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Schubert structure operators

Rebecca Goldin[1*] and Allen Knutson[2†]

[1] *George Mason University, Fairfax, VA 22030, USA*
[2] *Cornell University, Ithaca, NY 14853, USA*

Abstract. We use operators to reformulate the Andersen-Jantzen-Soergel/Billey formula for the point restrictions of equivariant Schubert classes of the cohomology of G/B. We introduce new operators whose coefficients compute Schubert structure constants (in a manifestly polynomial, but not positive, way), resulting in a formula much like and generalizing the positive AJS/Billey formula. Our proof involves Bott-Samelson manifolds, and in particular, the cohomology basis dual to the homology basis of classes of sub-Bott-Samelson manifolds.

Keywords: Schubert calculus, nil Hecke algebra

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1 Introduction and the main theorem

Fix a complex reductive Lie group G and maximal torus $T \leq G$, for example $G = GL_n(\mathbb{C})$ and T the diagonal matrices. Fix opposed Borel subgroups B, B_- with intersection T. This choice results in a length function ℓ on $W = N(T)/T$ and a set $\{\alpha_i\}$ of simple roots. The quotient G/B is the associated **flag manifold** and the left T-action on G/B has isolated fixed points $\{wB/B \,:\, w \in W\}$, where $W := N(T)/T$ is the **Weyl group**.

In the case that $G = GL_n(\mathbb{C})$ and B = upper-triangular matrices, G/B is (uniquely) G-isomorphic to the set of complete flag manifolds $Fl(\mathbb{C}^n)$. The fixed points $N(T)B/B$

* rgoldin@gmu.edu
† allenk@math.cornell.edu

of the T-action correspond, under that isomorphism, to coordinate flags in $Fl(\mathbb{C}^n)$. In particular, there are $n!$ such flags, corresponding to elements in the Weyl group $W \cong S_n$, the symmetric group on n letters.

We denote by H_T^* the T-equivariant cohomology of a point with coefficients in $\mathbb{Z}$, and recall that H_T^* is the polynomial ring $Sym(T^*)$ over $\mathbb{Z}$ in the weight lattice $T^* := Hom(T, \mathbb{C}^\times)$. The equivariant cohomology $H_T^*(G/B)$ is a free H_T^*-module with a basis given by Schubert classes (recalled below). Our references for equivariant (co)homology are [3, 8, 9].

Let $\mathbb{Z}[\partial]$ denote the **nil Hecke algebra** with $\mathbb{Z}$-basis $\{\partial_w \ :\ w \in W\}$, whose products are defined by

$$\partial_w \partial_v := \begin{cases} \partial_{wv} & \text{if } \ell(vw) = \ell(v) + \ell(w) \\ 0 & \text{otherwise, i.e. if } \ell(vw) < \ell(v) + \ell(w). \end{cases}$$

These $\{\partial_w\}$ act on the polynomial ring H_T^* as follows: for each root α with simple reflection r_α, the **divided difference operator** $\partial_{r_\alpha} := \partial_\alpha$ is defined by

$$\partial_\alpha \cdot f := \frac{f - r_\alpha f}{\alpha}$$

The nil Hecke algebra acts on the first factor in the tensor product $H_T^* \otimes_{\mathbb{Z}} H_T^*$, and this action descends to the quotient $H_T^* \otimes_{(H_T^*)^W} H_T^*$. This latter ring has a well-defined map $\lambda \otimes \mu \mapsto \lambda c_1(\mathcal{L}_\mu) \in H_T^*(G/B)$ called the **equivariant Borel presentation** of $H_T^*(G/B)$, which is a rational (and for $G = GL_n$, an integral) isomorphism. (Here $\mathcal{L}_\mu$ is the Borel-Weil line bundle $G \times^B \mathbb{C}_\mu$, where $\mathbb{C}_\mu$ is the 1-dimensional representation of B, neither of which will be using again.)

Since our interest is in cohomology not homology, we privilege codimension over dimension and define $X^v := \overline{BvB}/B \subseteq G/B$ to be an **opposite Schubert variety** with equivariant homology class $[X^v] \in H_*^T(G/B)$. As these $\{[X^v]\}$ form an H_T^*-basis and G/B enjoys equivariant Poincaré duality, we can define the dual basis $\{S_w \in H_T^*(G/B)\}$ of **Schubert classes** by $\langle S_w, [X^v] \rangle = \delta_{wv}$. Here $\langle , \rangle$ denotes the Alexander pairing, of (equivariant) cap-product followed by pushforward to a point. In fact S_w is the Poincaré dual to the subvariety $\overline{B_- wB}/B$.

The nil Hecke algebra $\mathbb{Z}[\partial]$ acts on the basis $\{S_v\}_{v \in W}$: in particular, $\partial_w \cdot S_{w_0} = S_{ww_0}$ for each $w \in W$ (since we act on the left factor in the Borel presentation), though we won't use this recursion.

