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From multiline queues to Macdonald polynomials via the exclusion process

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Abstract. Recently James Martin introduced *multiline queues*, and used them to give a combinatorial formula for the stationary distribution of the multispecies asymmetric simple exclusion exclusion process (ASEP) on a circle. The ASEP is a model of particles hopping on a one-dimensional lattice, which has been extensively studied in statistical mechanics, probability, and combinatorics. In this article we give an independent proof of Martin's result, and we show that by introducing additional statistics on multiline queues, we can use them to give a new combinatorial formula for both the symmetric Macdonald polynomials $P_\lambda(\mathbf{x}; q, t)$, and the nonsymmetric Macdonald polynomials $E_\lambda(\mathbf{x}; q, t)$, where λ is a partition. This formula is rather different from others that have appeared in the literature (Haglund-Haiman-Loehr'05, Ram-Yip'11, and Lenart'09). Our proof uses results of Cantini, de Gier, and Wheeler who recently linked the multispecies ASEP on a circle to Macdonald polynomials.

Keywords: asymmetric simple exclusion process, Macdonald polynomials

1 Introduction and results

Introduced in the late 1960s [21, 27], the *asymmetric simple exclusion process* (ASEP) is a model of interacting particles hopping left and right on a one-dimensional lattice of n sites. There are many versions of the ASEP: the lattice might be a lattice with open boundaries, or a ring, among others; and we may allow multiple species of particles with different "weights". In this article, we will be concerned with the multispecies ASEP on a ring, where the rate of two adjacent particles swapping places is either 1 or t, depending on their relative weights. Recently James Martin [23] gave a combinatorial formula in terms of *multiline queues* for the stationary distribution of this multispecies ASEP on a ring, building on his earlier joint work with Ferrari [11].

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On the other hand, recent work of Cantini, de Gier, and Wheeler [4] gave a link between the multispecies ASEP on a ring and *Macdonald polynomials*. Symmetric Macdonald polynomials $P_\lambda(\mathbf{x};q,t)$ [19] are a family of multivariable orthogonal polynomials indexed by partitions, whose coefficients depend on two parameters q and t; they generalize multiple important families of polynomials, including Schur polynomials (at $q=t=0$) and Hall-Littlewood polynomials (at $q=0$). *Nonsymmetric Macdonald polynomials* [8, 20] were introduced shortly after the introduction of Macdonald polynomials, and defined in terms of *Cherednik operators*; the symmetric Macdonald polynomials can be constructed from their nonsymmetric counterparts.

There has been a lot of work devoted to understanding Macdonald polynomials from a combinatorial point of view. Haglund-Haiman-Loehr [14, 13] gave a combinatorial formula for the *transformed Macdonald polynomials* $\tilde{H}_\mu(\mathbf{x};q,t)$ (which are connected to the geometry of the Hilbert scheme [16]) as well as for the *integral forms* $J_\mu(\mathbf{x};q,t)$, which are scalar multiples of the classical monic forms $P_\mu(\mathbf{x};q,t)$. They also gave a formula for the nonsymmetric Macdonald polynomials [15]. Building on work of Schwer [26], Ram and Yip [25] gave general-type formulas for both the Macdonald polynomials $P_\lambda(\mathbf{x};q,t)$ and the nonsymmetric Macdonald polynomials; however, their type A formulas have many terms. Lenart [18] showed how to “compress" the Ram-Yip formula in type A to obtain a Haglund-Haiman-Loehr type formula for the polynomials $P_\lambda(\mathbf{x};q,t)$ for λ with all parts distinct. Finally, Ferreira [12] and Alexandersson [2] gave Haglund-Haiman-Loehr type formulas for *permuted basement Macdonald polynomials*, which generalize the nonsymmetric Macdonald polynomials.

