Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #99, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

Finiteness theorems for matroid complexes with prescribed topology

Federico Castillo[1] and José Alejandro Samper[2]

[1]*University of Kansas*
[2]*University of Miami*

Abstract. It is known that there are finitely many simplicial complexes (up to isomorphism) with a given number of vertices. Translating to the language of h-vectors, there are finitely many simplicial complexes of bounded dimension with $h_1 = k$ for any natural number k. In this paper we study the question at the other end of the h-vector: given d and k there are only finitely many $d-1$-dimensional independence complexes, broken circuit complexes, and order complexes of geometric lattices (without coloops) with $h_d = k$. This suggests new upper/lower bound programs for these types of simplicial complexes.

Resumen. Es conocido que hay finitos complejos simpliciales (módulo isomorfismo) con un número de vertices fijo. En el lenguage de h-vectores, hay finitos complejos simpliciales que satisfacen $h_1 = k$ para k fijo. En esta nota estudiamos la pregunta al extremo opuesto del h-vector: dados d y k hay finitos complejos de independencia, complejos de circuitos cortados, y complejos de orden, con dimension $d-1$ y $h_d = k$. Esto sugiere nuevos programas para determinar cotas inferiores y superiores para estos tipos de complejos.

Keywords: Matroids, broken circuit complexes, geometric lattices.

1 Introduction

This paper aims to present a new approach to the study of matroids from the perspective of the topology of various simplicial complexes. In the survey [5], Björner presented the story of three complexes associated to a matroid: the independence complex, the broken circuit complex, and the order complex of its lattice of flats. We introduce a program that aims to study, for each of the three associated complexes, all matroids whose complex has a fixed homotopy type.

The program is cast in the language of h-numbers, and their equivalent relatives f-numbers. These invariants have been extensively studied and are the subject of widely celebrated new results and old conjectures. For instance, the recent resolution of the Rota-Herron-Welsh conjecture by Adiprasito, Huh and Katz [1] can be interpreted as a set of inequalities on f-vectors of broken circuit complexes. Other recent breakthroughs include the proof, by Ardila, Denham and Huh [3], that the h-vector of any broken circuit complex is log concave, and the solutions of Brändén-Huh [6] and Anari-Lui-Oreis Gharan-Vinzant [2] of the strongest version of Mason's conjecture for f-vectors of independence complexes.

From the work of Chari [7] (for independence complexes), Nyman and Swartz [11] (for order complexes of geometric lattices), and Juhnke-Kubitzke and Van Dihn [9] (for broken circuit complexes) we now know that the h-vector in all these cases is *flawless*. In terms of the entries it says that if $h = (h_0, \dots h_s)$ is the h-vector of a complex, with $h_s \neq 0$ and $\delta = \lfloor \frac{s}{2} \rfloor$, then $h_0 \leq h_1 \leq \cdots \leq h_\delta$. and $h_i \leq h_{s-i}$ for $i \leq \delta$.

If $i < d$ and the h-vector is flawless, then $h_i \geq h_1 = f_0 - d$, where f_0 is the number of vertices. It follows that, after fixing k and d, the number of (isomorphism types of) complexes of rank d with $h_i = k$ and no cone vertices is finite. This is however, far away from the case if we consider h_d instead: the g-theorem [13, Theorem 1.1 Section III] implies that the h-vector of the boundary of any $(d-1)$-dimensional simplicial polytope is flawless and has $h_d = 1$.

Surprisingly for independence complexes, broken circuit complexes, and geometric lattices, the restriction for h_d still implies finiteness.

1.1 Independence complexes

Theorem 1. *Let d, k be positive integers. There are finitely many isomorphism classes of loopless rank d matroids M whose independence complex satisfies $h_d(\mathcal{I}(M)) = k$.*

This result should be surprising at first sight. However, it is a natural consequence of several results that exist in the literature, some dating back to 1980.

Theorem 1 implies that there are upper bounds on all h-numbers in terms of h_d. On the other hand, lower bounds exist from the fact that the h-vector is an O-sequence. Thus it seems reasonable to launch a program to understand extremal matroids for upper and lower bounds for matroid independence complexes with fixed rank and topology.

Another natural path to follow is trying to estimate the size of the set $\Psi_{d,k}$ of all isomorphism classes of loopless matroids of rank d with $h_d = k$. It is a priori not clear that such a set is not empty, but we provide several examples in each class. Furthermore, we provide non-trivial upper and lower bounds for the cardinality of $|\Psi_{d,1}|$. In particular, we extend a result of Chari, who showed that $|\Psi_{d,1}| = p(d)$, the number of integer partitions of d.

