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Multiplicities of Schubert varieties in the symplectic flag variety

Dave Anderson*[1], Takeshi Ikeda†[2], Minyoung Jeon‡[1], and Ryotaro Kawago§[2]

[1]*Department of Mathematics, The Ohio State University, Columbus, OH 43210 USA*
[2]*Department of Applied Mathematics, Okayama University of Science, 1-1 Ridai-cho, Okayama, Japan*

Abstract. Let Ω_w be a Schubert variety in the symplectic flag variety, and let $e_v \in \Omega_w$ be a torus fixed point. We give a combinatorial formula for the Hilbert-Samuel multiplicity of Ω_w at the point e_v, in the case where w is a vexillary signed permutation. Our formula is phrased in terms of excited Young diagrams, extending results by Ghorpade-Raghavan and Ikeda-Naruse for Grassmannians, as well as Li-Yong for vexillary Schubert varieties in type A flag manifolds.

Résumé. Soit Ω_w une variété de Schubert dans la variété de drapeaux symplectiques, et $e_v \in \Omega_w$ un point fixé par le tore. Nous donnons une formule combinatoire pour la multiplicité Hilbert-Samuel de Ω_w à le point e_v, dans le cas d'une permutation vexillaire w. Notre théorème est formulé en termes de diagrammes de Young excités, et généralise les résultats de Ghorpade-Raghavan et Ikeda-Naruse pour grassmanniennes, et de Li-Yong pour variétés de Schubert vexillaires dans type A.

Keywords: Schubert variety, multiplicity, vexillary permutation

1 Introduction

The singularities of Schubert varieties in a generalized flag variety have been studied intensively for decades; see, e.g., [4] for a survey. An interesting problem is to give a manifestly positive combinatorial rule for the Hilbert-Samuel multiplicity of a torus fixed point in a Schubert variety. For the ordinary Grassmannian, this has been solved; see for example [10, 11, 12, 15]. For the Lagrangian (or symplectic) Grassmannian, there are formulas by Ghorpade and Raghavan [7] and Ikeda and Naruse [9]. These authors also

*anderson.2804@math.osu.edu. DA was partially supported by NSF Grant DMS-1502201.

†ike@xmath.ous.ac.jp. TI was partially supported by JSPS KAKENHI Grant Numbers 18K03261, 17H02838, 16H03920.

‡jeon.163@osu.edu.

§kawago3@gmail.com.

gave similar formulas for Grassmannians of other classical types, but in this extended abstract, we focus on the symplectic case.

The combinatorics of *excited (shifted) Young diagrams* provide an appealing way to express these multiplicity formulas. In the Lagrangian Grassmannian, Schubert varieties Ω_λ and fixed points e_μ are both indexed by shifted Young diagrams, with $e_\mu \in \Omega_\lambda$ if and only if $\mu \supseteq \lambda$. An excitation of μ is a way of moving the boxes of μ inside λ, according to certain rules; the multiplicity formula says $\mathrm{mult}_{e_\mu}(\Omega_\lambda)$ counts the number of excited Young diagrams.

The corresponding story for complete flag varieties is less well understood. Even for ordinary (type A) flags, there is no known manifestly positive combinatorial rule for multiplicities of points on general Schubert varieties. However, when the Schubert variety is associated with a *vexillary*[1] permutation, Li and Yong [13] proved a remarkable formula for the multiplicities, which can be stated in terms of excited Young diagrams. The aim of our work is to prove an analogous combinatorial rule for the symplectic flag variety. In particular, we give an excited Young diagram formula when the Schubert variety is associated to a *vexillary signed permutation*. We expect that our method of proof will generalize to other classical types, including a simpler proof of the Li-Yong formula.

In this extended abstract, we will state the result precisely and give a sketch of the proof. We conclude by presenting computational evidence for a conjecture about the equations defining certain Schubert varieties, and discussing several natural extensions of our results.

Notation. Consider the symplectic group $Sp_{2n} = Sp_{2n}(\mathbb{C})$. Fix a Borel subgroup B and a maximal torus T of Sp_{2n} such that $T \subset B$ and consider the symplectic flag variety $X = Sp_{2n}/B$.

The Weyl group is isomorphic to the group $W_n := S_n \ltimes \{\pm 1\}^n$ of signed permutations. We will realize these as follows. Let $\mathcal{I}_n = \{\overline{n}, \dots, \overline{1}, 1, \dots, n\}$, where the barred numbers should be interpreted as negative integers. Thus $\mathcal{I}_n$ is naturally ordered $\overline{n} < \cdots < \overline{1} < 1 \cdots < n$. A signed permutation $w \in W_n$ is a bijection from $\mathcal{I}_n$ onto itself such that $\overline{w(i)} = w(\bar{\imath})$ for all $i \in \mathcal{I}_n$. Typically we write signed permutations in one-line notation, by recording the values $w(1)\, w(2) \, \cdots \, w(n)$. For example, $w = \overline{2}\,3\,\overline{1}$ is in W_3.