The main goal of this article is to define some polynomials combinatorially in terms of multiline queues which simultaneously compute the stationary distribution of the multispecies ASEP and also symmetric Macdonald polynomials $P_\lambda(\mathbf{x};q,t)$. More specifically, we introduce some polynomials $F_\mu(x_1,\dots,x_n;q,t) = F_\mu(\mathbf{x};q,t) \in \mathbb{Z}(q,t)[x_1,\dots,x_n]$ which are certain weight-generating functions for multiline queues with bottom row μ, where $\mu = (\mu_1,\dots,\mu_n)$ is an arbitrary weak composition. We show that these polynomials have the following properties:

1. When $x_1 = \cdots = x_n = 1$ and $q=1$, $F_\mu(\mathbf{x};q,t)$ is proportional to the steady state probability that the multispecies ASEP is in state μ. (This recovers a result of Martin [23], but we give an independent proof.)

2. When μ is a partition, $F_\mu(\mathbf{x};q,t)$ is equal to the nonsymmetric Macdonald polynomial $E_\mu(\mathbf{x};q,t)$.

3. For any partition λ, the quantity $Z_\lambda(\mathbf{x};q,t) := \sum_\mu F_\mu(\mathbf{x};q,t)$ (where the sum is over all distinct compositions obtained by permuting the parts of λ) is equal to the symmetric Macdonald polynomial $P_\lambda(\mathbf{x};q,t)$.

In the remainder of the introduction we will make the above statements more precise.

1.1 The multispecies ASEP

We start by defining the multispecies ASEP or the L-ASEP as a Markov chain on the cycle $\mathbb{Z}_n$ with L classes of particles as well as holes. The L-ASEP on a ring is a natural generalization for the two-species ASEP; for the latter, solutions were given using a matrix product formulation in terms of a quadratic algebra similar to the matrix ansatz described in [9].

For the L-ASEP when $t = 0$ (i.e. particles only hop in one direction), Ferrari and Martin [11] proposed a combinatorial solution for the stationary distribution using multiline queues. This construction was restated as a matrix product solution in [10] and was generalized to the partially asymmetric case (t generic) in [24]. In [3] the authors explained how to construct an explicit representation of the algebras involved in the L-ASEP. Finally James Martin [23] gave an ingenious combinatorial solution for the stationary distribution of the L-ASEP when t is generic, using more general multiline queues and building on ideas from [11] and [10].

Definition 1.1. *Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be a partition with greatest part $\lambda_1 = L$, and let t be a constant such that $0 \leq t \leq 1$. Let* $\mathrm{States}(\lambda)$ *be the set of all weak compositions of length n obtained by permuting the parts of λ. We consider indices modulo n; i.e. if $\mu = \mu_1 \ldots \mu_n$ is a composition, then $\mu_{n+1} = \mu_1$. The* multispecies asymmetric simple exclusion process $\mathrm{ASEP}(\lambda)$ *on a ring is the Markov chain on* $\mathrm{States}(\lambda)$ *with transition probabilities:*

- *If $\mu = AijB$ and $\nu = AjiB$ are in* $\mathrm{States}(\lambda)$ *(here A and B are words in the parts of λ), then $P_{\mu,\nu} = \frac{t}{n}$ if $i > j$ and $P_{\mu,\nu} = \frac{1}{n}$ if $i < j$.*
- *Otherwise $P_{\mu,\nu} = 0$ for $\nu \neq \mu$ and $P_{\mu,\mu} = 1 - \sum_{\mu \neq \nu} P_{\mu,\nu}$.*

We think of the 1's, 2's, ..., L's as representing various types of particles of different weights; each 0 denotes an empty site. See Figure 1.

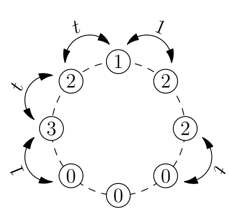

Figure 1: The multispecies ASEP on the lattice $\mathbb{Z}_8$. There is one particle of type 3, three particles of type 2, one particle of type 1, and three holes (0's), so we refer to this Markov chain as $\mathrm{ASEP}(3, 2, 2, 2, 1, 0, 0, 0)$.

1.2 Multiline queues

We now define ball systems and multiline queues. These concepts are due to Ferrari and Martin [11] for $t = 0$ and $q = 1$ and to Martin [23] for t general and $q = 1$.