Theorem 2. *Let $d, k > 0$ and let $T_{d,k}$ be the number of matroids of rank at most d with at most k bases. Then*

$$2^d k T_{d,k} \geq |\Psi_{d,k}| \geq |\Psi_{d,1}| = p(d).$$

The bounds above are far from tight. Nonetheless we expect the asymptotics to be close to the upper bound. It is not even clear that the cardinality of $\Psi_{d,k}$ increases as d

or k increase. Furthermore, restricting to the subset $\Sigma_{d,k}$ of $\Psi_{d,k}$ that consists of simple matroids one observes the following: $|\Sigma_{2,1}| = 1 > 0 = |\Sigma_{2,2}|$. Hence a different behavior in the case of simple matroids is expected.

1.2 Broken circuit complexes

A natural question that follows after studying independence complexes is that of broken circuit complexes. They arise naturally in the study of hyperplane arrangements and are a meaningful generalization of matroids: every matroid is a reduced broken circuit complex.

Theorem 3. *Let d, k be positive integers. The number of isomorphism classes of simple connected, rank d ordered matroids M whose reduced broken circuit complex satisfies $h_{d-1}(\overline{BC_<(M)}) = k$ is finite.*

It is known that h-vectors of broken circuit complexes properly contain the h-vectors of matroids (see [13]). The real reason for the difference is not fully understood. There are examples of broken circuit complexes whose h-vector is not a pure O-sequence and others which do not admit convex ear decompositions. However, numerical inequalities known to be satisfied by h-vectors of matroids are also known to hold for broken circuit complexes after the recent work of Ardila, Denham and Huh.

1.3 Geometric lattices

Interest in geometric lattices has flourished significantly in the last two decades due to their connection with tropical geometry. They are connected to tropical linear spaces via the Bergman fan of M. After after intersecting the fan with a unit sphere, the remaining cellular complex is triangulated a geometric realization of the order complex of the lattice of flats of M. See for instance [4]. It is also crucial in the study of the Chow ring of a matroid and its Hodge structure [1]. Even more, Huh and Wang [8] recently proved Dowling's top heavy conjecture for representable geometric lattices: a theorem on numerical invariants of the lattice, by studying again elements of Hodge theory. It is therefore desirable to get a better grasp of aforementioned invariants from a different point of view, which as a way to complement the new results.

Hidden in one of the exercises in [14, Problem 100.(d) Ch. 3] is the following result: the number of isomorphism classes of simple, loop and coloop free matroids whose geometric lattice is homotopy equivalent to a wedge of k spheres (independently of dimension!) is finite. This is much stronger than the result for independence complexes and can be expressed in terms of Euler characteristics, Möbius functions or the top nonzero h-number of the order complex of the proper part of the lattice. Even though the result is stated in Stanley's book, there seems to be no published proof.

Theorem 4. *Let d, k be positive integers. The number of isomorphism classes of simple matroids M of rank d whose geometric lattice, $\mathcal{L}(M)$, satisfies $|\mu(\mathcal{L}(M))| = k$ is finite. Furthermore if we restrict to coloopless matroids, we can drop the rank condition.*

2 Definitions and notation

This section is devoted to defining, summarizing and relating various aspects of matroid theory that appear in the arguments of this paper. See [13] for more detailed definitions.

2.1 Simplicial complexes

A simplicial complex Δ is a collection of subsets of a finite set E that is closed under inclusion. Any simplicial complex admits a geometric realization, a topological space whose different aspects (geometric and topological) encode the information about the complex. The topology of a simplicial complex refers to the topology of its geometric realization. Throughout this paper we use reduced simplicial homology with rational coefficients.

Let Δ_1 and Δ_2 be simplicial complexes on disjoint ground sets E_1 and E_2, the join $\Delta_1 * \Delta_2$ is the complex on the ground set $E_1 \cup E_2$ whose faces are unions of faces of Δ_1 and Δ_2. Joins of several complexes are defined in the natural straightforward way. The join of two spheres is again a sphere and the join of a sphere and a ball yields another ball. A simplicial complex Δ is said to be *join irreducible* if it is not equal to the join of two non-trivial subcomplexes.

