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Birational antichain toggling and rowmotion

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Abstract. We define a new combinatorial dynamical system on node-labelings of a poset P by rational functions called *birational antichain rowmotion* (*BAR-motion* for short). It is analogous to the previously-studied *birational rowmotion* action (here called *birational order rowmotion*, or *BOR-motion*). The latter action is detropicalized from an extension of combinatorial rowmotion on order ideals of P to Stanley's *order polytope* $\mathcal{OP}(P)$. Analogously BAR-motion detropicalizes the extension of rowmotion on *antichains* of P to the *chain polytope* $\mathcal{C}(P)$.

We study BAR-motion by defining a birational antichain *toggle group* generated by involutions called *toggles*. This lifts Striker's toggle group on the set of antichains of a poset to the birational realm. We construct an explicit isomorphism between this group and Einstein and Propp's group of birational order toggles, lifting an analogous one between the toggle groups of order ideals and antichains at the combinatorial level.

For certain nice families of posets, the order of BOR-motion is known to be finite, and can often be easily computed. We lift Stanley's transfer map between $\mathcal{C}(P)$ and $\mathcal{OP}(P)$ to a birational transfer map, which allows us to easily deduce certain properties of one kind of birational rowmotion from the other. We take advantage of this to derive the periodicity and order of BAR-motion on certain root and minuscule posets. We also lift a refined homomesy result of Propp and the second author from the combinatorial setting to the birational one, using an analogue of the "Stanley–Thomas" word, which cyclically rotates equivariantly with BAR-motion.

Keywords: birational rowmotion, dynamical algebraic combinatorics, homomesy, periodicity, poset, toggling.

1 Introduction

Combinatorial rowmotion is a particular permutation of the set of order ideals $\mathcal{J}(P)$ of a finite poset P or of the set of antichains $\mathcal{A}(P)$ of P. It was first studied as a map on $\mathcal{A}(P)$ by Brouwer and Schrijver [2], and goes by several names; in recent literature, the name "rowmotion," due to Striker and Williams [19] (who summarize the history), has stuck.

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Rowmotion has proven to be of great interest in dynamical algebraic combinatorics. On several "nice" posets (e.g., positive root poset or minuscule posets, such as a product of two chains), rowmotion exhibits various phenomena including *periodicity* (of a relatively small order), *cyclic sieving* (as defined by Reiner, Stanton, and White [13]), and *homomesy* (where a natural statistic, e.g. cardinality, has the same average over every orbit) [1, 8, 10, 12, 15, 20]. Rowmotion is related to Auslander–Reiten translation on certain quivers [21]. Quite surprisingly, some of these features extend to the piecewise-linear (order polytope) level and can be lifted further to the birational level. [4]. One sometimes gets periodicity of the same order as the combinatorial map, and homomesy extends as well. Birational rowmotion is related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [14, §4.4] and to the *R-systems* of Galashin–Pylyavskyy [6].

The lifting of order ideal rowmotion (herein denoted $\rho_{\mathcal{J}}$) to BOR-motion (birational order rowmotion) proceeds by first writing $\rho_{\mathcal{J}}$ as a product of involutions called *toggles*, which act on $\mathcal{J}(P)$, the set of order ideals of a poset. These toggles are then extended to Stanley's *order polytope* $\mathcal{OP}(P)$, which can then be lifted (via detropicalization) to toggles at the birational level [4]. Striker defined *antichain toggles* that act on $\mathcal{A}(P)$, the set of antichains of a poset [18], as part of a broader study of toggling in general. The first author gave an explicit isomorphism between these two different toggle groups for the same poset P, and extended these results to the piecewise-linear level, where $\mathcal{A}(P)$ extends to Stanley's *chain polytope* $\mathcal{C}(P)$ [9]. In this work we lift this as before to obtain a new operation called *Birational Antichain Rowmotion* or *BAR-motion* for short.

We show that *BAR-motion* is periodic on $P = [a] \times [b]$ (product of two chains), with the same order as BOR-motion (or indeed order ideal rowmotion), and lift a homomesy result of [12] for fibers of the poset $P = [a] \times [b]$ (product of two chains) to BAR-motion, which we conjecture holds for all a, but currently only have a proof for $a \leq 2$. Our main tools for doing this include a lifting of Stanley's transfer map between $\mathcal{O}(P)$ and $\mathcal{C}(P)$, and a generalization of the Stanley–Thomas word, which provides an equivariant projection of BAR-motion to cyclic rotation. (See Section 3.)

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1.1 Rowmotion in the combinatorial realm

We assume familiarity with basic notions from the theory of posets (chains, antichains, order ideals, order filters, linear extensions, etc.) as given in [17, Ch. 3]. Throughout this paper P will denote a finite poset.