Definition 1.2. *Fix positive integers L and n. A* ball system *B is an $L \times n$ array in which each of the Ln positions is either empty or occupied by a ball. We number the rows from bottom to top from 1 to L, and the columns from left to right from 1 to n. Moreover we require that there is at least one ball in the top row, and that the number of balls in each row is weakly increasing from top to bottom.*

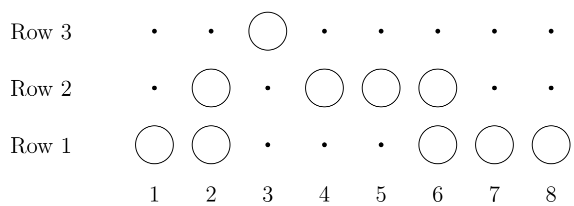

Figure 2: A ball system.

Definition 1.3. *Given an $L \times n$ ball system B, a multiline queue Q (for B) is, for each row r where $2 \leq r \leq L$, a matching of balls from row r to row $r-1$. A ball b may be matched to any ball b' in the row below it; we connect b and b' by a shortest strand that travels either straight down or from left to right (allowing the strand to wrap around the cylinder if necessary). We refer to two balls being matched by a* pairing*, with pairings obtained by the following algorithm:*

- *We start by matching all balls in row L to a collection of balls (their partners) in row $L-1$. We then match those partners in row $L-1$ to new partners in row $L-2$, and so on. This determines a set of balls, each of which we label by L.*
- *We then take the unmatched balls in row $L-1$ and match them to partners in row $L-2$. We then match those partners in row $L-2$ to new partners in row $L-3$, and so on. This determines a set of balls, each of which we label by $L-1$.*
- *We continue in this way, determining a set of balls labeled $L-2$, $L-3$, and so on, and finally we label any unmatched balls in row 1 by 1.*
- *If at any point there's a free (unmatched) ball b' directly underneath the ball b we're matching, we must match b to b'. We say that b and b' are* trivially paired.

Let $\mu = (\mu_1, \ldots, \mu_n) \in \{0, 1, \ldots, L\}^n$ be the labeling of the balls in row 1 at the end of this process (where an empty position is denoted by 0). We then say that Q is a multiline queue of type *μ. See Figure 3 for an example.*

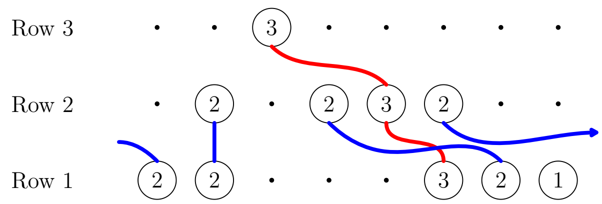

Figure 3: A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

Remark 1.4. *Note that the induced labeling on the balls satisfies the following properties:*

- *If ball b with label i is directly above ball b' with label j, then we must have $i \leq j$.*
- *Moreover if $i = j$, then those two balls are matched to each other (a trivial pairing).*

We now define the weight of each multiline queue. Here we generalize Martin's ideas [23] by adding parameters q and $x_1, \dots, x_n$.

Definition 1.5. *Given a multiline queue Q, we let m_i be the number of balls in column i. We define the x-*weight *of Q to be* $\mathrm{wt}_x(Q) = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$.

*We also define the q, t-*weight *of Q by associating a weight to each nontrivial pairing p of balls. These weights are computed in order as follows. Consider the nontrivial pairings between rows r and $r-1$. We read the balls in row r in decreasing order of their label (from L to r); within a fixed label, we read the balls from right to left. As we read the balls in this order, we imagine placing the strands pairing the balls one by one. The balls that have not yet been matched in row $r-1$ are considered* free. *If pairing p matches ball b in row r and column c to ball b' in row $r-1$ and column c', then the free balls in row $r-1$ and columns $c+1, c+2, \dots, c'-1$ (indices considered modulo n) are considered* skipped. *Note that the balls which are trivially paired between rows r and $r-1$ are not considered free. Let i be the label of balls b and b'. We then associate to pairing p the weight*