The set X^T of T-fixed points is naturally identified with W_n, and we will write $e_w \in X$ for the fixed point corresponding to $w \in W_n$. The Schubert variety Ω_w associated with $w \in W_n$ is defined to be the Zariski closure of the B_--orbit of $e_w \in X$, where B_- is the opposite Borel subgroup.

Let V be a complex vector space of dimension $2n$ equipped with a nondegenerate skew-symmetric form $\langle \cdot, \cdot \rangle$. A subspace $E \subset V$ is isotropic if the form vanishes identi-

[1] Li and Yong discuss *co-vexillary* Schubert varieties, but our conventions exchange Schubert varieties and opposite Schubert varieties, and correspondingly exchange vexillary with co-vexillary. A permutation $w \in S_n$ is co-vexillary if and only if $w_\circ w$ is vexillary.

cally on $E \times E$; that is, if $E \subseteq E^{\perp}$. An isotropic flag in V is a chain of subspaces

$$E_{\bullet} : E_n \subset \cdots \subset E_1 \subset V,$$

where $\dim E_p = n + 1 - p$ and each E_p is isotropic. The symplectic flag variety $X = Sp_{2n}/B$ naturally parametrizes isotropic flags $E_{\bullet}$ in V.

The Hilbert polynomial of a Noetherian local ring $(R, \mathfrak{m})$ (as a function of n) has leading term $(e/d!)n^d$, where d is the Krull dimension of R, and e is an integer, called the **multiplicity** of R. The multiplicity of a point y on a variety Y, written $\mathrm{mult}_y(Y)$, is defined to be that of the corresponding local ring $\mathcal{O}_{Y,y}$.

2 Vexillary signed permutations

Vexillary elements of W_n were first considered by Billey and Lam in the context of symmetric functions [5], and were revisited by Anderson and Fulton in the context of degeneracy loci [3]. We will follow the latter point of view.

Definition 1 (Triples)**.** A *triple* is $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, where

$$\boldsymbol{k} \colon 0 < k_1 < \cdots < k_s, \quad \boldsymbol{p} \colon p_1 \geq \cdots \geq p_s > 0, \quad \text{and} \quad \boldsymbol{q} \colon q_1 \geq \cdots \geq q_s > 0$$

satisfy

$$(p_i - p_{i+1}) + (q_i - q_{i+1}) > k_{i+1} - k_i$$

for all $i = 1, \ldots, s-1$.

For example, $\boldsymbol{\tau} = (\ 2\ 3\ 4\ ,\ 4\ 2\ 1\ ,\ 4\ 2\ 1\ )$ is a triple.

There are many equivalent characterizations of vexillary (ordinary) permutations in S_n (see [14] and [6]). For example, $v \in S_n$ is vexillary if it avoids the pattern 2 1 4 3, or if the boxes of its essential set proceed from southwest to northeast. Similarly, there are several ways of describing vexillary signed permutations in W_n. We will start with one closely related to essential sets.

Given a triple $\boldsymbol{\tau}$, construct a signed permutation $w(\boldsymbol{\tau})$ as follows. Starting at position p_1, place k_1 consecutive negative (barred) integers in increasing order, so that the largest is equal to $-q_1$. Next, starting at position p_2, place $k_2 - k_1$ negative entries—consecutive among those not yet used, and as large as possible such that the largest is no greater than $-q_2$. Repeat this s times, so that a total of k_s negative entries have been placed. Finally, fill any gaps by an increasing sequence of positive entries (consecutive among those whose absolute values have not been used).

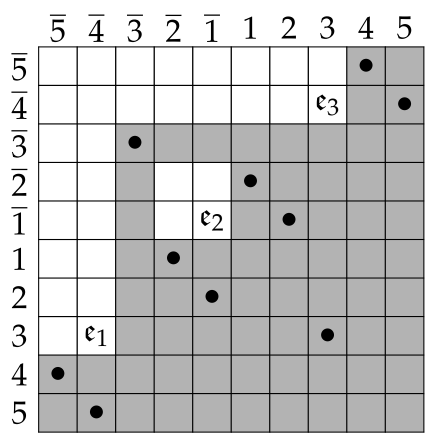
For example, with $\boldsymbol{\tau} = (\,2\,3\,4\,,\,4\,2\,1\,,\,4\,2\,1\,)$, this produces

$$\begin{aligned} &\cdot\ \cdot\ \cdot\ \overline{5}\,\overline{4} \\ &\cdot\ \overline{2}\ \cdot\ \overline{5}\,\overline{4} \\ &\overline{1}\,\overline{2}\ \cdot\ \overline{5}\,\overline{4} \\ w(\boldsymbol{\tau}) =&\overline{1}\,\overline{2}\,3\,\overline{5}\,\overline{4}. \end{aligned}$$