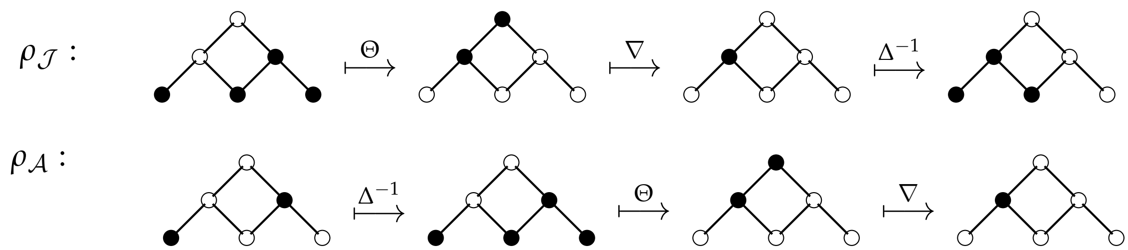
To define rowmotion, following the notation of Einstein-Propp [4], we define the following natural bijections between the sets $\mathcal{J}(P)$ of all *order ideals* of P, $\mathcal{F}(P)$ of all *order filters* of P, and $\mathcal{A}(P)$ of all *antichains* of P.

- The map $\Theta : 2^P \to 2^P$ where $\Theta(S) = P \setminus S$ is the complement of S (so Θ sends order ideals to order filters and vice versa).

- The **up-transfer** $\Delta : \mathcal{J}(P) \to \mathcal{A}(P)$, where $\Delta(I)$ is the set of maximal elements of I. For an antichain $A \in \mathcal{A}(P)$, $\Delta^{-1}(A) = \{x \in P : x \leq y \text{ for some } y \in A\}$.
- The **down-transfer** $\nabla : \mathcal{F}(P) \to \mathcal{A}(P)$, where $\nabla(F)$ is the set of minimal elements of F. For an antichain $A \in \mathcal{A}(P)$, $\nabla^{-1}(A) = \{x \in P : x \geq y \text{ for some } y \in A\}$.

Definition 1.1. Order ideal rowmotion is the map $\rho_{\mathcal{J}} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the composition $\rho_{\mathcal{J}} = \Delta^{-1} \circ \nabla \circ \Theta$. **Antichain rowmotion** is the map $\rho_{\mathcal{A}} : \mathcal{A}(P) \to \mathcal{A}(P)$ given by the composition $\rho_{\mathcal{A}} = \nabla \circ \Theta \circ \Delta^{-1}$.

Example 1.2. Below we show examples of $\rho_{\mathcal{J}}$ and $\rho_{\mathcal{A}}$ on the positive root poset $\Phi^+(A_3)$. In each step, the elements of the subset of the poset are given by the filled-in circles.



1.2 The order ideal toggle group

The map $\rho_{\mathcal{J}}$ can also be written a composition of involutions on $\mathcal{J}(P)$ called *toggles*, as first shown by Cameron and Fon-Der-Flaass [3]. They also proved that for any finite connected poset P, the toggle group $\mathrm{Tog}_{\mathcal{J}}(P)$ generated by $\{t_e : e \in P\}$ is either the symmetric or alternating group on the set $\mathcal{J}(P)$.

Definition 1.3 ([3])**.** For $e \in P$, the **order ideal toggle** corresponding to e is the map $t_e : \mathcal{J}(P) \to \mathcal{J}(P)$ defined by $t_e(I) = \begin{cases} I \cup \{e\} & \text{if } e \notin I \text{ and } I \cup \{e\} \in \mathcal{J}(P), \\ I \setminus \{e\} & \text{if } e \in I \text{ and } I \setminus \{e\} \in \mathcal{J}(P), \\ I & \text{otherwise.} \end{cases}$
Let $\mathrm{Tog}_{\mathcal{J}}(P)$ denote the **toggle group** of $\mathcal{J}(P)$ generated by the toggles $\{t_e : e \in P\}$.

The toggle t_e either adds or removes e from the order ideal if the resulting set is still an order ideal, and otherwise does nothing. The following proposition presents basic properties of order ideal toggles.

Proposition 1.4 ([3])**.** *Each toggle t_x is an involution (i.e., t_x^2 is the identity). Two toggles t_x, t_y commute if and only if neither x nor y covers the other.*

Proposition 1.5 ([3])**.** *For any linear extension $(x_1, x_2, \ldots, x_n)$ of P, order ideal rowmotion is given by $\rho_{\mathcal{J}} = t_{x_1} t_{x_2} \cdots t_{x_n}$.*

1.3 The antichain toggle group

Toggling makes sense in a broader context, as formalized by Striker [18]. We can define *antichain toggles* on $\mathcal{A}(P)$, by replacing $\mathcal{J}(P)$ with $\mathcal{A}(P)$ in the definition. Removing any element from an antichain always yields an antichain, giving a simpler second case.