$$\mathrm{wt}_{q,t}(p) = \begin{cases} \frac{(1-t)t^{\#\,\mathrm{skipped}}}{1-q^{i-r+1}t^{\#\,\mathrm{free}}} \cdot q^{i-r+1} & \text{if } c' < c \\ \frac{(1-t)t^{\#\,\mathrm{skipped}}}{1-q^{i-r+1}t^{\#\,\mathrm{free}}} & \text{if } c' > c. \end{cases}$$

Note that the extra factor q^{i-r+1} appears precisely when the strand connecting b to b' wraps around the cylinder.

Having associated a q, t-weight to each nontrivial pairing of balls, we define the q, t-weight of the multiline queue Q to be

$$\mathrm{wt}_{q,t}(Q) = \prod_p \mathrm{wt}_{q,t}(p),$$

where the product is over all nontrivial pairings of balls in Q.

Finally the weight *of Q is defined to be*

$$\mathrm{wt}(Q) = \mathrm{wt}_x(Q)\,\mathrm{wt}_{q,t}(Q).$$

Example 1.6. *In Figure 3, the x-weight of the multiline queue Q is* $x_1x_2^2x_3x_4x_5x_6^2x_7x_8$.

The weight of the unique pairing between row 3 and row 2 is $\frac{(1-t)t}{1-qt^4}$. *The weight of the pairing of balls labeled 3 between row 2 and 1 is* $\frac{(1-t)}{1-q^2t^5}$, *and the weights of the pairings of balls labeled 2 are* $\frac{(1-t)t^2}{1-qt^3}\cdot q$ *and* $\frac{1-t}{1-qt^2}$. *Therefore*

$$\mathrm{wt}(Q) = x_1x_2^2x_3x_4x_5x_6^2x_7x_8 \cdot \frac{(1-t)t}{1-qt^4} \cdot \frac{(1-t)}{1-q^2t^5} \cdot \frac{(1-t)t^2}{1-qt^3} \cdot q \cdot \frac{1-t}{1-qt^2}.$$

We now define the weight-generating function for multiline queues of a given type, as well as the *combinatorial partition function* for multiline queues.

Definition 1.7. *Let* $\mu = (\mu_1, \dots, \mu_n) \in \{0, 1, \dots, L\}^n$ *be a composition with largest part L. Set*

$$F_\mu = F_\mu(x_1, \dots, x_n; q, t) = F_\mu(\mathbf{x}; q, t) = \sum_Q \mathrm{wt}(Q),$$

where the sum is over all $L \times n$ *multiline queues of type* μ.

Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ *be a partition with n parts and largest part L. We set*

$$Z_\lambda = Z_\lambda(x_1, \dots, x_n; q, t) = Z_\lambda(\mathbf{x}; q, t) = \sum_\mu F_\mu(x_1, \dots, x_n; q, t),$$

where the sum is over all distinct compositions μ *obtained by permuting the parts of* λ. *We call* Z_λ *the* combinatorial partition function.

Remark 1.8. *Recently Aas-Grinberg-Scrimshaw [1] studied multiline queues in the case that* $t = 0$, *putting in "spectral weights" (which correspond to our x-weight); they then gave a connection to tensor products of KR-crystals.*

1.3 The main result

The goal of this article is to show that with the refined statistics given in Definition 1.5, we can use multiline queues to give formulas for Macdonald polynomials.

Proposition 1.9. *Let* λ *be a* partition. *Then the nonsymmetric Macdonald polynomial* $E_\lambda(\mathbf{x}; q, t)$ *is equal to the quantity* $F_\lambda(\mathbf{x}; q, t)$ *from Definition 1.7.*

Theorem 1.10. *Let* λ *be a partition. Then the symmetric Macdonald polynomial* $P_\lambda(\mathbf{x}; q, t)$ *is equal to the quantity* $Z_\lambda(\mathbf{x}; q, t)$ *from Definition 1.7.*

See Figure 4 for an example illustrating Proposition 1.9.