2.2 PS ear decompositions.

The *full d-simplex* Γ_d is the simplicial complex whose faces are all the subsets of a set with $d+1$ elements: it is homeomorphic to a d-dimensional ball. The *boundary of the d-simplex* $\hat{\Gamma}_d$ is the set of proper subsets of a set with $d+1$ elements: it is homeomorphic to a $(d-1)$-sphere. A PS-*sphere* is a join of boundaries of simplices $\hat{\Gamma}_{d_1} * \hat{\Gamma}_{d_2} * \cdots * \hat{\Gamma}_{d_k}$. It is homeomorphic to a sphere of dimension $d_1 + d_2 + \cdots + d_k - 1$.

Lemma 5. *Let Δ be any* PS*-sphere of dimension $d-1$. For every $1 \le i \le d$, the following inequality holds:*

$$h_i(\Delta) \le \binom{d}{i}. \tag{2.1}$$

Consequently, $f_{d-1}(\Delta) \le 2^d$.

Proof. The join operation on simplicial complexes has the effect of multiplying the respective h-polynomials. We have that $h(\hat{\Gamma}_d, t) = 1 + t + \cdots + t^d$, and $h(\hat{\Gamma}_1^d, t) = (1+t)^d$, where $\hat{\Gamma}_1^d$ is the join of d boundaries of segments. This implies that, coefficient by coefficient, we have $h(\hat{\Gamma}_d, t) \leq h(\hat{\Gamma}_1^d, t)$.

For a general PS-sphere we have $h(\hat{\Gamma}_{d_1} * \hat{\Gamma}_{d_2} * \cdots * \hat{\Gamma}_{d_k}, t) = h(\hat{\Gamma}_{d_1}, t)h(\hat{\Gamma}_{d_2}, t) \cdots h(\hat{\Gamma}_{d_k}, t) \leq h(\hat{\Gamma}_1^{d_1}, t)h(\hat{\Gamma}_1^{d_2}, t) \cdots h(\hat{\Gamma}_1^{d_k}, t) = h(\hat{\Gamma}_1^d, t)$, where $d = d_1 + \cdots + d_k$, showing the inequality we wanted. The combinatorially unique maximizer is $\hat{\Gamma}_1^d$ and it is equal to $\partial\Diamond_d$, the boundary of a d-dimensional crosspolytope. □

A PS-ball is a complex of the form $\Sigma * \Gamma_\ell$, where Σ is a PS-sphere. This is a cone over $\hat{\Sigma}$ with apex the whole ball Γ_ℓ. The (topological) boundary of such a PS-ball is the PS-sphere $\Sigma * \hat{\Gamma}_\ell$. Notice that, unless $\ell = 0$, the vertices of a PS-ball are all in the boundary. In the special case $\ell = 0$ the PS ball has one interior vertex.

Definition 6. *Let Δ be a simplicial complex and $K \cong \Sigma * \Gamma_\ell$ a PS-ball with* $\dim(\Delta) = \dim(K)$ *and such that $\Delta \cap K = \partial K$. The complex $\Delta' = \Delta \cup K$ is said to be obtained from Δ by* attaching a PS ear.

Lemma 7. *Under the conditions of Definition 6 above we have the following relation of h-polynomials:*

$$h(\Delta', t) = h(\Delta, t) + t^{l+1}h(\partial K, t).$$

Proof. This is the polynomial version of Lemma 3 [7] together with the Dehn-Sommerville relations for simplicial spheres. □

Definition 8. *A $(d-1)$-dimensional simplicial complex Δ is said to be* PS-ear decomposable *if there is $k \geq 0$ and a sequence $\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k = \Delta$ of complexes, such Δ_0 is a PS-sphere and for $0 \leq j \leq k-1$ the complex Δ_{j+1} is obtained from Δ_j by attaching a PS-ear.*

Remark 9. *Each time we attach an ear the top Betti number goes up by one and hence if we attach $k-1$ PS ears, the resulting complex has $|\tilde{\chi}(\Delta)| = k$.*

2.3 Matroids

A *matroid* is a pair $M = (E, r)$, where E is a finite set and $r : 2^E \to \mathbb{Z}$ is a function on subsets of E that is nonnegative, nondecreasing, and submodular. An *independent set* $I \subset E$ is a subset such that $r(I) = |I|$. Independent sets form a simplicial complex denoted by $\mathcal{I}(M)$. A matroid is said to be *connected*, if $\mathcal{I}(M)$ is join irreducible. Maximal independent sets are called *bases* and we denote the set of bases of matroid $\mathcal{B}(M)$. Minimally dependent (that is, not independent) sets are called *circuits*.