Definition 1.6 ([18])**.** Let $e \in P$. Then the **antichain toggle** corresponding to e is the map $\tau_e : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_e(A) = \begin{cases} A \cup \{e\} & \text{if } e \notin A \text{ and } A \cup \{e\} \in \mathcal{A}(P), \\ A \setminus \{e\} & \text{if } e \in A, \\ A & \text{otherwise.} \end{cases}$$

Let $\text{Tog}_{\mathcal{A}}(P)$ denote the **toggle group** of $\mathcal{A}(P)$ generated by the toggles $\{\tau_e : e \in P\}$.

Proposition 1.7 ([18])**.** *Each antichain toggle τ_x is an involution. Two toggles τ_x, τ_y commute if and only if $x = y$ or x and y are incomparable.*

The first author constructed an explicit isomorphism between $\text{Tog}_{\mathcal{J}}(P)$ and $\text{Tog}_{\mathcal{A}}(P)$ [9].

Definition 1.8. For $e \in P$, we define the following.

- Set $t_e^* \in \text{Tog}_{\mathcal{A}}(P)$ by $t_e^* := \tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}\tau_e\tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}$ where $e_1, \dots, e_k$ are the elements of P covered by e. (If e is a minimal element of P, then $k = 0$ and $t_e^* = \tau_e$.)
- Set $\eta_e \in \text{Tog}_{\mathcal{J}}(P)$ by $\eta_e := t_{x_1}t_{x_2}\cdots t_{x_k}$ where $(x_1, x_2, \dots, x_k)$ is a linear extension of the subposet $\{x \in P : x < e\}$ of P.[1]
- Let $\tau_e^* := \eta_e t_e \eta_e^{-1} \in \text{Tog}_{\mathcal{J}}(P)$.

The following theorem gives an explicit isomorphism from $\text{Tog}_{\mathcal{A}}(P)$ to $\text{Tog}_{\mathcal{J}}(P)$ via $\tau_e \mapsto \tau_e^*$, with inverse given by $t_e \mapsto t_e^*$.

Theorem 1.9 ([9, Theorems 2.15 and 2.19])**.** *The following diagrams commute.*

$$\begin{array}{ccc} \mathcal{J}(P) & \xrightarrow{\ t_e\ } & \mathcal{J}(P) \\ \Delta\downarrow & & \downarrow\Delta \\ \mathcal{A}(P) & \xrightarrow[\ t_e^*\ ]{} & \mathcal{A}(P) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{J}(P) & \xrightarrow{\ \tau_e^*\ } & \mathcal{J}(P) \\ \Delta\downarrow & & \downarrow\Delta \\ \mathcal{A}(P) & \xrightarrow[\ \tau_e\ ]{} & \mathcal{A}(P) \end{array}$$

Note that in Theorem 1.9, the left commutative diagram together with the basic properties of toggles (Proposition 1.4 and 1.7) are enough to prove the right commutative diagram using group theory alone. A consequence of this isomorphism is that antichain rowmotion is also a product of antichain toggles in an order specified by a linear extension, but in the *opposite* order from order ideal rowmotion.

[1] Any two linear extensions of a poset differ by a sequence of transpositions between adjacent incomparable elements [5]. So η_e is well-defined.

Proposition 1.10 ([9, Prop. 2.24]). *For any linear extension $(x_1, x_2, \ldots, x_n)$ of P, antichain rowmotion is given by $\rho_{\mathcal{A}} = \tau_{x_n} \cdots \tau_{x_2}\tau_{x_1}$.*

1.4 Birational dynamics

The toggling perspective allows us to extend these maps from the combinatorial realm (on finite sets) to the piecewise-linear realm (polytopes whose vertices correspond to these sets), and then lift to the birational realm by detropicalizing the operations [4]. Piecewise-linear dynamics begin with two polytopes introduced by Stanley [16], the *order polytope* and the *chain polytope* of P. The vertices of these polytopes are the sets $\mathcal{F}(P)$ of order filters and $\mathcal{A}(P)$ of antichains (associating a subset of P with its indicator function labeling). Einstein and Propp defined piecewise-linear toggle operations on the order polytope that match the order ideal toggle t_e when restricted to the vertices (though here we use order *filters* instead of order ideals) [4, §3, 4].

This paper focuses on the generalization to the birational realm. Let $\mathbb{K}$ be a field. To define birational toggles, we "detropicalize" the operations from the piecewise-linear toggles, by replacing addition with multiplication and the max operation with addition. The additive identity 0 lifts to 1, and 1 lifts to a generic fixed constant $C \in \mathbb{K}$. This gives the following definition of the birational order toggle. (The definition in [4] is slightly more general than we need here; setting $\alpha = 1$ and $\omega = C$ in their version gives ours.)