Definition 2 (Vexillary signed permutations)**.** A signed permutation $w \in W_n$ is *vexillary* if $w = w(\boldsymbol{\tau})$ for some triple $\boldsymbol{\tau}$. We will write $W_n^\# \subseteq W_n$ for the set of vexillary signed permutations.

An equivalent characterization is this: considering the action of W_n on the $2n$ elements of $\mathcal{I}_n$, one obtains a natural embedding $\iota\colon W_n \hookrightarrow S_{2n}$. Then a signed permutation $w \in W_n$ is vexillary if and only if the corresponding ordinary permutation $\iota(w) \in S_{2n}$ is vexillary (see [3, Theorem 2.5]).

Example 3. Consider $w = \overline{2}\ \overline{1}\ 3\ \overline{5}\ \overline{4} \in W_5$. The ordinary Rothe diagram $D^A_{\iota(w)}$ of $\iota(w)$ as an element of S_{10} is shown below. Our convention is that we place a dot in position $(p, q) \in \mathcal{I}_n \times \mathcal{I}_n$ if $p = w(q)$. (The diagram $D^A_{\iota(w)}$ is the collection of boxes that remain after striking out the hooks which extend south and east of each dot; see [6].)



The essential boxes $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$ go from southwest to northeast, so $\iota(w) \in S_{10}$ is a vexillary permutation, and therefore $w \in W_5$ is a vexillary signed permutation.

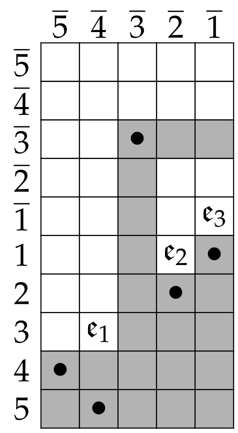
The *extended diagram* is $\widetilde{D}_w \subset \Xi_n := \mathcal{I}_n \times \mathcal{I}_n^-$ is the left half of $D^A_{\iota(w)}$, where $\mathcal{I}_n^- = \{\overline{n}, \ldots, \overline{1}\}$ (see [1], [3]).

Definition 4 (Essential set)**.** Let $w \in W_n^\#$ be a vexillary signed permutation. An element $\mathfrak{e} \in \mathcal{I}_n \times \mathcal{I}_n^-$ is an *essential box* of w if $\mathfrak{e}$ is not in the set $\{(q, \overline{1}) \mid q \leq \overline{2}\}$ and is a south-east corner of $\widetilde{D}_w$. Let $\mathscr{E}ss(w)$ be the set of essential boxes of w.

Remark 5. For general signed permutations, the definition of essential sets in [1] is slightly more complicated. Here we use a simplified version which is valid in the vexillary case (see [3, p. 8, lines 16–17]).

Lemma 6 ([3])**.** *For $w = w(\boldsymbol{\tau}) \in W_n^{\#}$, the essential boxes are precisely those in positions $(q_i - 1, \overline{p_i})$, for $i = 1, \ldots, s$.*

Example 7. The signed permutation $w = \overline{1}\ \overline{2}\ 3\ \overline{5}\ \overline{4}$ is vexillary, with diagram shown below.



Note that the position $(\overline{4}, \overline{1})$ is not an essential box.

Define a particular strict partition $\rho_n = (2n-1, 2n-3, \cdots, 3, 1)$. Let $\mathcal{SP}_{\rho_n}$ denote the set of strict partitions λ such that $\lambda \subset \rho_n$.

For $(q, p) \in \mathcal{I}_n \times \mathcal{I}_n^-$, we define the *rank function* by

$$r_w(q, p) := \#\{i \in \mathcal{I}_n \mid i \leq p,\ w(i) \leq q\}.$$

Note that $r_w(q, p)$ is the number of dots lying (weakly) northwest of position (q, p) in the diagram of w. (This notation is different from what is used in [1], [3].) Let us also define an *incidence function*

$$\begin{aligned} k_w(q, p) &:= \#\{i \in \mathcal{I}_n \mid i \leq p,\ w(i) > q\} \\ &= n + 1 + p - r_w(q, p). \end{aligned}$$

This is the number of dots strictly south and weakly west of position (q, p).