A *flat* is a subset $F \subset E$ such that $r(F) < r(F \cup \{x\})$ for any $x \notin F$. If we have a total order $<$ on E, a *broken circuit* is a circuit with its smallest element removed. A basis is called an *nbc basis* if it does not contain any broken circuit.

An *ordered matroid* $(M, <)$ is a matroid together with an ordering on its ground set. Given an ordered matroid M, a basis B and $b \in B$, say that b is *internally passive* if there is $b' < b$ such that $(B \backslash \{b\}) \cup \{b'\} \in \mathcal{B}(M)$, i.e., it can be replaced by a smaller element to obtain another basis of M. The set of all internally passive elements of a basis B is denoted by $IP(B)$ and it is called the internally passive set of B.

In the paper [5] Bjorner studies three simplicial complexes associated with a matroid M. The first one is the independence complex defined above. The other two are defined here:

Definition 10. *Let $M = (E, r)$ be a matroid of rank d, i.e., $r(E) = d$. We define the following complexes:*

- *The **broken circuit complex** $BC_<(M)$, whenever $(M, <)$ is an ordered matroid, consists of the ground set E with faces given by sets that do not contain broken circuits.*
- *The **order complex of the lattice of flats** $\mathcal{L}(M)$ is the order complex of poset given by flats of M ordered by inclusion (see the precise definitions below).*

All of these complexes have dimension $d - 1$.

In [5] it is shown that all three complexes are *shellable*, a concept we will not define but only state the consequence we need. A shellable simplicial complex Δ of dimension $d - 1$ is homotopy equivalent to the wedge product of k spheres of dimension $d - 1$, where $k = h_d(\Delta) = |\tilde{\chi}(\Delta)|$. Hence, its homotopy type depends on just two parameters: $\dim(\Delta)$ and $\tilde{\chi}(\Delta)$ (or alternatively $h_d(\Delta)$).

The broken circuit complex turns out to be a cone over a non-contractible space: the number of cone points equals the number of connected components of the matroid as shown in [5]. The **reduced** broken circuit complex $\overline{BC}_<(M)$ is the complex that results from removing the cone points of the broken circuit complex. For simplicity we only work with connected matroids, i.e matroids whose independence complex cannot be decomposed as a join of two non-trivial complexes.

The following theorem provides one topological difference between independence and broken circuit complexes. Indeed, it follows from the work of Swartz [15] that it is *false* for broken circuit complexes.

Theorem 11 ([7, Theorem 3])**.** *For any matroid M, the independence complex $\mathcal{I}(M)$ is PS-ear decomposable.*

2.4 Geometric lattices

For any matroid M we have a partially ordered set (by inclusion) on the set of flats. These posets are characterized by certain extra properties, they are precisely the *geometric lattices*.

Theorem 12. *Assigning the poset $\mathcal{L}(M)$ to each matroid M induces a one-to-one correspondence between geometric lattices and simple matroids.*

Every poset $\mathbb{P}$ gives a simplicial complex $\mathcal{O}(\mathbb{P})$, called the order complex of $\mathbb{P}$, in the following way: Its elements are the elements of $\mathbb{P}$ and the faces are the chains ordered by inclusion. As mentioned before, the order complex of a geometric lattice L is shellable.

3 Independence Complexes

This section is devoted to various proofs of Theorem 1. Quite surprisingly the result is a simple consequence of several standard (yet deep) theorems in matroid theory.

Definition 13. *Let $\Psi_{d,k}$ be the set of all isomorphism classes of* loopless *matroids M such that* $\dim(\mathcal{I}(M)) = d-1$ *and* $|\tilde{\chi}(\mathcal{I}(M))| = k$.

Theorem 1 implies the existence of upper bounds for each entry of the h-vectors and f-vectors of a matroid in terms of its dimension and its Euler characteristic. We start by providing tight bounds. These bounds were first proved in Swartz [16] with a complicated inductive argument. The advantage of our approach is that we can completely classify equality cases.

