R}I$ to $\mathbb{R}J$ relabelling the basis vectors $\{e_i : i \in I\}$ of $\mathbb{R}I$ according to σ.

We have seen that orbit polytopes up to normal equivalence are in bijection with compositions. It is interesting to consider the species of orbit polytopes up to normal equivalence.

Definition 4.6. The *set species of normal equivalence classes of orbit polytopes*, denoted $\overline{\mathrm{OP}}$, maps a finite set I to the set $\overline{\mathrm{OP}}[I]$ of normal equivalence classes of finite products of orbit polytopes in $\mathrm{OP}[I]$. In other words, a general element of $\overline{\mathrm{OP}}[I]$ has the form

$$\mathcal{O}_{\alpha_1,S_1} \cdot \cdots \cdot \mathcal{O}_{\alpha_k,S_k}$$

where $I = S_1 \sqcup \cdots \sqcup S_k$ and α_i is a composition of $|S_i|$ for all i.

Let Comp be the set species of compositions where $\mathrm{Comp}[I]$ is the set of integer compositions of $|I|$ and $\mathrm{Comp}[\sigma] = \mathrm{id}$ for all bijections $\sigma : I \to J$. Let $\widehat{\mathrm{Comp}}$ be the result of removing compositions with one part from $\mathrm{Comp}[I]$ when $|I| \geq 2$, so $\widehat{\mathrm{Comp}}[I]$ for $|I| \geq 2$ is the set of compositions of $|I|$ with more than one part. Define $\widehat{\mathrm{Comp}}[\varnothing] := \varnothing$.

Proposition 4.7. *The species $\overline{\mathrm{OP}}$ is isomorphic to $\mathrm{E} \circ \widehat{\mathrm{Comp}}$, where E is the exponential species and $\circ$ denotes composition of species (see [2]).*

The first few terms of the generating function for $\overline{\mathrm{OP}}$ are

$$\overline{\mathrm{OP}}(t) = 1 + t + 2\frac{t^2}{2!} + 7\frac{t^3}{3!} + 29\frac{t^4}{4!} + 136\frac{t^5}{5!} + \dots$$

4.4 Hopf Monoid

Definition 4.8. A *Hopf monoid in set species* is a set species H equipped with a product $\mu = \{\mu_{S,T} : \mathrm{H}[S] \times \mathrm{H}[T] \to \mathrm{H}[I]\}$ and a coproduct $\Delta = \{\Delta_{S,T} : \mathrm{H}[I] \to \mathrm{H}[S] \times \mathrm{H}[T]\}$ where S and T are any pair of disjoint finite sets and $I = S \sqcup T$. These operations must satisfy naturality, unitality, associativity, and compatibility axioms (see [2, 1]). A Hopf monoid in set species is *connected* if $|\mathrm{H}[\varnothing]| = 1$.

Proposition 4.9 (OP is a Hopf submonoid of GP)**.** *Define a product and coproduct on* OP *as follows:*

- *The product is a collection of maps* $\mu = \{\mu_{S,T} : \mathrm{OP}[S] \times \mathrm{OP}[T] \to \mathrm{OP}[I]\}$ *for all ordered partitions of a finite set* I *into finite sets* S *and* T*. If* $\mathcal{O} \in \mathrm{OP}[S]$ *and* $\mathcal{O}' \in \mathrm{OP}[T]$*, then their product is*
$$\mu_{S,T}(\mathcal{O}, \mathcal{O}') := \mathcal{O} \cdot \mathcal{O}' \in \mathrm{OP}[I]$$
as defined in Definition 4.1.