Definition 1.11 ([4, Definition 5.1]). Let $\mathbb{K}^P$ be the set of $\mathbb{K}$-labelings of the elements of P. We extend P to the poset $\hat{P}$ by adding a minimal element $\hat{0}$ and maximal element $\hat{1}$. For $e \in P$, the **birational order toggle** at e is the birational map $T_e : \mathbb{K}^P \dashrightarrow \mathbb{K}^P$ given by

$$(T_e(f))(x) = \begin{cases} f(x) & \text{if } x \neq e \\ \dfrac{\sum\limits_{y \in \hat{P}, y \lessdot x} f(y)}{f(e) \sum\limits_{y \in \hat{P}, y \gtrdot x} \frac{1}{f(y)}} & \text{if } x = e \end{cases}$$

where we set $f(\hat{0}) = 1$ and $f(\hat{1}) = C$. The notation $y \lessdot x$ means "x covers y'.

The birational order toggles $\{T_e : e \in P\}$ generate a group, which we will call $\mathrm{BTog}_O(P)$. These are involutions whose commutation is like with order ideal toggles.

Proposition 1.12. *Each toggle T_x is an involution (i.e., T_x^2 is the identity). Two toggles T_x, T_y commute if and only if neither x nor y covers the other.*

Definition 1.13 ([4, Definition 5.2]). Let $(x_1, x_2, \ldots, x_n)$ be any linear extension of P. The birational analogue of order ideal rowmotion, which we will call **birational order rowmotion** (or **BOR-motion**), is $\mathrm{BOR} = T_{x_1} T_{x_2} \cdots T_{x_n}$. (Compare with Proposition 1.5.)

Any property of periodicity or homomesy satisfied by birational toggling and rowmotion also holds in the piecewise-linear realm (by tropicalization) and furthermore in the combinatorial realm (by restriction). What is more surprising, however, is that for certain families of posets (e.g. type A and B root posets and products of two chains), many properties (particularly periodicity) that hold in the combinatorial realm extend to the birational realm; see [4, 7, 11]. Furthermore, studying birational rowmotion can shed light on some patterns in the combinatorial realm, such as why the posets where rowmotion has nice properties are usually graded posets; see [7, §3–6].

1.5 Birational transfer maps

Stanley defined a bijection between the order and chain polytopes of P called the *transfer map*. By detropicalizing the operations, we get the birational analogue of the up-transfer map. Einstein and Propp prove [4, §6] that $\mathrm{BOR} = \Theta \circ \Delta^{-1} \circ \nabla$ where the birational transfer maps are defined below. The order of composition differs from order ideal toggling since birational rowmotion $\rho_{\mathcal{J}}$ is defined through the order filter perspective.

Definition 1.14 ([4, §6])**.** For a labeling $f \in \mathbb{K}^P$ and element $x \in P$,

- $(\Theta f)(x) = \frac{C}{f(x)}$,
- $(\nabla f)(x) = \frac{f(x)}{\sum\limits_{y \lessdot x} f(y)}$ (with $f(\hat{0}) = 1$),
- $(\Delta^{-1} f)(x) = \sum \{ f(y_1) f(y_2) \cdots f(y_k) : x = y_1 \lessdot y_2 \lessdot \cdots \lessdot y_k \lessdot \hat{1} \}$.

The maps Θ, Δ, and ∇ are called **complementation**, **up-transfer**, and **down-transfer**. They generalize those above Definition 1.1. We use the same symbols in both the combinatorial and birational realms, allowing context to clarify which is meant.

2 Birational antichain toggling and rowmotion

2.1 Birational antichain rowmotion

Now we will combine the different generalizations of toggling and study a birational analogue of antichain toggling and rowmotion.

Definition 2.1. Birational antichain rowmotion (or **BAR-motion**) is the birational map $\mathrm{BAR} = \nabla \circ \Theta \circ \Delta^{-1}$ where $\nabla, \Theta, \Delta^{-1}$ are defined in Definition 1.14.

Example 2.2. Figure 1 shows the poset $P = [2] \times [3]$ and a generic labeling $g \in \mathbb{K}^P$. Figure 2 show one iteration of BAR-motion on this labeling.