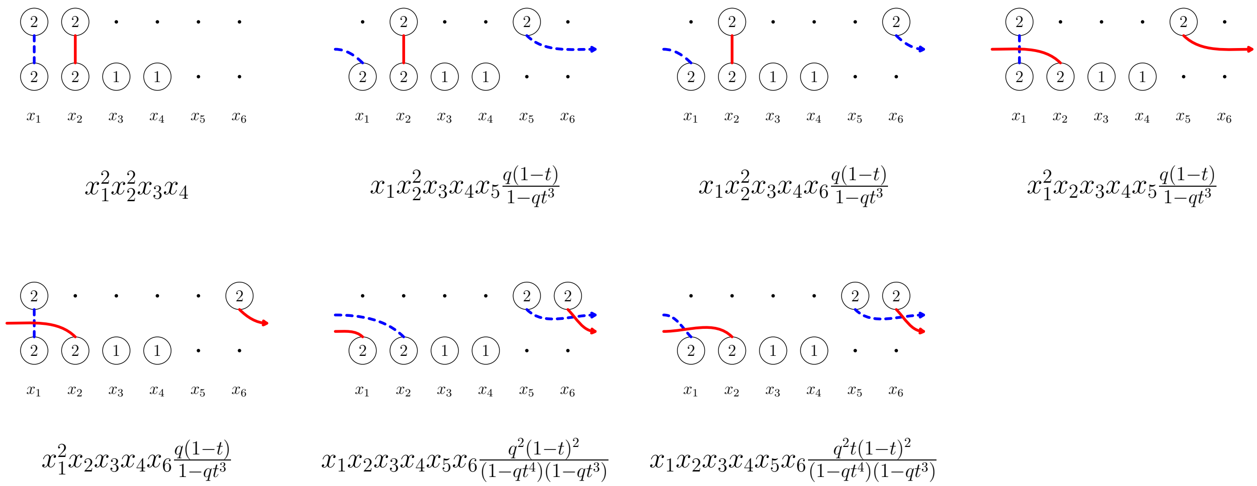

Figure 4: The generating function for the multiline queues of type $(2,2,1,1,0,0)$ give an expression for the nonsymmetric Macdonald polynomial $E_{(2,2,1,1,0,0)}(\mathbf{x};q,t)$

2 The Hecke algebra and its connection to ASEP and Macdonald polynomials

To explain the connection between the ASEP and Macdonald polynomials, and explain how we prove Proposition 1.9 and Theorem 1.10, we need to introduce the Hecke algebra and recall some notions from [17] and Cantini-deGier-Wheeler [4].

Definition 2.1. *The* Hecke algebra *of type A_{n-1} is the algebra with generators T_i for $1 \le i \le n-1$ and parameter t which satisfies the following relations:*

$$(T_i - t)(T_i + 1) = 0, \qquad T_iT_{i\pm1}T_i = T_{i\pm1}T_iT_{i\pm1}, \qquad T_iT_j = T_jT_i \text{ when } |i-j| > 1. \tag{2.1}$$

There is an action of the Hecke algebra on polynomials $f(x_1, \ldots, x_n)$ which is defined as follows:

$$T_i = t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i) \text{ for } 1 \le i \le n-1, \tag{2.2}$$

where s_i acts by

$$s_i f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) := f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n). \tag{2.3}$$

One can check that the operators (2.2) satisfy the relations (2.1).

We also define the shift operator ω via $(\omega f)(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1})$. Given a composition $\mu = (\mu_1, \ldots, \mu_n)$, we let $|\mu| := \sum \mu_i$ and define $s_i\mu := s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n)$ for $1 \le i \le n-1$.

The following notion of *qKZ family* was introduced in [17], also explaining the relationship of such polynomials to nonsymmetric Macdonald polynomials. We use the conventions of [5, Definition 2], also [4, Section 1.3], and [4, (23)].