- *The coproduct is a collection of maps* $\Delta = \{\Delta_{S,T} : \mathrm{OP}[I] \to \mathrm{OP}[S] \times \mathrm{OP}[T]\}$ *for all ordered partitions of a finite set* I *into finite sets* S *and* T*. If* $\mathcal{O} \in \mathrm{OP}[I]$*, then its coproduct is*
$$\Delta_{S,T}(\mathcal{O}) := (\mathcal{O}|_S, \mathcal{O}/_S) \in \mathrm{OP}[S] \times \mathrm{OP}[T],$$
where $\mathcal{O}|_S$ *and* $\mathcal{O}/_S$ *are the restriction and contraction discussed in Proposition 4.3.*

These operations turn the set species OP *into a connected Hopf submonoid of* GP*, where* GP *is the Hopf monoid of generalized permutahedra defined in [1].*

Proposition 4.10 ([1])**.** *Taking normal equivalence classes respects the product and coproduct of* OP *defined in Proposition 4.9.*

Corollary 4.11. *The set species* $\overline{\mathrm{OP}}$ *of normal equivalence classes of orbit polytopes forms a connected Hopf monoid under the induced product and coproduct from* OP*.*

As a consequence of Proposition 4.2, we get that $\overline{\mathrm{OP}}$ is a free commutative Hopf monoid generated under multiplication by elements $\mathcal{O}_{\alpha,I}$ where I is some finite set and α is an integer composition of $|I|$ that has more than one part if $|I| > 1$. This characterizes the product of $\overline{\mathrm{OP}}$. The coproduct of $\overline{\mathrm{OP}}$ also has a very nice formulation in terms of compositions. This formulation uses two standard operations.

Definition 4.12. The *concatenation* of compositions $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ is the composition
$$\beta \cdot \gamma := (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_\ell).$$

Definition 4.13. The *near-concatenation* of nonempty compositions $\beta = (\beta_1, \ldots, \beta_k)$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ is the composition
$$\beta \odot \gamma := (\beta_1, \ldots, \beta_k + \gamma_1, \ldots, \gamma_\ell).$$

Proposition 4.14. *Let* I *be a finite set with* $|I| = n$ *and let* α *be an integer composition of* n*, so* $\mathcal{O}_\alpha \in \overline{\mathrm{OP}}[I]$*. Then if* $I = S \sqcup T$ *we have*
$$\Delta_{S,T}(\mathcal{O}_\alpha) = (\mathcal{O}_\beta, \mathcal{O}_\gamma)$$
where β *and* γ *are the unique pair of compositions satisfying*

(i) β *is a composition of* $|S|$ *and* γ *is a composition of* $|T|$*, and*

(ii) either $\beta \cdot \gamma = \alpha$ *or* $\beta \odot \gamma = \alpha$*.*

4.5 Hopf Algebra

Given a Hopf monoid in set species, we can obtain a Hopf algebra by applying first a linearization functor and then a Fock functor [2, §15]. The first Fock functor produces the Hopf algebra

$$\bigoplus_{n\geq 0} \mathrm{Span}\{\text{isomorphism classes of elements of } \mathrm{H}[I] \text{ where } |I| = n\}$$

where isomorphisms are given by relabeling maps in the species. Applying this construction to $\overline{\mathrm{OP}}$ results in a Hopf algebra of normal equivalence classes of orbit polytopes. Like $\overline{\mathrm{OP}}$, this Hopf algebra can be described in terms of compositions. We will use the notation $|\alpha|$ to denote the sum of the parts of a composition α.

Theorem 4.15.

(i) Let A be the set containing all integer compositions with more than one part and the unique composition of 1. Consider the commutative algebra Comp generated freely by A. Let $\alpha \in A$ and define a coproduct $\Delta : Comp \to Comp \otimes Comp$ by

$$\Delta(\alpha) := \sum_{\substack{\beta\cdot\gamma=\alpha \\ \text{or} \\ \beta\odot\gamma=\alpha}} \binom{|\alpha|}{|\beta|} \beta \otimes \gamma.$$

This makes Comp into a graded Hopf algebra.

(ii) Comp is isomorphic to the Hopf algebra of normal equivalence classes of orbit polytopes.

5 The Character Group of $\overline{\mathrm{OP}}$

Studying the character group of a Hopf monoid can lead to surprising connections, as seen for the cases of permutahedra and associahedra in [1]. To begin, let H be a connected Hopf monoid in set species, and let $\Bbbk$ be a field.