The structure constants $c_{uv}^w \in H_T^*$ are defined by the relation in $H_T^*(G/B)$

$$S_u S_v = \sum_w c_{uv}^w S_w \tag{1.1}$$

These polynomials c_{uv}^w are known to be positive in the following sense [6]: when written (uniquely) as a sum of monomials in the simple roots $\{\alpha_i\}$, each monomial has a non-negative coefficient. It is a very famous problem to compute these in a manifestly positive way, solved in special cases such as $u, v \in W^P$ where G/P is a Grassmannian or 2-step flag manifold [7, 5]. Another solved case is $u = w$, in which case c^w_{wv} is computed positively by the AJS/Billey formula [1, 2] (recalled below) for the point restrictions $S_w|_v = c^v_{wv}$ of Schubert classes. In this abstract, we prove a formula for the $\{c^w_{uv}\}$ in terms of a certain composition of operators in the nil Hecke algebra, applied to 1. Along the way, we re-prove the AJS/Billey formula; more specifically, our nonpositive formula reduces to the positive AJS/Billey formula in the special case $u = w$.

Theorem 1. *Let Q be a reduced word for w. Then*

$$c^w_{uv} = \sum_{\substack{P,R\subseteq Q \text{ reduced} \\ \prod P=u,\ \prod R=v}} \prod_Q \left(\alpha_q{}^{[q\in P,R]}\ \partial_q{}^{[q\notin P,R]}\ r_q\right)\cdot 1$$

where the exponent "$[\sigma]$" is 1 if the statement σ is true, 0 if false.

Example. Let $Q = 1\,2\,1$ so $w = r_1r_2r_1$, $u = r_1$, $v = r_1r_2$ all in S_3 the Weyl group of GL_3. Then $P \in \{1--, --1\}$, $R = 1\,2\,-$ as subwords of $1\,2\,1$, in our sum

$$c^{r_1r_2r_1}_{r_1,\,r_1r_2} = (\alpha_1 r_1\, r_2\, \partial_1 r_1)\cdot 1 + (r_1\, r_2\, r_1)\cdot 1 = 0 + 1$$

whereas if we change v to r_2r_1 so $R = -\,2\,1$, then

$$c^{r_1r_2r_1}_{r_1,\,r_2r_1} = (r_1\, r_2\, r_1)\cdot 1 + (\partial_1 r_1\, r_2\, \alpha_1 r_1)\cdot 1 = 1 + \partial_1\cdot\alpha_2 = 0.$$

Example. Let $Q = 1\,2\,3\,1\,2$, so $w = r_1r_2r_3r_1r_2 = [3421]$ in one-line notation, and take $u = r_2r_3r_2 = [1432]$, $v = r_1r_2r_1 = [3214]$. Then $P = -23-2$ and $R \in \{12-1-, -2-12\}$ so we have

$$\begin{aligned} c^w_{uv} &= (r_1\, \alpha_2 r_2\, r_3\, r_1\, r_2 + \partial_1 r_1\, \alpha_2 r_2\, r_3\, r_1\, \alpha_2 r_2)\cdot 1 \\ &= (\alpha_1+\alpha_2)\cdot 1 + \partial_1(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)\cdot 1 \\ &= \alpha_1+\alpha_2+\partial_1(\alpha_1+\alpha_2)\alpha_2\cdot 1 + \partial_1(\alpha_1+\alpha_2)\alpha_3\cdot 1 \\ &= \alpha_1+\alpha_2+0+\alpha_3. \end{aligned}$$

We now recall the AJS/Billey formula. The T-invariant inclusion i of T-fixed points into G/B results in a map in equivariant cohomology:

$$i^* : H_T(G/B) \longrightarrow \bigoplus_{w\in W} H^*_T(wB/B) \cong \bigoplus_{w\in W} H^*_T \tag{1.2}$$

and i is known to be an *injection.* The inclusion $i_w : wB/B \hookrightarrow G/B$ induces the projection to the w-term in this sum, so we may write $i^* = \oplus_{w\in W}\, i^*_w$.

For any $v, w \in W$, the **point restriction** $S_v|_w \in H^*_T$ is defined by $i^*_w(S_v)$, i.e. the image of S_v under the map i^* in (1.2), then projected to the w summand. Since (1.2) is an inclusion, each Schubert class S_v is described fully by the list $\{i_w^*(S_v) : w \in W\}$ of these restrictions. Note that $S_w|_u \neq 0$ implies $uB/B \in \overline{B_- wB}/B$, i.e. $u \geq w$ in **Bruhat order**, and in fact the converse is also true. This **upper triangularity of the support** will be useful just below.

In the case $u = w$, the relation (1.1) and this upper triangularity imply that $c_{uv}^w = S_v|_w$. After choosing Q a reduced word for w, the only choice of reduced word P for u is Q itself. The formula thus simplifies to

$$S_v|_w = \sum_{\substack{R \subseteq Q \text{ reduced} \\ P=Q,\ \prod R = v}} \prod_Q \left({\alpha_q}^{[q \in R]}\, r_q \right) \cdot 1,$$

which is just a restatement of the AJS/Billey formula.

After describing our geometric proof, we give an algebraic interpretation of Theorem 1 as a coefficient of the product of certain *Schubert structure operators*. Let $H_T^*[\partial]$ denote the smash product of H_T^* with $\mathbb{Z}[\partial]$, the algebra consisting of the free H_T^*-module $H_T^* \otimes_{\mathbb{Z}} \mathbb{Z}[\partial]$ with product given by, for $p, q \in H_T^*$,

$$(p \otimes \partial_v) \cdot (q \otimes \partial_w) = p(\partial_v q) \otimes \partial_v \partial_w$$

and extended linearly. This smash product was first introduced by Kostant and Kumar in [8]. Since r_α acts on $H_T^*(G/B)$ equivalently to $1 - \alpha\partial_\alpha$, we will abuse notation and denote by $r_\alpha \in H_T^*[\partial]$ the operator $1 - \alpha\partial_\alpha$.