Theorem 14. *Let $M \in \Psi_{d,k}$ we have the following inequalities:*

1. $h_i(\mathcal{I}(M)) \leq \binom{d}{i} + (k-1)\binom{d-1}{i-1}$*, for* $0 \leq i \leq d$.
2. $f_i(\mathcal{I}(M)) \leq \binom{d}{i+1}2^{i+1} + (k-1)\binom{d-1}{i}2^i$*, for* $-1 \leq i \leq d-1$.

Furthermore, these inequalities are tight and equality holds only if M is isomorphic to the matroid $V_{d,k}$ defined below.

Proof. We begin with the first part. We will use Theorem 11, i.e., the fact that $\mathcal{I}(M)$ is PS ear decomposable. To begin with, there is a unique h-vector maximizer among the PS spheres Δ_0; namely it is the boundary of a d-dimensional crosspolytope and its h-vector is given by the binomial coefficients (Lemma 5). By Lemma 7, together with Lemma 5, the way to attach a PS ear with maximal resulting h-vector is by attaching a PS ball whose boundary is isomorphic to $\partial\Diamond_{d-1}$. We now show that this maximal bound can be attained.

Set Δ_0 to be $\partial\Diamond_d$. Fix a vertex $v \in \Delta_0$ and attach an ear using the PS ball $\Sigma * \Gamma_0$, where Σ is the link of v (which isomorphic to $\partial\Diamond_{d-1}$) and Γ_0 is just a single new vertex. We can repeat this process k times, always using the same link of the original vertex v. The simplicial complex obtained in this way is the independence complex of matroid. Our choice of Δ_0 is the independence complex of the graphical matroid of a path with each edge doubled. Each ear attachment corresponds to adding parallel elements to a fixed edge. We denote this matroid by $V_{d,k}$.

The second part follows from the fact that $V_{d,k}$ also maximizes each entry of the f-vector. This is because the f-vector is a positive combination of the h-vector. □

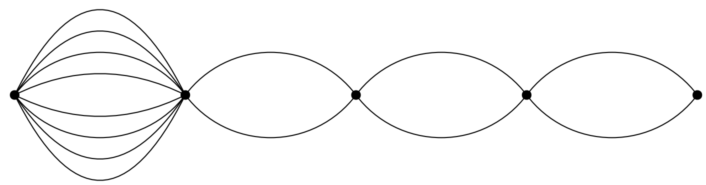

Figure 1: The graphical matroid $V_{4,7}$.

Now we can give another proof of Theorem 1.

Proof of Theorem 1. We have $h_d(\mathcal{I}(M)) = |\tilde{\chi}((\mathcal{I}(M)))|$, so Theorem 14 gives $f_0(\mathcal{I}(M)) \leq 2d + h_d(\mathcal{I}(M)) - 1$. Fixing $h_d(\mathcal{I}(M))$ and d bounds the number of vertices $(\mathcal{I}(M))$ can have, whence the result follows. □

In contrast to the case of the Upper Bound Theorem for spheres (see [12]), $V_{d,k}$ is the unique maximizer up to isomorphism. However, $V_{d,k}$ is perhaps not very interesting from the matroid theoretic perspective (for instance lattice of flats of $V_{d,k}$ is the boolean lattice B_d). A relevant variant, which we expect to be harder, is the analogous question over the family of **simple** matroids.

Question 15. *What is the maximal value of $h_j(\mathcal{I}(M))$ for where M ranges over all simple matroids of $\Psi_{d,k}$? Is there a single simple matroid that simultaneously maximizes all the h-vector entries? What if we further restrict to the class of simple connected matroids?*

In light of the above question, we notice that for simple matroids, the number of vertices is strictly less than $2d + h_d(\mathcal{I}(M)) - 1$ which is the tight upper bound for general matroids.

Corollary 16. *If M is a matroid with $f_0(\mathcal{I}(M)) = 2d + h_d(\mathcal{I}(M)) - 1$, then M is isomorphic to $V_{d,k}$.*

To finish, we present a proof of the main theorem which allows us to say something about the size of $\Psi_{d,k}$.

Second proof of Theorem 1. Given a matroid M and a basis B, Corollary 3.5 in [10] shows that the h-polynomial of the independence complex of M can be decomposed as:

$$h(\mathcal{I}, x) = \sum_I x^{|I|} h(\mathrm{link}_{\mathcal{I}}(I)|_B, x).$$

The sum is taken over the independent sets I of M that are disjoint from B. Lemma 3.8 in [10] shows that all maximal such I under inclusion, i.e the bases of the induced matroid on $E \backslash B$, satisfy that $h_{d-|I|}(\mathrm{link}_{\mathcal{I}}(I)|_B) \neq 0$. It follows that $h_d(M)$ is bounded below by the number of bases of $M|_{E-B}$. This implies that there are at most k maximal bases.