Definition 2.2. *Fix a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. We say that a family $\{f_{\mu=\lambda\circ\sigma}\}_{\sigma\in S_n}$ of homogeneous degree $|\lambda|$ polynomials in n variables $\mathbf{x} = (x_1, \dots, x_n)$, with coefficients which are rational functions of q and t, is a* qKZ family *if they satisfy*

$$T_i f_\mu(\mathbf{x}; q, t) = f_{s_i\mu}(\mathbf{x}; q, t), \text{ when } \mu_i > \mu_{i+1}, \tag{2.4}$$

$$T_i f_\mu(\mathbf{x}; q, t) = t f_\mu(\mathbf{x}; q, t), \text{ when } \mu_i = \mu_{i+1}, \tag{2.5}$$

$$q^{\mu_n} f_\mu(\mathbf{x}; q, t) = f_{\mu_n, \mu_1, \dots, \mu_{n-1}}(q x_n, x_1, \dots, x_{n-1}; q, t). \tag{2.6}$$

The following lemma explains the relationship of the f_μ's to the ASEP.

Lemma 2.3. *[5, Corollary 1]. Consider the polynomials f_μ from Definition 2.2. When $q = x_1 = \cdots = x_n = 1$, $f_\mu(1, \dots, 1; 1, t)$ is proportional to the steady state probability that the multispecies ASEP is in state μ.*

As we will explain in Lemma 2.6 and Lemma 2.7, the polynomials f_μ are also related to Macdonald polynomials. We first quickly review the relevant definitions.

Definition 2.4. *Let $\langle \cdot, \cdot \rangle$ denote the Macdonald inner product on power sum symmetric functions [19, Chapter VI, (1.5)], where $<$ denotes the dominance order on partitions. Let λ be a partition. The (symmetric)* Macdonald polynomial *$P_\lambda(x_1, \dots, x_n; q, t)$ is the unique homogeneous symmetric polynomial in $x_1, \dots, x_n$ which satisfies $\langle P_\lambda, P_\mu \rangle = 0$, $\lambda \neq \mu$ and*

$$P_\lambda(x_1, \dots, x_n; q, t) = m_\lambda(x_1, \dots, x_n) + \sum_{\mu < \lambda} c_{\lambda,\mu}(q, t) m_\mu(x_1, \dots, x_n).$$

The following definition can be found in [20] (also [22] for a nice exposition).

Definition 2.5. *For $1 \le i \le n$, define the q-*Dunkl *or* Cherednik operators *[6, 7] by*

$$Y_i = T_i^{-1} \dots T_{n-1}^{-1} \omega T_1 \dots T_{i-1}.$$

The Cherednik operators commute pairwise, and hence possess a set of simultaneous eigenfunctions, which are (up to scalar) the nonsymmetric Macdonald polynomials*. We index the nonsymmetric Macdonald polynomials $E_\mu(\mathbf{x}; q, t)$ by compositions μ so that*

$$E_\mu(\mathbf{x}; q, t) = \mathbf{x}^\mu + \sum_{\nu < \mu} b_{\mu\nu} \mathbf{x}^\nu.$$

In particular, when $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$ is a partition, we have that for $1 \le i \le n$,

$$Y_i E_\lambda = y_i(\lambda) E_\lambda \tag{2.7}$$

where

$$y_i(\lambda) = q^{\lambda_i} t^{\#\{j<i | \lambda_j = \lambda_i\} - \#\{j>i | \lambda_j = \lambda_i\}}.$$

In the following Lemmas 2.6 and 2.7, let $\{f_{\mu=\lambda\circ\sigma}\}_{\sigma\in S_n}$ be a set of homogeneous degree $|\lambda|$ polynomials as in Definition 2.2. Note that Lemma 2.6 below essentially appears in [17, Section 3.3]. We thank Michael Wheeler for his explanations.