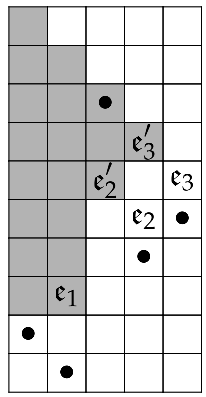
Definition 8. [Shape of a vexillary signed permutation] Let $\boldsymbol{\tau}$ be a triple. The *shape* of $\boldsymbol{\tau}$ (or of the corresponding vexillary signed permutation $w(\boldsymbol{\tau})$) is the smallest strict partition $\lambda = \lambda(\boldsymbol{\tau})$ such that $\lambda_{k_i} = p_i + q_i - 1$ for each $i = 1, \ldots, s$.

In other words, the strict partition $\lambda(\boldsymbol{\tau}) = (\lambda_1 > \cdots > \lambda_{k_s} > 0)$ has parts $\lambda_k = p_i + q_i - 1 + k - k_i$ whenever $k_{i-1} < k \leq k_i$. (We use the convention $k_0 = 0$.) For example, with $\boldsymbol{\tau} = (\ 2\ 3\ 4\ ,\ 4\ 2\ 1\ ,\ 4\ 2\ 1\ )$, we obtain $\lambda(\boldsymbol{\tau}) = (8, 7, 3, 1)$.

It is not hard to see that if $w(\boldsymbol{\tau}) \in W_n$, then $\lambda(\boldsymbol{\tau}) \in \mathcal{SP}_{\rho_n}$. In fact, the longest element $w_\circ \in W_n$ is vexillary, having ρ_n as its shape.

Lemma 9. *Let $\boldsymbol{\tau}$ be a triple, with corresponding vexillary signed permutation $w = w(\boldsymbol{\tau})$ and strict partition $\lambda = \lambda(\boldsymbol{\tau})$. For each $\mathfrak{e} \in \mathscr{E}ss(w)$, let $\mathfrak{e}'$ be the box obtained by moving $\mathfrak{e}$ diagonally north-west by $r_w(\mathfrak{e})$ units. Then λ is the smallest strict partition containing all the $\mathfrak{e}'$ with $\mathfrak{e} \in \mathscr{E}ss(w)$. (Here we view $\lambda \subseteq \rho_n$ as a subset of Ξ_n, as above.)*

Example 10. The vexillary signed permutation $w = \overline{1}\,\overline{2}\,3\,\overline{5}\,\overline{4}$ has the shape $\lambda = (8, 7, 3, 1)$. This can be seen from the diagram:



The Schubert variety associated to a vexillary signed permutation $w = w(\boldsymbol{\tau})$ can be defined by the conditions

$$\Omega_w = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) \geq k_i \text{ for } i = 1, \ldots, s\},$$

where $F_\bullet$ is the isotropic flag fixed by the (opposite) Borel B_-. (See [1, 3].) One characterization of Bruhat order says that $v \geq w$ in W_n if and only if $e_v \in \Omega_w$. It follows that

$$\begin{aligned} v \geq w &\Leftrightarrow k_v(q,p) \geq k_w(q,p) \text{ for all } q, p \\ &\Leftrightarrow k_v(q_i', p_i') \geq k_w(q_i', p_i') = k_i \text{ for all essential boxes } (q_i', p_i') \in \mathscr{E}ss(w) \\ &\Leftrightarrow r_v(q_i', p_i') \leq r_w(q_i', p_i') = r_i \text{ for all essential boxes } (q_i', p_i') \in \mathscr{E}ss(w). \end{aligned}$$

(By Lemma 6, the essential boxes are in positions (q_i', p_i'), with $q_i' = q_i + 1$ and $p_i' = \overline{p_i}$.)

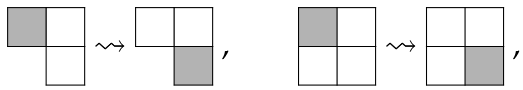
3 Theorem on multiplicities

Let $w = w(\boldsymbol{\tau}) \in W_n^\#$ be a vexillary signed permutation, and let $v \in W_n$ be such that $w \leq v$. Let $\lambda = \lambda(\boldsymbol{\tau}) \in \mathcal{SP}_{\rho_n}$ be the shifted Young diagram associated with w. A key construction is the following (cf. Lemma-Definition 5.1 in [13]).

Definition 11 (Outer shape)**.** Given a pair $w \leq v$ as above, for each $\mathfrak{e} \in \mathscr{E}ss(w)$, let $\mathfrak{e}'$ be the box obtained by moving $\mathfrak{e}$ diagonally north-west by $r_v(\mathfrak{e})$ units. Let $\mu \in \mathcal{SP}_{\rho_n}$ be the smallest shifted diagram containing all the $\mathfrak{e}'$ with $\mathfrak{e} \in \mathscr{E}ss(w)$.