Together with the fact that the rank of the restriction is bounded above by d, this implies that the number of possible restrictions is finite. The missing independent sets consist of a subset of B together with an element of the restriction, thus the number of matroids with $h_d = k$ is bounded above by $2^d k T_{d,k}$, where $T_{d,k}$ is the number of matroids of rank at most d with at most k bases. □

In general, it follows from Chari's Theorem 11 that $|\Psi_{d,1}| = p(d)$, the number of integer partitions of d. Consequently, the best kind of formula we can expect for the cardinality of $\Psi_{d,k}$ is asymptotic. It is unclear that the value of $\Psi_{d,k}$ is monotone in either of the parameters. At least the construction of $V_{d,k}$ shows that $\Psi_{d,k} \neq \emptyset$. Using the same ideas we can say a little more.

Lemma 17. $|\Psi_{d,1}| \leq |\Psi_{d,k}|$ *for every positive integer d*

Proof. Since every matroid in $\Psi_{d,1}$ is a PS-sphere, we can choose any vertex v and replicate the construction of $V_{d,k}$ to get an inclusion $\Psi_{d,1} \to \Psi_{d,k}$. □

Notice that the previous argument is not strong enough to prove that $\Psi_{d,k} \leq \Psi_{d,k+1}$ in general (if $d = 1$ the number of all such matroids is one). In particular, it would be interesting to find a matroid operation that increases $h_d(\mathcal{I}(M))$ by one in general. The previous construction relies heavily on having a vertex of the independence complex whose link is a sphere. This is, presumably, almost never the case.

4 Broken Circuit complexes

Theorem 3 is a natural extension of Theorem 1. The proof is a consequence of Theorem 5.4 in [16]. Notice that if M is a connected rank-d simple matroid without coloops, then the upper bounds for h_i in terms of h_{d-1} coincide with the tight ones found for matroids.

Thus the numerical inequalities found for broken circuit complexes are not stronger than the ones found for matroids. However, the lack of flexibility of the approach leads to some potential differences: for instance, while the independence complex maximizer is unique, none of the arguments that apply to broken circuit complexes. In particular Swartz [15] provided examples of broken circuit complexes such that the Artinian reduction of the Stanley-Reisner ring admits no g-element. This means that some broken complexes do not admit convex ear decompositions (even after increasing the family of allowable convex spheres and balls). As a result it follows that the proof using PS-ear decomposition cannot be extended, and the equality classification fails.

An alternative approach, which is part of a current research project of the second author, comes from studying the $\text{Int}_<(M)$ poset when restricted to the facets of $BC_<(M)$. As evidence that an argument along these lines may be reasonable, we provide a new structural theorem about the subposet of $\text{Int}_<(M)$ that consists of nbc bases: it is an order ideal. By studying the structure of the subposet carefully we expect to be able to classify upper bounds and hopefully obtain significant quantitative differences between the two families of h-vectors.

5 Order complexes of geometric lattices.

A careful look at Exercise 100(d) in Chapter 3 of [14] gives a much stronger result than an analogous of Theorem 1. The level of the problem in the ranking [3-], but unlike most problems in the book, the solution is not written down. To the best of our knowledge, it is not anywhere in the literature, so we include it here for the sake of completeness.

Theorem 18. *Fix a natural number k. There exist finitely many geometric lattices $L_1, \cdots, L_m$ such that if L is any finite geometric lattice satisfying $|\tilde{\chi}(\mathcal{O}(L))| = k$ then $L = L_i \times B_d$ for some i, d.*

Proof. Notice that the simple matroid associated to $L \times B_d$ is the join of the matroid of L with the full $d-1$-simplex Γ_{d-1}. Thus it suffices to show that there are finitely many simple coloop free matroids M whose lattice of flats has Euler characteristic equal to k.

Assume that M is such a matroid and L is the associated geometric lattice. By [5, Proposition 7.4.5] the Euler characteristic of $\tilde{\chi}(\mathcal{O}(L))$ equals the number of facets of $BC_{<}(M)$. And there are finitely many isomorphism classes of such broken circuit complexes with k facets.