Let

$$K^\alpha := (\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)$$

in $H_T^*[\partial] \otimes \mathbb{Z}[\partial] \otimes \mathbb{Z}[\partial]$. The **Schubert structure operators** K^α braid and commute appropriately (in the simply and doubly laced cases; we conjecture but haven't checked the remaining G_2 case), and square to 0. They act on $H_T^*(G/B) \otimes H_T^*(G/B) \otimes H_T^*(G/B)$, resulting in another way (in Section 5) to obtain the coefficients c_{uv}^w. It seems likely that further analysis of them would give a purely algebraic proof of Theorem 1. As an application of Theorem 1, we derive two recursive formulas for structure constants.

2 Ingredients of the proof

Recall that the **Bott-Samelson manifold** associated to a word $Q = r_{\alpha_{i_1}} r_{\alpha_{i_2}} \cdots r_{\alpha_{i_\ell}}$ in simple reflections is given by

$$BS^Q = P_{\alpha_{i_1}} \times^B P_{\alpha_{i_2}} \times^B \cdots \times^B P_{\alpha_{i_\ell}}/B$$

where $P_{\alpha_{i_j}}$ is the minimal parabolic associated to the simple reflection r_{i_j} and the quotient results in an equivalence of elements given by $(g_1, g_2, \ldots, g_\ell) \sim (g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{\ell-1}^{-1} g_\ell b_\ell)$. We denote the resulting equivalence classes with square brackets, i.e. $[g_1, g_2, \ldots, g_\ell] \in BS^Q$.

There is an action by T on the left of BS^Q with $2^{\#Q}$ fixed points; more specifically the set of sequences $(g_1, g_2, \ldots, g_\ell) \in P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_\ell}}$ such that $\forall j,\ g_j \in \{1, s_j\}$ maps bijectively to the fixed point set $(BS^Q)^T$. In this way we index the fixed points by subsets $L \subseteq \{1, \ldots, \ell\}$, but instead of writing "L is the $\{2,3\}$ subword of (r_1, r_2, r_1)" we will write "L is the subword $-r_2r_1$ of (r_1, r_2, r_1)", allowing e.g. distinction between the $r_1 - -$ and $- - r_1$ subwords. In addition, the inclusion of the fixed points induces a map in equivariant cohomology

$$H^*_T(BS^Q) \longrightarrow \bigoplus_{L \subseteq Q} H^*_T \tag{2.1}$$

which is known to be an injection.

For any subword $L = s_{t_1} \cdots s_{t_k}$ of Q, there is a corresponding copy of BS^L obtained as a submanifold of BS^Q by

$$BS^L = \left\{ [g_1, \cdots, g_\ell] \in BS^Q \mid g_j = 1 \text{ if } j \notin L \right\}.$$

The submanifolds BS^L are T-invariant, and each $BS^L_\circ := BS^L \setminus \bigcup_{M \subsetneq L} BS^M$ contains a unique T-fixed point $[g_1, \ldots, g_\ell] \in BS^L$, the one we also corresponded to L.

The equivariant homology classes $\{[BS^L] \ : \ L \subseteq Q\}$ form a basis of $H^T_*(BS^Q)$ as a (free) module over H^*_T. There exists a dual basis $\{T_J\}_{J \subseteq Q}$ of $H^*_T(BS^Q)$, again defined by the H^*_T-valued Alexander pairing $\langle, \rangle$; we compute its point restrictions in Lemma 2.

Consider the natural map $\pi_R :\ BS^R \to G/B$ that multiplies the terms, $[g_1, \ldots, g_\ell] \mapsto (\prod_i g_i)B/B$. The image is B-invariant, irreducible, and closed, so necessarily some X^w (but w may not be $\prod R$). However $\dim BS^R = \dim X^w$ if and only if R is a reduced word, in which case the top homology class of BS^R pushes forward to that of X^w. The pushforward sends the homology class of BS^R to that of X^w in G/B whenever R is a reduced word for w, and otherwise sends it to 0. These statements are true both for singular homology and also, since the varieties involved are T-invariant, for equivariant homology [9, 3].