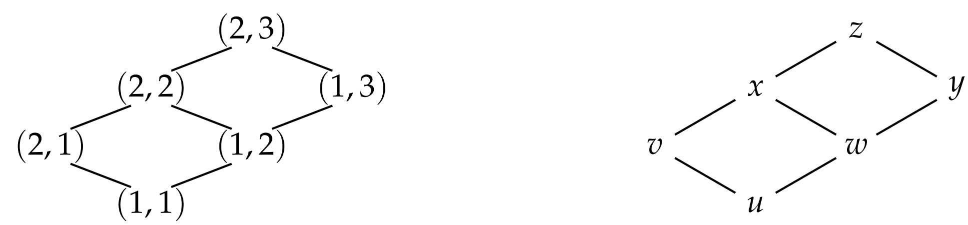


Figure 1: Left: The poset $[2] \times [3]$. Right: A generic labeling on $[2] \times [3]$, used in examples throughout this abstract.

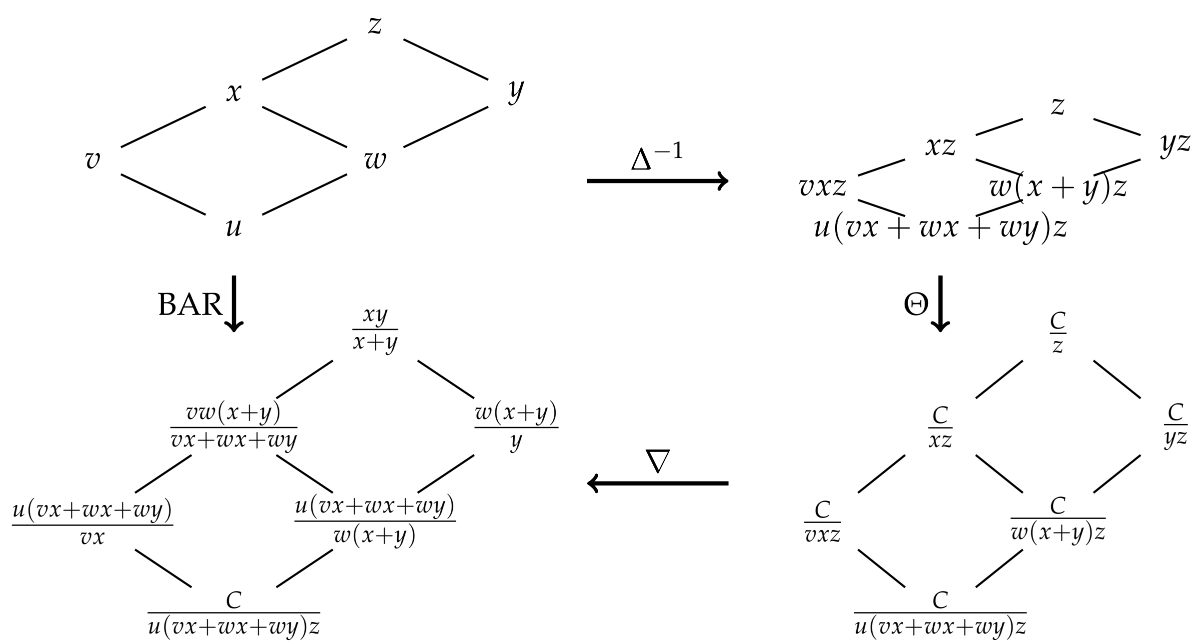


Figure 2: One iteration of BAR-motion on $[2] \times [3]$.

2.2 Birational antichain toggle group

As in the combinatorial realm, we can obtain an equivalent expression of BAR-motion through toggle operations, which we now introduce.

Definition 2.3. Let $e \in P$ and $g \in \mathbb{K}^P$. Then we define $\mathrm{MC}_e(P)$ as the set of all maximal chains of P through e and $\mathrm{Y}_e g = \sum_{(y_1,\dots,y_k)\in \mathrm{MC}_e(P)} g(y_1)\cdots g(y_k)$.

Definition 2.4. Let $e \in P$. The **birational antichain toggle** is the rational map $\tau_e : \mathbb{K}^P \dashrightarrow \mathbb{K}^P$ defined as follows:

$$(\tau_e(g))(x) = \begin{cases} \dfrac{C}{\sum\limits_{(y_1,\dots,y_k)\in \mathrm{MC}_e(P)} g(y_1)\cdots g(y_k)} & \text{if } x = e \\ g(x) & \text{if } x \neq e \end{cases} = \begin{cases} \dfrac{C}{\mathrm{Y}_e g} & \text{if } x = e \\ g(x) & \text{if } x \neq e. \end{cases}$$

Let $\mathrm{BTog}_A(P)$ denote the group generated by the toggles $\{\tau_e : e \in P\}$. We call $\mathrm{BTog}_A(P)$ the **birational antichain toggle group**.

The definition of the birational antichain toggle comes directly from detropicalizing the operations in the toggles on the chain polytope (as defined in [9, §3]), which correspond to the antichain toggles when restricting to the vertices of the chain polytope. Note that we use the same notation for (combinatorial) antichain toggles τ_e and birational antichain toggles. In the remainder of this abstract, we are working in the birational realm (except briefly in Section 3 when we discuss the combinatorial results we are lifting to the birational realm). It is not hard to verify the following basic properties of antichain toggles hold for the birational liftings too.