Let Δ be one such broken circuit complex. We claim that only finitely many matroids can have Δ as a broken circuit complex. To prove this we will bound the number of vertices of the independence complex of any such matroid. Let $C_1, C_2, \ldots C_s$ be the minimal nonfaces of Δ, that is, the broken circuits of any potential matroid. Let M be a simple ordered matroid that has Δ as a broken circuit complex. Assume that $C_i \cup x$ and $C_i \cup y$ are circuits of M. Pick an arbitrary $c \in C_i$. Note that by the circuit elimination axiom, the set $(C_i \cup \{x, y\}) \backslash \{c\}$ is a nonface. Since M is simple, $x < y$ are not parallel. Thus there is a circuit of M containing $\{x, y\}$. Such a circuit has to be equal to $C_j \cup \{x\}$ for some j or $C_j \cup \{z\}$ for some j and some other z in the groundset of M. In either case $y \in C_j$, and hence a vertex of Δ. Hence the number of vertices of M not in Δ that extend the broken circuit C_j is at most one, which leads to the inequality $f_0(\mathcal{I}(M)) \leq f_0(\Delta) + s$ as desired. □

Note that one cannot drop the dimension assumption from Theorem 1, since

$$\tilde{\chi}(\mathcal{I}(U_{d,d+1}))) = 1$$

for every d, where $U_{d,d+1}$ is the uniform matroid of rank d on $[d+1]$.

Acknowledgements

We thank Isabella Novik, Ed Swartz and Steven Klee for helpful conversations.

References

[1] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". *Ann. Math. (2)* **188**.2 (2018), pp. 381–452. Link.

[2] N. Anari, K. Liu, S. Oveis, and C. Vinzant. "Log-Concave Polynomials III: Mason's Ultra-Log-Concavity Conjecture for Independent Sets of Matroids". 2018. arXiv:1811.01600.

[3] F. Ardila. *Matroids and tropical geometry*. Slides from talk at MSRI. 2017. Link.

[4] F. Ardila and C. J. Klivans. "The Bergman complex of a matroid and phylogenetic trees". *J. Combin. Theory Ser. B* **96**.1 (2006), pp. 38–49. Link.

[5] A. Björner. "The homology and shellability of matroids and geometric lattices". *Matroid Applications*. Encyclopedia Math. Appl. 40. Cambridge Univ. Press, Cambridge, 1992, pp. 226–283.

[6] P. Brändén and J. Huh. "Hodge-Riemann relations for Potts model partition functions". 2018. arXiv:1811.01696.

[7] M. K. Chari. "Two decompositions in topological combinatorics with applications to matroid complexes". *Trans. Amer. Math. Soc.* **349**.10 (1997), pp. 3925–3943. Link.

[8] J. Huh and B. Wang. "Enumeration of points, lines, planes, etc". *Acta Math.* **218**.2 (2017), pp. 297–317. Link.

[9] M. Juhnke-Kubitzke and D. V. Le. "Flawlessness of h-vectors of broken circuit complexes". *Int. Math. Res. Not. IMRN* 5 (2018), pp. 1347–1367. Link.

[10] S. Klee and J. A. Samper. "Lexicographic shellability, matroids, and pure order ideals". *Adv. in Appl. Math.* **67** (2015), pp. 1–19. Link.

[11] K. Nyman and E. Swartz. "Inequalities for the h-vectors and flag h-vectors of geometric lattices". *Discrete Comput. Geom.* **32**.4 (2004), pp. 533–548. Link.

[12] R. P. Stanley. "The upper bound conjecture and Cohen-Macaulay rings". *Studies in Appl. Math.* **54**.2 (1975), pp. 135–142. Link.

[13] R. P. Stanley. *Combinatorics and Commutative Algebra*. 2nd ed. Progress in Mathematics 41. Boston, MA: Birkhäuser Boston, Inc., 1996.

[14] R. P. Stanley. *Enumerative Combinatorics. Vol. 1*. 2nd ed. Cambridge Studies in Advanced Mathematics 49. Cambridge University Press, Cambridge, 2012.

[15] E. Swartz. "g-elements of matroid complexes". *J. Combin. Theory Ser. B* **88**.2 (2003), pp. 369–375. Link.

[16] E. Swartz. "Lower bounds for h-vectors of k-CM, independence, and broken circuit complexes". *SIAM J. Discrete Math.* **18**.3 (2004/05), pp. 647–661. Link.