We are interested in the transpose map in equivariant cohomology, where we have the dual bases $\{T_J\}$, $\{S_w\}$ of $H^*_T(BS^Q)$, $H^*_T(G/B)$ respectively. Since $(\pi_Q)_*([BS^R]) = [X^w]$ in equivariant homology, the transpose statement is the lemma:

Lemma 1. *Let $\pi_Q : BS^Q \to G/B$ be the product map. Then*

$$\pi_Q^*(S_w) = \sum_{\substack{R \subseteq Q \text{ reduced} \\ \prod R = w}} T_R.$$

Proof. Let $[BS^L]$, $[X^w]$ denote the equivariant homology classes, and $\langle, \rangle_M$ denote the perfect H^*_T-valued pairing between $H^T_*(M)$ and $H^*_T(M)$ for M a smooth compact oriented

T-manifold. Then

$$\begin{aligned}\langle \pi_Q^*(S_w), [BS^L]\rangle_{BS^Q} &= \langle S_w, (\pi_Q)_*([BS^L])\rangle_{G/B} \\ &= \begin{cases} \langle S_w, [X^v]\rangle & \text{if } L \text{ is reduced, with product } v \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } L \text{ is reduced, with product } w \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Since the $\{T_R\}$ are defined so that $\langle T_R, [BS^L]\rangle = \delta_{RL}$, we conclude that $\pi_Q^*(S_w) = \sum_{\substack{R\subseteq Q \text{ reduced} \\ \prod R = w}} T_R$. □

We pull back the equation $S_u S_v = \sum_{x\in W} c_{uv}^x S_x$ along $\pi_Q : BS^Q \to G/B$ and simplify the right hand side of the equation:

$$\pi_Q^*(S_u)\, \pi_Q^*(S_v) = \sum_{x\in W} c_{uv}^x\, \pi_Q^*(S_x) = \sum_{x\in W} c_{uv}^x \sum_{\substack{R\subseteq Q \text{ reduced} \\ \prod R = x}} T_R = \sum_{R\subseteq Q \text{ reduced}} c_{uv}^{\prod R}\, T_R. \tag{2.2}$$

By expanding the left hand side in a similar fashion, we obtain

$$\pi_Q^*(S_u)\pi_Q^*(S_v) = \sum_{\substack{R\subseteq Q \text{ reduced} \\ \prod R = u}} T_R \sum_{\substack{S\subseteq Q \text{ reduced} \\ \prod S = v}} T_S = \sum_{\substack{R,S\subseteq Q \text{ reduced} \\ \prod R = u, \prod S = v}} T_R T_S.$$

Define b_{RS}^J to be the structure constants for the multiplication in $H_T^*(BS^Q)$ in the basis $\{T_J\}$, defined by the relationship

$$T_R T_S = \sum_{J\subset Q} b_{RS}^J T_J.$$

Thus we have shown

$$\pi_Q^*(S_u)\pi_Q^*(S_v) = \sum_{\substack{R,S\subseteq Q \text{ reduced} \\ \prod R = u, \prod S = v}} \sum_{J\subset Q} b_{RS}^J T_J. \tag{2.3}$$

Now we take Q to be reduced with product w and look at the coefficient of T_Q in (2.2) and (2.3):

$$c_{uv}^w = \sum_{\substack{R,S\subset Q \text{ reduced} \\ \prod R = u, \prod S = v}} b_{RS}^Q. \tag{2.4}$$

Theorem 2. *Let the equivariant intersection numbers b_{RS}^Q be defined as above. Then,*

$$b_{RS}^Q = \prod_{q\in Q} \left(\alpha_q^{[q\in R,S]} \partial_q^{[q\notin R,S]} r_q \right) \cdot 1$$

where the exponent $[q \in J]$ indicates inclusion of the factor only when $q \in J$.

Theorem 1 then follows directly from Theorem 2 and (2.4).

The proof of Theorem 2 is an inductive argument based on Lemma 2 below; both proofs will appear elsewhere.

As with Schubert classes, we define the point restriction $T_J|_L$ to be the restriction of $T_J \in H_T^*(BS^Q)$ under the map (2.1) to the fixed point $L \subseteq Q$. These restrictions can be computed explicitly:

Lemma 2. *The equivariant class $T_J \in H_T^*(BS^Q)$ has the following restriction to a T-fixed point L:*

$$T_J|_L = \begin{cases} \left(\prod_{m \in L} \alpha_m^{[m \in J]} r_m \right) \cdot 1 & \text{if } J \subseteq L \\ 0 & \text{if } J \not\subseteq L. \end{cases}$$

where the exponent $[m \in J]$ indicates inclusion of the factor only when $m \in J$.

In the remainder we present these coefficients in terms of some apparently natural families of operators, based on reflections and divided difference operators.

3 AJS/Billey operators

In the next two sections we interpret the AJS/Billey formula, and Theorem 1, in terms of certain operators; our results are that these operators satisfy the various (nil-)Coxeter relations. We hope someday to run the arguments backward and use the relations to give an algebraic proof of Theorem 1.

Let $H_T^*[W]$ be the smash product of H_T^* and the group algebra of W, i.e. the free H_T^*-module with basis W and multiplication $wp := (w \cdot p)w$. For each $w \in W$, we introduce an **AJS/Billey operator**

$$J_w := \sum_{v \leq w} (S_v|_w) w \otimes \partial_v \quad \in H_T^*[W] \otimes_{\mathbb{Z}} \mathbb{Z}[\partial] \tag{3.1}$$

so in particular

$$J_\alpha := J_{r_\alpha} = (r_\alpha \otimes 1) + (\alpha r_\alpha \otimes \partial_\alpha).$$

Note that these operators are homogeneous of degree 0, where the degrees of $\alpha, r_\alpha, \partial_\alpha$ are $+1, 0, -1$ respectively.

Theorem 3. *1. If Q is a reduced word for w, then $J_w = \prod_Q J_q$.*

2. If $\ell(w) + \ell(v) = \ell(wv)$, then $J_w J_v = J_{wv}$, and this fact is essentially equivalent to the AJS/Billey formula.