Definition 5.1. A *character* $\zeta : \mathrm{H} \to \Bbbk$ is collection of natural maps $\{\zeta_I : \mathrm{H}[I] \to \Bbbk\}$ for each finite set I such that

(i) $\zeta_\varnothing : \mathrm{H}[\varnothing] \to \Bbbk$ is the map sending $1 \in \mathrm{H}[\varnothing]$ to $1 \in \Bbbk$, and

(ii) if $I = S \sqcup T$, then for any $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$ we have

$$\zeta_I(\mu_{S,T}(x,y)) = \zeta_S(x)\zeta_T(y).$$

Let $\mathbb{X}(\mathrm{H})$ be the set of all characters on H.

Definition 5.2. The *convolution* of $\zeta, \psi \in \mathbb{X}(\mathrm{H})$ is defined for $x \in \mathrm{H}[I]$ to be

$$(\zeta \star \psi)_I(x) := \sum_{I = S \sqcup T} \zeta_S(x|_S)\psi_T(x/_S)$$

where the sum is taken over all ordered partitions of I into sets S and T.

This convolution product gives $\mathbb{X}(\mathrm{H})$ a group structure [1, Theorem 8.2]. Inverses of characters can be obtained by composing the character with the antipode map (Section 6). We will prove that the character group of $\overline{\mathrm{OP}}$ is related to the Hopf algebra of noncommutative symmetric functions (*NSym*). This is the graded dual of the Hopf algebra *QSym* of quasisymmetric functions. One basis for *NSym* is given by the *noncommutative ribbon functions* $\{R_\alpha\}$ which are indexed by integer compositions α. This basis is dual to the fundamental basis of *QSym* and has the product

$$R_\beta R_\gamma = R_{\beta \cdot \gamma} + R_{\beta \odot \gamma}. \tag{5.1}$$

A detailed explanation of the Hopf structures of *QSym* and *NSym* can be found in [4].

Definition 5.3. The *completion* $\overline{NSym}$ of *NSym* is the ring $\Bbbk[[\{R_\alpha\}]]$ of generating functions of the form $\sum_\alpha c_\alpha R_\alpha$ where the sum is over compositions α. The product in this ring is induced from the product of the R_α given in (5.1).

As with standard power series, one can show that an element of $\overline{NSym}$ is invertible if and only if $c_\varnothing \neq 0$, and that the invertible elements form a group under multiplication.

Definition 5.4. Define G to be the collection of invertible elements in $\overline{NSym}$ with the properties that $c_\varnothing = 1$ and $n! c_{(n)} = c_{(1)}^n$ for all $n > 1$.

It is straightforward to check that G is a subgroup of the invertible elements of $\overline{NSym}$.

Theorem 5.5. *The character group* $\mathbb{X}(\overline{\mathrm{OP}})$ *is isomorphic to* G.

Proof. (Details of the proof are omitted.) Let $\zeta \in \mathbb{X}(\overline{\mathrm{OP}})$. For a positive integer n and a composition α of n, let $b_\alpha = \zeta_{[n]}(\mathcal{O}_{\alpha,[n]})$. Then for each α, the corresponding b_α can have any value in $\Bbbk$ subject to the restrictions that $b_\varnothing = 1$ and $b_{(n)} = b_{(1)}^n$ for $n > 1$. The isomorphism is given by

$$\zeta \longmapsto \sum_\alpha \frac{b_\alpha}{|\alpha|!} R_\alpha,$$

where the sum is taken over all integer compositions α. □

6 The Antipode of $\overline{\mathbf{OP}}$

The antipode is an important map in the study of Hopf monoids and Hopf algebras. It is analogous to the group map sending g to g^{-1}. One application of the antipode is inversion of characters in the character group. In combinatorics, a nice interpretation of the antipode can lead to interesting reciprocity results, as in [1, §18].

A Hopf monoid in set species does not have enough structure to define an antipode, so we must introduce the notion of Hopf monoids in vector species. Let Vec be the category of vector spaces with linear maps.

Definition 6.1. A *vector species* $\mathbf{F}$ is a functor from $\mathsf{Set}^\times \to \mathsf{Vec}$. If I is a finite set, then $\mathbf{F}$ maps I to a vector space $\mathbf{F}[I]$. If $\sigma : I \to J$ is a bijection of finite sets, then $\mathbf{F}$ maps σ to a linear map $\mathbf{F}[\sigma] : \mathbf{F}[I] \to \mathbf{F}[J]$.