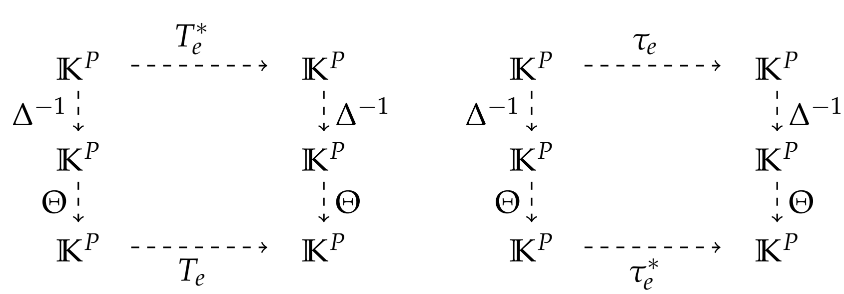
Proposition 2.5. *Each birational antichain toggle τ_x is an involution. Two toggles τ_x, τ_y commute if and only if $x = y$ or x and y are incomparable.*

We construct an isomorphism between the birational toggle groups $\mathrm{BTog}_O(P)$ and $\mathrm{BTog}_A(P)$ analogous to one for the combinatorial toggle groups $\mathrm{Tog}_{\mathcal{J}}(P)$ and $\mathrm{Tog}_{\mathcal{A}}(P)$.

Definition 2.6. For $e \in P$, we define the following.

- Set $T_e^* \in \mathrm{BTog}_A(P)$ by $T_e^* := \tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}\tau_e\tau_{e_1}\tau_{e_2}\cdots\tau_{e_k}$ where $e_1,\dots,e_k$ are the elements of P covered by e. (If e is a minimal element of P, then $k = 0$ and $t_e^* = \tau_e$.)
- Set $\eta_e \in \mathrm{BTog}_O(P)$ by $\eta_e := T_{x_1}T_{x_2}\cdots T_{x_k}$ where $(x_1, x_2, \dots, x_k)$ is a linear extension of the subposet $\{x \in P : x < e\}$ of P.
- Let $\tau_e^* := \eta_e T_e \eta_e^{-1} \in \mathrm{BTog}_O(P)$.

Theorem 2.7. *Let $e \in P$. Then the following diagrams commute on the domains in which the maps are defined. So there is an isomorphism from $\mathrm{BTog}_A(P)$ to $\mathrm{BTog}_O(P)$ given by $\tau_e \mapsto \tau_e^*$ with inverse given by $T_e \mapsto T_e^*$.*



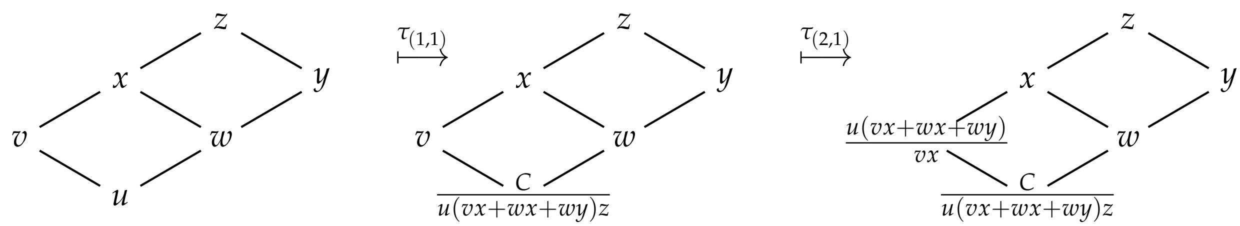


Figure 3: The effect of applying the toggles $\tau_{(1,1)}$ and then $\tau_{(2,1)}$, as in Example 2.9.

The proof of Theorem 2.7 is very similar to the piecewise-linear realm proof in [9, Thm. 3.19] but with detropicalized operations.

Using the isomorphism between the two toggle groups, we can prove that for a linear extension $(x_1, x_2, \ldots, x_n)$ of P, $\mathrm{BOR} = T_{x_1} T_{x_2} \cdots T_{x_n} = \tau^*_{x_n} \cdots \tau^*_{x_2} \tau^*_{x_1}$ using a purely group-theoretic proof analogous to the one found in [9, Thm. 3.21]. This gives the following birational generalization of Proposition 1.10.

Proposition 2.8. *Let $(x_1, x_2, \ldots, x_n)$ be any linear extension of a finite poset P. Then* $\mathrm{BAR} = \tau_{x_n} \cdots \tau_{x_2} \tau_{x_1}$.