3. $J_\alpha^2 = 1 \otimes 1$, so in fact any word Q for w suffices in (1), and $J_w J_v = J_{wv}$ for all w, v.

Proof. 1. Let Q be a reduced word for w. Then since $S_v|_{r_\alpha}$ is 0 unless $v = 1$ or $v = r_\alpha$,

$$\begin{aligned}\prod_Q J_q &= \prod_Q \sum_{v \leq r_q} (S_v|_{r_q}) r_q \otimes \partial_q = \prod_Q \left((r_q \otimes 1) + (\alpha_q r_q \otimes \partial_q) \right) \\ &= \sum_{R \subseteq Q} \left(\prod_Q \alpha_q^{[q\in R]} r_q \right) \otimes \prod_R \partial_r = \sum_v \sum_{\substack{R \subseteq Q \text{ reduced} \\ \prod R = v}} \left(\prod_Q \alpha_q^{[q\in R]} r_q \right) \otimes \partial_v\end{aligned}$$

as $\prod_R \partial_r = 0$ unless R is reduced. The AJS/Billey formula states that

$$\sum_{\substack{R \subseteq Q \text{ reduced} \\ \prod R = v}} \prod_Q \alpha_q^{[q\in R]} r_q = S_v|_w w,$$

from which it follows that

$$\prod_Q J_q = \sum_{v \leq w} (S_v|_w) w \otimes \partial_v = J_w.$$

2. From (1) the equality $J_w J_v = J_{wv}$ follows by concatenating words for w and v. Conversely, the equality implies $J_w = \prod_Q J_q$ when Q is a reduced word for w, which in turn implies the AJS/Billey formula by the calculation above.

3.

$$\begin{aligned}J_\alpha^2 &= ((r_\alpha \otimes 1) + (\alpha r_\alpha \otimes \partial_\alpha))^2 = ((r_\alpha \otimes 1) + (\alpha r_\alpha \otimes \partial_\alpha))\,((r_\alpha \otimes 1) + (\alpha r_\alpha \otimes \partial_\alpha)) \\ &= (1 \otimes 1) + (r_\alpha \alpha r_\alpha \otimes \partial_\alpha) + (\alpha \otimes \partial_\alpha) + (\alpha r_\alpha \alpha r_\alpha \otimes \partial_\alpha^2) = 1 \otimes 1\end{aligned}$$

□

Let $(G/B)_\Delta$ denote the diagonal copy of G/B in $(G/B)^2$, which is invariant under the diagonal T-action on $(G/B)^2$. The corresponding Poincaré dual class $D^{w_0} \in H_T^*((G/B)^2)$ of this submanifold can be described explicitly in terms of the Poincaré duals $S^v \in H_T^*(G/B)$ to the X^v. Under the isomorphism

$$H_T^*((G/B)^2) \cong H_T^*(G/B) \otimes_{H_T^*} H_T^*(G/B)$$

we have from [4] the factorization of the diagonal

$$D^{w_0} = \sum_v S_v \otimes S^v \quad = \sum_v S_v \otimes (\partial_v \cdot S^1) \tag{3.2}$$

Consider its restriction along $i_w \times Id : \{wB/B\} \times G/B \to (G/B)^2$:

$$D^{w_0} = \sum_v S_v \otimes (\partial_v \cdot S^1) \xmapsto{(i_w \times Id)^*} \sum_v (S_v|_w) \otimes (\partial_v \cdot S^1) = J_w \cdot (S_1 \otimes S^1).$$

While we won't directly use this suggestive calculation of the $S_v|_w$, it will inform a similar operator-theoretic calculation of the c_{uv}^w in the next section. Towards that end we rephrase the equation above using the equivariant Euler class $e(T\,G/B)$ of the tangent bundle:

$$(e(T\,G/B) \otimes 1)\ D^{w_0} = \sum_{w \in W} (i_w \times Id)_* \big(J_w \cdot (S_1 \otimes S^1) \big) \tag{3.3}$$

4 Schubert structure operators

Analogously to $J_\alpha \in H_T^*[W] \otimes \mathbb{Z}[\partial]$, we introduce in $H_T^*[\partial] \otimes \mathbb{Z}[\partial] \otimes \mathbb{Z}[\partial]$ elements

$$K^\alpha := (\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha),$$

where $r_\alpha = (1 - \alpha\partial_\alpha) \in H_T^*[\partial]$. These are homogeneous of degree -1. Note that $r_\alpha \partial_\alpha = \partial_\alpha = -\partial_\alpha r_\alpha$.

Lemma 3. $(K^\alpha)^2 = 0$.