Definition 6.2. The *vector species of normal equivalence classes of orbit polytopes*, denoted $\overline{\mathbf{OP}}$, maps a finite set I to the $\Bbbk$-vector space $\overline{\mathbf{OP}}[I]$ consisting of formal linear combinations of products of normal equivalence classes of orbit polytopes in $\mathbb{R}I$. In other words, this is the linearization of the set species $\overline{\mathrm{OP}}$.

Analogous to Definition 4.8 is the notion of a *Hopf monoid in vector species*. The product and coproduct of $\overline{\mathrm{OP}}$ extend linearly to a product and coproduct on $\overline{\mathbf{OP}}$, which makes $\overline{\mathbf{OP}}$ into a Hopf monoid in vector species. Recall that Proposition 4.2 implies that $\overline{\mathbf{OP}}[I]$ has a basis given by products of the form

$$\mathcal{O}_{\alpha_1,S_1} \cdot \ldots \cdot \mathcal{O}_{\alpha_k,S_k}$$

as given in Definition 4.6, with the constraint that if $|S_i| > 1$, then α_i must have more than one part.

Definition 6.3 (Takeuchi's formula)**.** The *antipode* of a Hopf monoid $\mathbf{H}$ in vector species is a collection of maps $\{s_I : \mathbf{H}[I] \to \mathbf{H}[I]\}$ given by

$$s_I(x) = \sum_{k=1}^{|I|} \left(\sum_{(S_1,\ldots,S_k)\vDash I} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(x) \right)$$

where $x \in \mathbf{H}[I]$ and the sum is taken over ordered partitions $(S_1, \ldots, S_k)$ of I such that all of the S_i are nonempty.

Definition 6.3 can be interpreted for orbit polytopes using compositions.

Proposition 6.4. *Let $\mathcal{O}_\alpha \in \overline{\mathbf{OP}}[I]$, so α is an integer composition of $|I|$. Then we have*

$$s_I(\mathcal{O}_\alpha) = \sum_{k=1}^{|I|} \left(\sum_{(S_1,\ldots,S_k)\vDash I} (-1)^k \mathcal{O}_{\beta_1,S_1} \cdot \ldots \cdot \mathcal{O}_{\beta_k,S_k} \right).$$

For each $(S_1, \ldots, S_k) \vDash I$, the compositions $\beta_1, \ldots, \beta_k$ are the unique compositions satisfying *(i)* β_i *is a composition of* $|S_i|$ *for all* i*, and*

(ii) α *can be obtained from the* β_i*'s by some sequence of concatenations and near-concatenations, that is,*

$$\alpha = \beta_1 \square \ldots \square \beta_k$$

where each occurrence of $\square$ *is replaced with either concatenation* $\cdot$ *or near-concatenation* $\odot$*.*

Ardila and Aguiar showed that for generalized permutahedra, the antipode has the cancellation-free and grouping-free formula

$$s_I(P) = (-1)^{|I|} \sum_{F \leq P} (-1)^{\dim F} F$$

where $P \subset \mathbb{R}I$ is a generalized permutahedron and the sum is taken over all faces F of P [1]. One future direction of work could be to interpret this formula for orbit polytopes, and to find a grouping-free formula for the antipode.

7 Acknowledgements

The author would like to thank Federico Ardila for the valuable guidance, Mario Sanchez for helpful conversations, and the FPSAC referees for useful feedback.

References

[1] M. Aguiar and F. Ardila. "Hopf monoids and generalized permutahedra". 2017. arXiv:1709.07504.

[2] M. Aguiar and S. Mahajan. *Monoidal Functors, Species, and Hopf Algebras*. American Mathematical Society, 2010.

[3] S. Fujishige. *Submodular Functions and Optimization*. 2nd ed. Elsevier, Amsterdam, 2005.

[4] D. Grinberg and V. Reiner. "Hopf Algebras in Combinatorics". 2014. arXiv:1409.8356.

[5] A. Postnikov. "Permutohedra, associahedra, and beyond". *Int. Math. Res. Not.* **2009**.6 (2009), pp. 1026–1106. Link.