Example 12. Let $w = \overline{1}\ \overline{2}\ 3\ \overline{5}\ \overline{4}$ and $v = w_\circ$ be the longest element in W_5. Then the corresponding outer shape μ is $(8,7,4,3,1)$.

Definition 13 (Excited Young diagram)**.** Let D and D' be subsets of the shifted diagram of a strict partition μ. We say D' is obtained from D by *elementary excitations* if D' is obtained from D by modifying a part of D in one of the following ways:



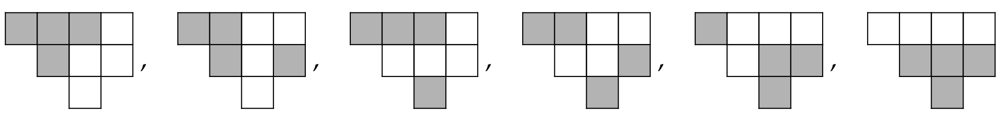
where in the left figure the modification occurs in the diagonal. Let $\mathcal{E}_\mu(\lambda)$ denote the set of all subsets of the shifted diagram of a strict partition μ obtained from $\lambda \subset \mu$ by successive applications of elementary excitations.

Here is our main theorem.

Theorem 14. *Let $w \in W_n^\#$ be vexillary, and let $v \in W_n$ such that $w \leq v$. Let $\lambda = \lambda(w)$ be the shape of w, and let μ be the outer shape determined by the pair (w, v). Then the Hilbert-Samuel multiplicity is computed by*

$$\mathrm{mult}_{e_v}(\Omega_w) = \#\mathcal{E}_\mu(\lambda).$$

Example 15. Let $w = \overline{1}\,\overline{2}\,3\,4$, $v = \overline{2}\,\overline{3}\,\overline{4}\,1$. We have $\mathrm{mult}_{e_v}(\Omega_w) = 6$:



4 Multiplicities in Lagrangian Grassmannians

Our main strategy for proving Theorem 14 is to reduce the theorem to a corresponding fact for the Lagrangian Grassmannian. Let $LG(V)$ be the Lagrangian Grassmannian of maximal isotropic subspaces in the $2n$-dimensional vector space V. Here Schubert varieties and fixed points are indexed by strict partitions, with $e_\mu \in \Omega_\lambda$ if and only if $\mu \supseteq \lambda$. In this context, a formula for multiplicities was given by Ghorpade-Raghavan and Ikeda-Naruse:

Theorem 16 ([7, 9])**.** *For a point e_μ on a Schubert variety Ω_λ in the Lagrangian Grassmannian $LG(V)$, we have*

$$\mathrm{mult}_{e_\mu}(\Omega_\lambda) = \#\mathcal{E}_\mu(\lambda).$$

Recall that the Schubert variety Ω_λ associated to $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$ may be defined by the conditions

$$\Omega_\lambda = \{L \in LG(V) \mid \dim(L \cap F_{\lambda_i}) \geq i \text{ for } i = 1, \ldots, r\},$$

where $F_\bullet$ is the B_--fixed isotropic flag in V. In fact, Ω_λ is the closure of the locus where equalities $\dim(L \cap F_{\lambda_i}) = i$ hold in the above conditions.

We also observe the following. Fix a choice of strictly increasing indices $\boldsymbol{k}$ and strictly decreasing indices $\boldsymbol{q}$, so that $q_i - q_{i+1} \geq k_{i+1} - k_i$ for all i. Let $Q \subseteq Sp_{2n}$ be the parabolic subgroup stabilizing the partial flag $F_{q_1} \subset \cdots \subset F_{q_s}$.

Lemma 17. *The locally closed subset*

$$Z^\circ = \{L \mid \dim(L \cap F_{q_i}) = k_i \text{ for } i = 1, \ldots, s\} \subseteq LG(V)$$

is a Q-orbit in $LG(V)$. In fact, letting μ be the smallest strict partition having $\mu_{k_i} = q_i$, we have $Z^\circ = Q \cdot e_\mu$. Moreover, Z° contains the fixed point $e_{\mu'}$ if and only if $\mu'_{k_i} = \mu_{k_i} = q_i$.

In the notation of the lemma, Z° is the disjoint union of the Schubert cells which are B_--orbits of the points $e_{\mu'}$.