Lemma 2.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. Then f_λ is a scalar multiple of the nonsymmetric Macdonald polynomial E_λ.*

Lemma 2.7 ([5, Lemma 1])**.** *Let λ be a partition. Then the Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$ is a scalar multiple of*

$$\sum_\mu f_\mu(x_1, \ldots, x_n; q, t),$$

where μ ranges over all distinct compositions which can be obtained by permuting the parts of λ.

The strategy of our proof of Theorem 1.10 is very simple. Our main task is to show that the F_μ's satisfy the following properties.

Theorem 2.8.

$$T_i F_\mu(\mathbf{x}; q, t) = F_{s_i\mu}(\mathbf{x}; q, t), \text{ when } \mu_i > \mu_{i+1}, \tag{2.8}$$

$$T_i F_\mu(\mathbf{x}; q, t) = tF_\mu(\mathbf{x}; q, t), \text{ when } \mu_i = \mu_{i+1}, \tag{2.9}$$

$$q^{\mu_n} F_\mu(\mathbf{x}; q, t) = F_{\mu_n,\mu_1,\ldots,\mu_{n-1}}(qx_n, x_1, \ldots, x_{n-1}; q, t). \tag{2.10}$$

We prove (2.10) directly using multiline queues. We prove (2.9) by showing F_μ is symmetric in x_i and x_{i+1} when $\mu_i = \mu_{i+1}$. Finally we prove (2.8) using multiline queues by induction on the number of rows.

Next, we verify the following lemma by comparing the coefficients of x^λ.

Lemma 2.9. *For any partition λ,*

$$F_\lambda(\mathbf{x}; q, t) = E_\lambda(\mathbf{x}; q, t),$$

where E_λ is the nonsymmetric Macdonald polynomial.

Then Theorem 2.8, Lemma 2.9, and Lemma 2.7 implies Theorem 1.10, that our sum over multiline queues equals the symmetric Macdonald polynomial P_λ.

3 Comparisons to other Macdonald polynomial formulas

In this paper we used multiline queues to give a new combinatorial formula for the Macdonald polynomial P_λ and the nonsymmetric Macdonald polynomial E_λ when λ is a partition. We note that these new combinatorial formulas are quite different from the combinatorial formulas given by Haglund-Haiman-Loehr [13, 14, 15], or Ram-Yip [25], or Lenart [18].

While it is not obvious combinatorially, we show algebraically in Proposition 3.1 that the polynomials F_μ (for μ an arbitrary composition) are equal to certain *permuted basement Macdonald polynomials* $E^\sigma_\alpha(\mathbf{x};q,t)$, which were introduced in [12] and further studied in [2] as a generalization of nonsymmetric Macdonald polynomials (where $\sigma \in S_n$ and α is a composition with n parts). They have the property that the nonsymmetric Macdonald polynomial E_μ is equal to $E^{w_0}_{\text{rev}(\mu)}$, where $\text{rev}(\mu)$ denotes the reverse composition $(\mu_n, \mu_{n-1}, \dots, \mu_1)$ of $\mu = (\mu_1, \dots, \mu_n)$ and $w_0 = (n, \dots, 2, 1)$.

Proposition 3.1. *For $\mu = (\mu_1, \dots, \mu_n)$, define $\text{inc}(\mu)$ to be the sorting of the parts of μ in increasing order. Then*

$$F_\mu = E^\sigma_{\text{inc}(\mu)}$$

where $\sigma\mu = \text{inc}(\mu)$, i.e. σ is any permutation such that $\mu_{\sigma(1)} \le \mu_{\sigma(2)} \le \cdots \le \mu_{\sigma(n)}$.

The permuted basement Macdonald polynomials can be described combinatorially using *nonattacking fillings* of certain diagrams [12, 2][1], which we call *permuted basement tableaux*. (Note that these permuted basement tableaux generalize the nonattacking fillings from [15]). In light of this, one may wonder if there is a bijection between multiline queues and these permuted basement tableaux. This is the case when the compositions have distinct parts. However, for general compositions, the number of permuted basement tableaux is different than the number of MLQs (there are more permuted basement tableaux). We conjecture that there is a way to group permuted basement tableaux so that the weight in a group equals the weight of one MLQ.

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