Example 2.9. Reconsider the poset $P = [2] \times [3]$ and labeling of Figure 1. We perform BAR along the linear extension $\Big((1,1),(2,1),(1,2),(2,2),(1,3),(2,3)\Big)$. There are three maximal chains through the bottom element $(1,1)$ of P:

- $(1,1) \lessdot (2,1) \lessdot (2,2) \lessdot (2,3)$ with product of labels $uvxz$
- $(1,1) \lessdot (1,2) \lessdot (2,2) \lessdot (2,3)$ with product of labels $uwxz$
- $(1,1) \lessdot (1,2) \lessdot (1,3) \lessdot (2,3)$ with product of labels $uwyz$

For $\mathrm{Y}_{(1,1)}g$, we add up the products of the labels on these three maximal chains, and get $\mathrm{Y}_{(1,1)}g = uvxz + uwxz + uwyz = u(vx + wx + wy)z$. Then to apply the toggle $\tau_{(1,1)}$, we change the label of $(1,1)$ from u to $\frac{C}{u(vx+wx+wy)z}$. This is shown in Figure 3. Now we apply $\tau_{(2,1)}$ to $\tau_{(1,1)}g$. There is only one maximal chain through $(2,1)$. The product of labels on that maximal chain is $\mathrm{Y}_{(2,1)}(\tau_{(1,1)}g) = \frac{C}{u(vx+wx+wy)z}vxz$. Thus we change the label of $(2,1)$ from v to $\frac{C}{\frac{C}{u(vx+wx+wy)z}vxz} = \frac{u(vx+wx+wy)}{vx}$ as in Figure 3.

To complete one iteration of BAR we must then apply the toggles $\tau_{(1,2)}$, $\tau_{(2,2)}$, $\tau_{(1,3)}$, and $\tau_{(2,3)}$ in that order, giving the same result as in Figure 2.

Iterating BAR five times to the generic labeling on $[2] \times [3]$ returns the original labeling again. We can explain this on a general product of two chains poset $[a] \times [b]$ with element set $\{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq a, 1 \leq j \leq b\}$ with $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$. For the poset $[a] \times [b]$, Grinberg and the second author proved that BOR-motion has order $a + b$ [7]. From the down-transfer, we can conclude the periodicity of BAR from that of BOR because $\mathrm{BOR} = \nabla^{-1} \circ \mathrm{BAR} \circ \nabla$.

Theorem 2.10. *On $P = [a] \times [b]$, the order of BAR-motion is $a + b$.*

3 Homomesy of fiber products for BAR-motion on $[2] \times [b]$

Let $\mathcal{S}$ be a collection of combinatorial objects, and $f : \mathcal{S} \to \mathbb{K}$ a "statistic" on $\mathcal{S}$. We call f *homomesic* with respect to an invertible action $w : \mathcal{S} \to \mathcal{S}$ if the (arithmetic) average of f over every orbit is the same [12]. In the birational setting, to avoid dealing with taking nth roots, this manifests itself as certain products across an orbit equaling a fixed constant, independent of the initial labels. See [12] for a more careful treatment of combinatorial homomesy and [4, 11] for birational homomesy.

Propp and the second author proved a homomesy result for antichain rowmotion on $[a] \times [b]$ in terms of **fibers** [12], which we lift to the birational realm.

Definition 3.1 ([12, §3.3.2])**.** Fix $a, b \in \mathbb{Z}^+$. For $1 \le k \le a$, the subset $\{(k, \ell) : 1 \le \ell \le b\}$ of $[a] \times [b]$ is called the ***k*th positive fiber**. For $1 \le \ell \le b$, the subset $\{(k, \ell) : 1 \le k \le a\}$ of $[a] \times [b]$ is called the **ℓth negative fiber**.

On $P = [a] \times [b]$, Stanley and Thomas defined an $(a + b)$-tuple $w(A)$ corresponding to an antichain $A \in \mathcal{A}(P)$ described as

$$w_i = \begin{cases} 1 & \text{if } 1 \le i \le a \text{ and } A \text{ has an element in the } a\text{th positive fiber,} \\ 1 & \text{if } a+1 \le i \le a+b \text{ and } A \text{ has NO element in the } (i-a)\text{th negative fiber,} \\ -1 & \text{otherwise.} \end{cases}$$

They proved that $A \mapsto w(A)$ is a bijection between antichains of P and the possible tuples that can occur. Furthermore, applying antichain rowmotion to A cyclically shifts the **Stanley–Thomas word** $w(A)$. This proves rowmotion (in the combinatorial realm) has order $a + b$ and also that the statistics $p_i : \mathcal{A}(P) \to \mathbb{Z}$ and $n_i : \mathcal{A}(P) \to \mathbb{Z}$ where $p_i(A)$ (resp. $n_i(A)$) is 1 if A has an element in the ith positive fiber (resp. negative fiber) and 0 otherwise are homomesic with average $b(a + b)$ for p_i and $a(a + b)$ for n_i on any orbit. As cardinality of an antichain can be expressed as $p_1 + p_2 + \cdots + p_a$, we see that cardinality on $\mathcal{A}(P)$ is homomesic with average $ab/(a + b)$ [12, §3.3.2].