5 Sketch of proof of Theorem 14

To apply Theorem 16 to the proof of Theorem 14, we first need to relax the notion of triple to a *weak triple*, by allowing the defining inequalities to be nonstrict. That is, a weak triple is $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, where $\boldsymbol{k}$ is a weakly increasing sequence, $\boldsymbol{p}$ and $\boldsymbol{q}$ are weakly decreasing, and

$$(p_i - p_{i+1}) + (q_i - q_{i+1}) \geq k_{i+1} - k_i$$

for all $i = 1, \ldots, s-1$. (To avoid trivial redundancy, we also require $(p_i, q_i) \neq (p_{i+1}, q_{i+1})$.) Starting with a weak triple $\boldsymbol{\tau}$, one defines an associated vexillary signed permutation $w(\boldsymbol{\tau})$ exactly as before, but now the essential boxes are generally a subset of the positions $(q_i + 1, \overline{p_i})$.

Now, given a vexillary element $w = w(\boldsymbol{\tau})$, recall that $k_i = k_w(q_i + 1, \overline{p_i})$ for $i = 1, \ldots, s$. For $v \geq w$, let $k'_i = k_v(q_i + 1, \overline{p_i})$ for $i = 1, \ldots, s$, so $k'_i \geq k_i$.

Lemma 18. *Let $\boldsymbol{k}' = (0 < k'_1 \leq \cdots \leq k'_s)$, where $k'_i = k_v(q_i + 1, \overline{p_i})$ for $v \geq w(\boldsymbol{\tau})$, as above. Then $\boldsymbol{\tau}' = (\boldsymbol{k}', \boldsymbol{p}, \boldsymbol{q})$ is a weak triple.*

Now $v' = w(\boldsymbol{\tau}')$ is a vexillary permutation having $v \geq v' \geq w$. Given any weak triple $\boldsymbol{\tau}'$, we consider the locally closed set

$$\Omega^\circ_{\boldsymbol{\tau}'} = \{E_\bullet \mid \dim(E_{p_i} \cap F_{q_i}) = k'_i \text{ for } 1 \leq i \leq s\} \subseteq X,$$

where $F_\bullet$ is the B_--fixed isotropic flag. (In fact, $\Omega^\circ_{\boldsymbol{\tau}'}$ is a union of certain Schubert cells.) With $v \geq v' \geq w$ as above, we see that both e_v and $e_{v'}$ lie in $\Omega^\circ_{\boldsymbol{\tau}'}$. Below we will show that this implies the multiplicities of Ω_w at e_v and $e_{v'}$ are the same.

Next we employ a "diagonal trick" to reduce to the Lagrangian Grassmannian, as follows; cf. [2, p. 18]. We equip $V \oplus V$ with the symplectic form

$$\langle a \oplus b, a' \oplus b' \rangle = \langle a, a' \rangle - \langle b, b' \rangle,$$

where the pairing on the RHS is the symplectic form on V. The diagonal $\Delta = \{a \oplus a\} \subseteq V \oplus V$ is a Lagrangian subspace with respect to this form, and $E_p \oplus F_q$ is isotropic in $V \oplus V$ whenever E_p and F_q are isotropic in V.

We denote by $\overline{LG} = LG(V \oplus V)$ the Lagrangian Grassmannian and we let $w = w(\boldsymbol{\tau})$, $v \geq w$, and $v' = w(\boldsymbol{\tau}')$ be as before. For the triple $\boldsymbol{\tau} = (\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})$, we define a locus $\widehat{\Omega}_{\boldsymbol{\tau}} \subseteq X \times \overline{LG}$ by

$$\widehat{\Omega}_{\boldsymbol{\tau}} = \{(E_\bullet, L) \mid \dim(L \cap (E_{p_i} \oplus F_{q_i})) \geq k_i \text{ for } i = 1, \ldots, s\},$$

and similarly define $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'}$ by requiring equalities instead of inequalities. Thus $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'} \subset \widehat{\Omega}_{\boldsymbol{\tau}}$ is a locally closed subset. Let $\pi_1 \colon \widehat{\Omega}_{\boldsymbol{\tau}} \to X$ and $\pi_2 \colon \widehat{\Omega}_{\boldsymbol{\tau}} \to \overline{LG}$ be the projections, and write π_1', π_2' for the corresponding projections from $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'}$.

For any isotropic flag $E_\bullet$, we have $\pi_1^{-1}(E_\bullet) \cong \Omega_\lambda \subseteq \overline{LG}$, where $\lambda = \lambda(\boldsymbol{\tau})$ (see Definition 8, in particular $\lambda_{k_i} = p_i + q_i - 1$). That is, π_1 makes $\widehat{\Omega}_{\boldsymbol{\tau}}$ into a locally trivial bundle over X, with fibers being Schubert varieties in the Lagrangian Grassmannian.