Proof. At the end we use the equality of operators $\partial_\alpha \alpha + \alpha \partial_\alpha = 2$, derivable from the twisted Leibniz identity $\partial_\alpha \cdot (xy) = (\partial_\alpha \cdot x)y + (r_\alpha \cdot x)(\partial_\alpha \cdot y)$.

$$\begin{aligned}
(K^\alpha)^2 &= (\partial_\alpha r_\alpha \otimes 1 \otimes 1)\left((\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\right) \\
&+ (r_\alpha \otimes \partial_\alpha \otimes 1)\left((\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\right) \\
&+ (r_\alpha \otimes 1 \otimes \partial_\alpha)\left((\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\right) \\
&+ (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\left((\partial_\alpha r_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\right) \\
&= (\partial_\alpha r_\alpha \partial_\alpha r_\alpha \otimes 1 \otimes 1) + (\partial_\alpha r_\alpha r_\alpha \otimes \partial_\alpha \otimes 1) + (\partial_\alpha r_\alpha r_\alpha \otimes 1 \otimes \partial_\alpha) + (\partial_\alpha r_\alpha \alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha) \\
&+ (r_\alpha \partial_\alpha r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha r_\alpha \otimes \partial_\alpha \partial_\alpha \otimes 1) + (r_\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha) + (r_\alpha \alpha r_\alpha \otimes \partial_\alpha \partial_\alpha \otimes \partial_\alpha) \\
&+ (r_\alpha \partial_\alpha r_\alpha \otimes 1 \otimes \partial_\alpha) + (r_\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha) + (r_\alpha r_\alpha \otimes 1 \otimes \partial_\alpha \partial_\alpha) + (r_\alpha \alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha \partial_\alpha) \\
&+ (\alpha r_\alpha \partial_\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha) + (\alpha r_\alpha r_\alpha \otimes \partial_\alpha \partial_\alpha \otimes \partial_\alpha) + (\alpha r_\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha \partial_\alpha) + (\alpha r_\alpha \alpha r_\alpha \otimes \partial_\alpha \partial_\alpha \otimes \partial_\alpha \partial_\alpha) \\
&= 0 + (\partial_\alpha \otimes \partial_\alpha \otimes 1) + (\partial_\alpha \otimes 1 \otimes \partial_\alpha) - (\partial_\alpha \alpha \otimes \partial_\alpha \otimes \partial_\alpha) - (\partial_\alpha \otimes \partial_\alpha \otimes 1) + 0 + (1 \otimes \partial_\alpha \otimes \partial_\alpha) + 0 \\
&- (\partial_\alpha \otimes 1 \otimes \partial_\alpha) + (1 \otimes \partial_\alpha \otimes \partial_\alpha) + 0 + 0 - (\alpha \partial_\alpha \otimes \partial_\alpha \otimes \partial_\alpha) + 0 + 0 + 0 \\
&= -(\partial_\alpha \alpha \otimes \partial_\alpha \otimes \partial_\alpha) + (1 \otimes \partial_\alpha \otimes \partial_\alpha) + (1 \otimes \partial_\alpha \otimes \partial_\alpha) - (\alpha \partial_\alpha \otimes \partial_\alpha \otimes \partial_\alpha) \\
&= (2 - \alpha\partial_\alpha - \partial_\alpha \alpha) \otimes \partial_\alpha \otimes \partial_\alpha = 0.
\end{aligned}$$

□

Theorem 4. *The operators K^α obey the commutation and (simply- or doubly-laced) braid relations, and as such, we can define $K^w := \prod_Q K^q$ (for W simply- or doubly-laced) using any reduced word Q for w.*

Proof. The commutation operations are obvious. For braiding, we compute $K^\alpha K^\beta K^\alpha$ for the simple roots in SL_3.

$$\begin{aligned}
K^\alpha K^\beta K^\alpha &= \big(-(\partial_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\big) \\
&\quad \big(-(\partial_\beta \otimes 1 \otimes 1) + (r_\beta \otimes \partial_\beta \otimes 1) + (r_\beta \otimes 1 \otimes \partial_\beta) + (\beta r_\beta \otimes \partial_\beta \otimes \partial_\beta)\big) \\
&\quad \big(-(\partial_\alpha \otimes 1 \otimes 1) + (r_\alpha \otimes \partial_\alpha \otimes 1) + (r_\alpha \otimes 1 \otimes \partial_\alpha) + (\alpha r_\alpha \otimes \partial_\alpha \otimes \partial_\alpha)\big)
\end{aligned}$$

We group the 4^3 terms (15 of which vanish by $\partial_\alpha^2 = 0$) according to their second and third tensor factors Using the relations

$$\partial_\alpha \alpha = 2 - \alpha\partial_\alpha \qquad \partial_\beta \beta = 2 - \beta\partial_\beta \qquad \partial_\alpha \beta = -1 + \alpha\partial_\alpha + \beta\partial_\alpha \qquad \partial_\beta \alpha = -1 + \alpha\partial_\beta + \beta\partial_\beta$$

we can write each matrix entry uniquely as $\sum_w h_w \partial_w$, $h_w \in H_T^*$, to compare the two operators. We left the resulting comparison of > 1000 terms to a computer. The corresponding B_2 calculation involved closer to $140,000$ terms. □

We are confident that the K^α satisfy the G_2 braid relation but have not done the computation (having run out of memory at $3M+$ terms).

As a result of Theorem 4, we may define operators $d_{uv}^w \in H_T^*[\partial]$ by

$$K^w := \sum_{u,v} d_{uv}^w w \otimes \partial_u \otimes \partial_v.$$

The successive application of K^α for each reflection r_α in a reduced word for w then results in the statement that

$$d_{uv}^w w = \prod_Q \left(\alpha_q^{[q\in R,S]} \partial_q^{[q\notin R,S]} r_q \right)$$

As these operators applied to 1 are the terms appearing in Theorem 1, we deduce that

$$K^w(S_1 \otimes S^1 \otimes S^1) = \sum_{u,v} c_{uv}^w \otimes S^u \otimes S^v$$

which we now manipulate to get a K^α analogue of (3.3).