Now we define a birational analogue of the Stanley–Thomas word. One key difference, however, is that a $\mathbb{K}$-labeling of P is no longer uniquely determined by its Stanley–Thomas word. Therefore, it can no longer be used to prove periodicity (which we already proved in Theorem 2.10). However, we can use it to prove an analogue of fiber homomesy in any situation where BAR cyclically rotates the birational Stanley–Thomas word (which so far we have only proven for the case $a = 2$).

Definition 3.2. Let $a, b \in \mathbb{Z}^+$, $P = [a] \times [b]$, and $g \in \mathbb{K}^P$. The **Stanley–Thomas word** ST_g is the $(a + b)$-tuple given by

$$\mathrm{ST}_g(i) = \begin{cases} g(i,1)g(i,2)\cdots g(i,b) & \text{if } 1 \le i \le a, \\ C/\big(g(1,i-a)g(2,i-a)\cdots g(a,i-a)\big) & \text{if } a+1 \le i \le a+b. \end{cases}$$

Example 3.3. Let g be the generic labeling of $[2] \times [3]$ displayed in the right side of Figure 1 and the top left corner of Figure 2. Then

$$\mathrm{ST}_g = \big(\mathrm{ST}_g(1), \mathrm{ST}_g(2), \mathrm{ST}_g(3), \mathrm{ST}_g(4), \mathrm{ST}_g(5)\big) = \big(uwy, vxz, C/(uv), C/(wx), C/(yz)\big).$$

After applying BAR-motion to g (bottom left corner of Figure 2), the Stanley–Thomas word of $\mathrm{BAR}(g)$ is

$$\mathrm{ST}_{\mathrm{BAR}(g)} = \big(C/(yz), uwy, vxz, C/(uv), C/(wx)\big)$$

which is simply a rightward cyclic shift of ST_g.

Theorem 3.4. *Let $P = [2] \times [b]$. For a labeling $g \in \mathbb{K}^P$, $\mathrm{ST}_{\mathrm{BAR}(g)}(i) = \mathrm{ST}_g(i-1)$ for $2 \le i \le 2+b$ and $\mathrm{ST}_{\mathrm{BAR}(g)}(1) = \mathrm{ST}_g(2+b)$. Thus,*

$$\prod_{m=0}^{1+b} (\mathrm{BAR}^m\, g)(k,1)(\mathrm{BAR}^m\, g)(k,2)\cdots(\mathrm{BAR}^m\, g)(k,b) = C^b$$

for $1 \le k \le 2$ and $\prod_{m=0}^{1+b} (\mathrm{BAR}^m\, g)(1,\ell)(\mathrm{BAR}^m\, g)(2,\ell) = C^2$ for $1 \le \ell \le b$.

Proving this result follows from an explicit description of $\mathrm{BAR}(g)$ in terms of the labels in g. The details are omitted, but this can be determined on $[a] \times [b]$ by writing $\mathrm{BAR} = \tau_{1,b} \cdots \tau_{1,2}\tau_{1,1}\tau_{0,b} \cdots \tau_{0,2}\tau_{0,1}$ as in Proposition 2.8. We do not believe $a = 2$ is necessary in Theorem 3.4, and we conjecture the following, which has been verified for $a, b \le 3$.

Conjecture 3.5. *Let $P = [a] \times [b]$. For a labeling $g \in \mathbb{K}^P$, $\mathrm{ST}_{\mathrm{BAR}(g)}(i) = \mathrm{ST}_g(i-1)$ for $2 \le i \le a+b$ and $\mathrm{ST}_{\mathrm{BAR}(g)}(1) = \mathrm{ST}_g(a+b)$. Thus,*

$$\prod_{m=0}^{a+b-1} (\mathrm{BAR}^m\, g)(k,1)(\mathrm{BAR}^m\, g)(k,2)\cdots(\mathrm{BAR}^m\, g)(k,b) = C^b \text{ for } 1 \le k \le a, \text{ and}$$

$$\prod_{m=0}^{a+b-1} (\mathrm{BAR}^m\, g)(1,\ell)(\mathrm{BAR}^m\, g)(2,\ell)\cdots(\mathrm{BAR}^m\, g)(a,\ell) = C^a \text{ for } 1 \le \ell \le b.$$

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