Furthermore, the projection $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'} \to X$ has fibers isomorphic to the subset $Z^\circ \subset \overline{LG}$ of Lemma 17. (Compared with the notation of the lemma, here we replace n by $2n$ and the indices q_i by $p_i + q_i - 1$; the fixed partial flag is $E_{p_1} \oplus F_{q_1} \subset \cdots \subset E_{p_s} \oplus F_{q_s}$.) Lemma 17 says each such fiber is a Q-orbit, for some parabolic Q which also acts on Ω_λ.

It follows that the singularities of $\widehat{\Omega}_{\boldsymbol{\tau}}$ along $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'}$ are the same as those of Ω_λ along Z°. Since the point e_μ lies in Z°, where $\mu = \lambda(\boldsymbol{\tau}')$, we see in particular that the multiplicity of $\widehat{\Omega}_{\boldsymbol{\tau}}$ at any point of $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'}$ is equal to $\#\mathcal{E}_\mu(\lambda)$, by Theorem 16. (It is not hard to see that this μ is the same as the outer shape of the pair (w, v'), as defined in Definition 11.)

On the other hand, we have $\pi_2^{-1}(\Delta) = \Omega_w \subseteq X$, and the fiber of $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'} \to \overline{LG}$ over Δ is equal to the locus $\Omega^\circ_{\boldsymbol{\tau}'}$. The local equations of $\Delta \in \overline{LG}$ form a regular sequence, and it follows that the singularities of Ω_w along $\Omega^\circ_{\boldsymbol{\tau}'}$ are the same as those of $\widehat{\Omega}_{\boldsymbol{\tau}}$ along $\widehat{\Omega}^\circ_{\boldsymbol{\tau}'}$. Combined with the previous paragraph, this establishes Theorem 14.

6 Computations and a conjecture

For $v \geq w$ in W_n, we define X°_v to be $B \cdot e_v \cong \mathbb{A}^{\ell(v)}$, where $\ell(v)$ is the length of v, and let $\mathcal{N}_{w,v} = \Omega_w \cap X^\circ_v$. These varieties are often called *Kazhdan-Lusztig varieties* (see, e.g., [13]). We can calculate $\mathrm{mult}_{e_v}(\Omega_w)$ by the localization of the affine variety $\mathcal{N}_{w,v}$ at e_v.

On the other hand, the essential set of w suggests a natural set of equations defining $\mathcal{N}_{w,v}$ inside the affine space X°_v, namely, the minors associated to the essential rank conditions. We conjecture these minors in fact generate a radical ideal. The precise statement is given below.

As before, let Ξ_n denote the square of $2n$ by n grid with coordinates (i,j) such that $i \in \mathcal{I}_n$, $j \in \mathcal{I}_n^-$. Following [3], we define a subset D_w of $\widetilde{D}_w$. A position $(i,j) \in \Xi_n$ corresponds to a box of $\widetilde{D}_w \setminus D_w$ if there is $s < j$ such that $i = \overline{w(s)}$. (In the notation of [3], these are boxes in the diagram indicated by $\times$ sign.)

For each $v \in W_n$, we can construct a $2n \times n$ normalized matrix representing elements in X_v°. The first i columns span the i-dimensional component of an isotropic flag. We index the coordinate functions on $2n \times n$ matrix by elements in Ξ_n. If v is the longest element, for example, $X^\circ_{\bar{1}\bar{2}\bar{3}\bar{4}} \cong \mathbb{A}^{16}$, the corresponding matrix looks like

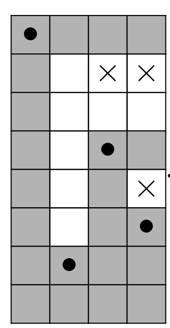
$$Z^{(v)} = \left(\begin{array}{cccc} z_{\bar{4},\bar{4}} & \boxed{z_{\bar{4},\bar{3}}} & \boxed{z_{\bar{4},\bar{2}}} & \boxed{z_{\bar{4},\bar{1}}} \\ z_{\bar{3},\bar{4}} & z_{\bar{3},\bar{3}} & \boxed{z_{\bar{3},\bar{2}}} & \boxed{z_{\bar{3},\bar{1}}} \\ z_{\bar{2},\bar{4}} & z_{\bar{2},\bar{3}} & z_{\bar{2},\bar{2}} & \boxed{z_{\bar{2},\bar{1}}} \\ z_{\bar{1},\bar{4}} & z_{\bar{1},\bar{3}} & z_{\bar{1},\bar{2}} & z_{\bar{1},\bar{1}} \\ \hline z_{1,\bar{4}} & z_{1,\bar{3}} & z_{1,\bar{2}} & 1 \\ z_{2,\bar{4}} & z_{2,\bar{3}} & 1 & 0 \\ z_{3,\bar{4}} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