Let $D_{12} \in H_T^*\left((G/B)^3\right)$ denote the Poincaré dual of the partial diagonal $\{(F_1, F_2, F_3) \in (G/B)^3 \ : \ F_1 = F_2\}$, and D_{13} denote that of $\{(F_1, F_2, F_3) \in (G/B)^3 \ : \ F_1 = F_3\}$ likewise. Then $D_{123} := D_{12} \cap D_{13}$ is the class of the full diagonal. By two applications of (3.2), we get

$$\begin{aligned} D_{123} &= D_{12} \cap D_{23} = \left(\sum_u (S_u \otimes S^u \otimes 1) \right) \left(\sum_v (S_v \otimes 1 \otimes S^v) \right) = \sum_{u,v} S_u S_v \otimes S^u \otimes S^v \\ &= \sum_{u,v} \left(\sum_w c_{uv}^w S_w \right) \otimes S^u \otimes S^v = \sum_w (S_w \otimes 1 \otimes 1) \sum_{u,v} (c_{uv}^w \otimes S^u \otimes S^v) \end{aligned}$$

Combined with the above equation, we get

$$D_{123} = \sum_w (S_w \otimes 1 \otimes 1)\, K^w(S_1 \otimes S^1 \otimes S^1), \tag{4.1}$$

a distinct echo of (3.3).

Question. What is a closed form for K^w, analogous to that of J^w in (3.1)?

5 Recursive formulas for structure constants

Corollary 1. *Fix a reflection r_α, and let $\bar{s}$ denote $r_\alpha s$ for $s \in W$. If $\overline{w} < w$, then*

$$c_{uv}^w = (\partial_\alpha r_\alpha) \cdot c_{uv}^{\overline{w}} + [\overline{u} < u] c_{\overline{u},v}^{\overline{w}} + [\overline{v} < v] c_{u,\overline{v}}^{\overline{w}} + [\overline{u} < u][\overline{v} < v] \alpha c_{\overline{u},\overline{v}}^{\overline{w}}$$

where $[\overline{s} < s]$ indicates 1 if $\overline{s} < s$, and 0 otherwise (i.e. $\overline{s} > s$).

Similarly, let $\underline{s}$ denote sr_α. If $\underline{w} < w$, then

$$c^w_{uv} \quad = \quad [\underline{u} < u]\left(c^{\underline{w}}_{\underline{u},v}\right) \quad + \quad [\underline{v} < v]\left(c^{\underline{w}}_{u,\underline{v}}\right) \quad + \quad [\underline{u} < u][\underline{v} < v]\left(d^{\underline{w}}_{\underline{u},\underline{v}} \cdot \alpha\right)$$

Proof. Suppose $w = r_\alpha r_{\alpha_1} \cdots r_{\alpha_k}$ is a reduced word expression for w. Then $K^w = K^\alpha K^{\overline{w}}$, where $\overline{w} = r_\alpha w$. In particular

$$\sum_{u,v} c^w_{uv} \otimes S^u \otimes S^v = K^w(S_1 \otimes S^1 \otimes S^1) = \left(K^\alpha \sum_{s,t} d^{\overline{w}}_{st}\overline{w} \otimes \partial_s \otimes \partial_t\right)(S_1 \otimes S^1 \otimes S^1)$$
$$= \sum_{s,t} \big(\partial_\alpha r_\alpha d^{\overline{w}}_{st}\overline{w} \otimes \partial_s \otimes \partial_t + r_\alpha d^{\overline{w}}_{st}\overline{w} \otimes \partial_\alpha\partial_s \otimes \partial_t + r_\alpha d^{\overline{w}}_{st}\overline{w} \otimes \partial_s \otimes \partial_\alpha\partial_t$$
$$+ \alpha r_\alpha d^{\overline{w}}_{st}\overline{w} \otimes \partial_\alpha\partial_s \otimes \partial_\alpha\partial_t\big)(S_1 \otimes S^1 \otimes S^1)$$

The term $c^w_{uv} \otimes S^u \otimes S^v$ on the left is obtained as the image of $S_1 \otimes S^1 \otimes S^1$ under those tensors with terms $\partial_u \otimes \partial_v$ in the second and third positions. Note that $\partial_\alpha\partial_s = \partial_{s'}$ exactly when $r_\alpha s = s'$ *and* $\ell(s') = \ell(s) + 1$. If $r_\alpha s = s'$ but $\ell(s') \neq \ell(s) + 1$, then $\partial_\alpha\partial_s = 0$. Let $\overline{v} = r_\alpha v$ and $\overline{u} = r_\alpha u$. By matching the terms,

$$c^w_{uv} \otimes S^u \otimes S^v = \big(\partial_\alpha r_\alpha d^{\overline{w}}_{uv}\overline{w} \otimes \partial_u \otimes \partial_v + r_\alpha d^{\overline{w}}_{\overline{u},v}\overline{w} \otimes \partial_\alpha\partial_{\overline{u}} \otimes \partial_v + r_\alpha d^{\overline{w}}_{u,\overline{v}}\overline{w} \otimes \partial_u \otimes \partial_\alpha\partial_{\overline{v}}$$
$$+ \alpha\partial_\alpha d^{\overline{w}}_{\overline{u},\overline{v}}\overline{w} \otimes \partial_\alpha\partial_{\overline{u}} \otimes \partial_\alpha\partial_{\overline{v}}\big)(S_1 \otimes S^1 \otimes S^1)$$
$$= \big(\partial_\alpha r_\alpha d^{\overline{w}}_{uv}\overline{w} \otimes \partial_u \otimes \partial_v + [\overline{u} < u]\, r_\alpha d^{\overline{w}}_{\overline{u},v}\overline{w} \otimes \partial_u \otimes \partial_v + [\overline{v} < v]\, r_\alpha d^{\overline{w}}_{u,\overline{v}}\overline{w} \otimes \partial_u \otimes \partial_v$$
$$+ [\overline{u} < u][\overline{v} < v]\, \alpha r_\alpha d^{\overline{w}}_{\overline{u},\overline{v}}\overline{w} \otimes \partial_u \otimes \partial_v\big)(S_1 \otimes S^1 \otimes S^1).$$

We evaluate the expression on the right and isolate the first tensor to obtain

$$c^w_{uv} = (\partial_\alpha r_\alpha d^{\overline{w}}_{uv}\overline{w}) \cdot 1 + [\overline{u} < u]\ (r_\alpha d^{\overline{w}}_{\overline{u},v}\overline{w}) \cdot 1 + [\overline{v} < v]\ (r_\alpha d^{\overline{w}}_{u,\overline{v}}\overline{w}) \cdot 1 + [\overline{u} < u][\overline{v} < v]\ (\alpha r_\alpha d^{\overline{w}}_{\overline{u},\overline{v}}\overline{w}) \cdot 1$$
$$= (\partial_\alpha r_\alpha d^{\overline{w}}_{uv}) \cdot 1 + [\overline{u} < u]c^{\overline{w}}_{\overline{u},v} + [\overline{v} < v]c^{\overline{w}}_{u,\overline{v}} + [\overline{u} < u][\overline{v} < v]\alpha c^{\overline{w}}_{\overline{u},\overline{v}}.$$

A similar proof holds for the second recursion. □

We finish with an example illustrating the use of the first recursive formula.

Example 1. We compute $c^w_{u,v}$ in the S_3 case, with $u = [312]$, $v = [132]$ and $w = w_0 = [321]$ in 1-line notation. First we use $\overline{w} = r_1 w$. Then $\overline{u} = r_1 u \not< u$ and $\overline{v} = r_1 v \not< v$. The three latter terms in the sum of the first recursion relationship drop out and we obtain

$$c^{[321]}_{[312],[132]} = c^w_{uv} = \partial_1 r_1 \cdot c^{\overline{w}}_{uv} = \partial_1 r_1 \cdot c^{[312]}_{[312],[132]}$$

We set about to compute $c^{\overline{w}}_{uv}$. Note that $r_2 r_1$ is a reduced word for $\overline{w}$. There is only one subword for u, mainly $r_2 r_1$, and one subword for v, mainly r_2-. Therefore $c^{\overline{w}}_{uv} = \alpha_2 r_2 r_1 \cdot 1$ and we obtain

$$c^w_{uv} = \partial_1 r_1 \alpha_2 r_2 r_1 \cdot 1 = \partial_1(r_1(\alpha_2)) = \partial_1(\alpha_1 + \alpha_2) = 1.$$

As a check on this result, we consider the recursion with r_2 instead of r_1, so $\overline{w} = r_2 w = [231]$. Then $\overline{u} = r_2 u = [213] < u$ and $\overline{v} = r_2 v = 1 \leq v$. In principle all four terms are nonzero:

$$c^w_{uv} = \partial_2 r_2 \cdot c^{\overline{w}}_{uv} + c^{\overline{w}}_{\overline{u},v} + c^{\overline{w}}_{u,\overline{v}} + \alpha c^{\overline{w}}_{\overline{u},\overline{v}}.$$

However $u \not\leq \overline{w}$, so the first and third terms $c^{\overline{w}}_{uv}$ and $c^{\overline{w}}_{u,\overline{v}}$ vanish. The last term $c^{\overline{w}}_{\overline{u},\overline{v}} = c^{[231]}_{[213],1} = 0$ because $S_{[213]} S_1 = S_{[213]}$. Thus $c^w_{uv} = c^{\overline{w}}_{\overline{u},v} = c^{[231]}_{[213],[132]}$ is the only remaining nonzero term. This smaller structure constant is easily seen to be 1, for instance by another application of same inductive formula with $r_1[231] = [132] < [231]$. Note that $r_1[132] \not< [132]$ which forces two terms in the recursive sum to be 0. We obtain

$$c^{[231]}_{[213],[132]} \quad = \quad \partial_1 r_1 \cdot c^{[132]}_{[213],[132]} \quad + \quad c^{[132]}_{1,[132]} \quad = \quad 0 \quad + \quad 1$$

where the last two equalities follow from $[213] \not\leq [132]$ and $S_1 S_{[132]} = S_{[132]}$.

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