By the isotropic conditions, the boxed entries are uniquely expressed in terms of the remaining entries. Note that the coordinate functions are in bijection with $\widetilde{D}_v$, and the boxed coordinates are in bijection with $\widetilde{D}_v \setminus D_v$. Let Ξ_v be the subset corresponding to D_v. The coordinate ring $\mathbb{C}[X_v^\circ]$ of the dual Schubert cell is

$$\mathbb{C}[X_v^\circ] = \mathbb{C}[z_{i,j} \mid (i,j) \in \Xi_v].$$

Example 19. Let us consider the vexillary signed permutation $v = 1\ \bar{3}\ \bar{2}\ \bar{4}\ \bar{5}$. The corresponding matrix giving coordinates of X_v° is the following:

$$Z^{(v)} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & z_{\bar{3},3} & \boxed{z_{\bar{3},\bar{1}}} & \boxed{z_{\bar{3},2}} \\ 0 & z_{\bar{2},3} & z_{\bar{2},\bar{1}} & z_{\bar{2},2} \\ 0 & z_{\bar{1},3} & 1 & 0 \\ \hline 0 & z_{1,3} & 0 & \boxed{z_{1,2}} \\ 0 & z_{2,3} & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

.

In computing, we first express boxed coordinates in terms of unboxed ones by using isotropic conditions. Note our convention for the skew symmetric form is $\langle \boldsymbol{a}, \boldsymbol{b} \rangle :=$

${}^t\boldsymbol{a}J\boldsymbol{b}$ with J the $2n \times 2n$ matrix having nonzero entries only on the anti-diagonal as $(1,\ldots,1,-1,\ldots,-1)$.

Let $Z^{(v)}_{\mathfrak{e}}$ denote the northwest submatrix of $Z^{(v)}$ whose southeast corner is $\mathfrak{e}$. Let $I_{v,w}$ be the ideal of $\mathbb{C}[X^\circ_v]$ generated by $r_w(\mathfrak{e})+1$ minors of $Z^{(v)}_{\mathfrak{e}}$ for all $\mathfrak{e} \in \mathscr{E}ss(w)$. Let $R_{v,w} = \mathbb{C}[X^\circ_v]/I_{v,w}$. Our conjecture is a direct analogue of a type A fact (due to work of Fulton [6, §6] and Woo-Yong [16, §3]).

Conjecture. *We have $\mathbb{C}[\mathcal{N}_{v,w}] = R_{v,w}$. That is, the ideal $I_{v,w}$ defines the Kazhdan-Lusztig variety.*

By the main results of [1], the minors from the essential set suffice to define $\mathcal{N}_{v,w}$ set-theoretically, so the conjecture is equivalent to the claim that $I_{v,w}$ is radical.

Using this description, we may calculate $\mathrm{mult}_0(R_{v,w})$, the local multiplicity of $R_{v,w}$ at the origin, by using **Singular** ([8]). We use a "local" term order **ds** (*negative degree reverse lexicographical ordering* in **Singular**, [8, page 16]) and obtain a *standard basis* ([8, §1]), an analogue of Groebner basis for local ring. See §5.1 of [8], in particular Proposition 5.5.7, and Proposition 5.5.12, and Example 5.5.13.

As evidence for the conjecture, we have verified

$$\mathrm{mult}_0(R_{v,w}) = \#\mathcal{E}_\mu(\lambda) = \mathrm{mult}_{e_v}(\Omega_w)$$

for all pairs (v,w) with w vexillary and $v \geq w$ in W_n, for $n \leq 4$.

7 Further directions

A natural next step is to study multiplicities of vexillary Schubert varieties in the orthogonal flag varieties of types B and D. In fact, our method seems to suggest a uniform approach to all classical types. Even in type A, we expect to be able to simplify the proof of Li-Yong's determinantal formula for the multiplicity of a vexillary Schubert variety.

The question of defining equations in other types appears to be more subtle. Already in type B_2, the naive analogue of the ideal $I_{v,w}$ generated by essential minors can fail to be radical: this happens with $v = \overline{1}\,\overline{2}$ and $w = \overline{1}\,2$. This suggests a question:

Question. For arbitrary $v \geq w$ in W_n, defining a Kazhdan-Lusztig variety $\mathcal{N}_{v,w}$ in type B, what are generators for the Kazhdan-Lusztig ideal $\ker(\mathbb{C}[X^\circ_v] \twoheadrightarrow \mathbb{C}[\mathcal{N}_{v,w}])$?

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