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# Limit shapes of evacuation and jeu de taquin paths in random square tableaux

Łukasz Maślanka*[1] and Piotr Śniady† [1]

[1] *Institute of Mathematics, Polish Academy of Sciences, Warszawa, Poland*

**Abstract.** We consider large random square Young tableaux and look for typical (in the sense of probability) *jeu de taquin paths* and *evacuation paths* in the asymptotic setting. We show that the probability distribution of such paths converges to a random *meridian* connecting the opposite corners of the square.

**Keywords:** random Young tableaux, limit shape, jeu de taquin, Schützenberger's evacuation, square Young diagrams

## 1 Introduction

A full version of this extended abstract will be published in [4].

### 1.1 Notation

For any $n \in \mathbb{N}$ let $\mathbb{Y}_n$ denote the *set of Young diagrams* with $n$ boxes. By $\square_N \in \mathbb{Y}_{N^2}$ we denote *the square Young diagram* of side $N$. For a Young diagram $\lambda \in \mathbb{Y}_n$ let $|\lambda| := n$ be the *size* of diagram $\lambda$ and by $\mathcal{T}_\lambda$ denote the *set of standard Young tableaux of shape* $\lambda$. On the set $\mathcal{T}_{\square_N}$ of square tableaux with $N^2$ boxes we consider *the uniform probability measure* $\mathbb{P}_N$.

We draw Young tableaux in the *French notation*. For a tableau $T$, by $\operatorname{pos}_k(T) = (x_k, y_k)$ we denote the *Cartesian coordinates of entry* $k$, i.e. $x_k$ is its column and $y_k$ is its row. The difference $u_k^T := x_k - y_k$ will be called *the u-coordinate* of the box $k$ (in the literature the name *content* is also used).

The irreducible representation of the symmetric group $\mathfrak{S}_n$ which corresponds to $\lambda \in \mathbb{Y}_n$ will be denoted by $V^\lambda$ and its character by $\chi^\lambda$.

*lmaslanka@impan.pl.
†psniady@impan.pl.

## 1.2 Jeu de taquin, jeu de taquin path, evacuation path

***Jeu de taquin*** is a transformation of tableaux that was introduced by Schützenberger [7]. It is a fundamental tool for studying the combinatorics of Young tableaux, Robinson–Schensted–Knuth (RSK) algorithm, and Littlewood–Richardson coefficients [2].

Jeu de taquin acts as follows (see Figure 1(a) and (b)): we start with the bottom-left corner box of the given tableau $T$, we erase it and obtain a *hole* in its place. Then we look at the neigboring two boxes: the one to the right and the one above the hole, and choose the smaller one. We slide the selected box into the hole; as a result the hole moves one box to the right or up. We continue sliding as long as there is some box to right or up to the hole. The resulting tableau will be denoted by $j(T)$. The zig-zag path traversed by the hole will be called ***jeu de taquin path***.

For a given tableau $T \in \mathcal{T}_\lambda$ with $n = |\lambda|$ boxes we may iterate jeu de taquin $n$ times until we end with the empty tableau. During each iteration the box with the biggest number $n$ either moves one node left or down, or stays put. Its trajectory

$$\operatorname{pos}_n(T),\ \operatorname{pos}_n\Big(j(T)\Big),\ \operatorname{pos}_n\Big(j^2(T)\Big),\ \dots,\ \operatorname{pos}_n\Big(j^{n-1}(T)\Big) \tag{1.1}$$

will be called ***evacuation path***.

## 1.3 The problem

If we draw the boxes of a given square tableau $T_N \in \mathcal{T}_{\square_N}$ as little squares of size $\frac{1}{N}$ then both the corresponding jeu de taquin path as well as the evacuation path is a zig-zag line connecting the opposite corners of the unit square $[0,1]^2$. We consider the case when $T_N \in \mathcal{T}_{\square_N}$ is a *random* square tableau sampled with the uniform distribution $\mathbb{P}_N$. **The goal of the current paper is to find asymptotics of the corresponding scaled random zig-zag lines in the limit as $N \to \infty$, see Figure 2a.**

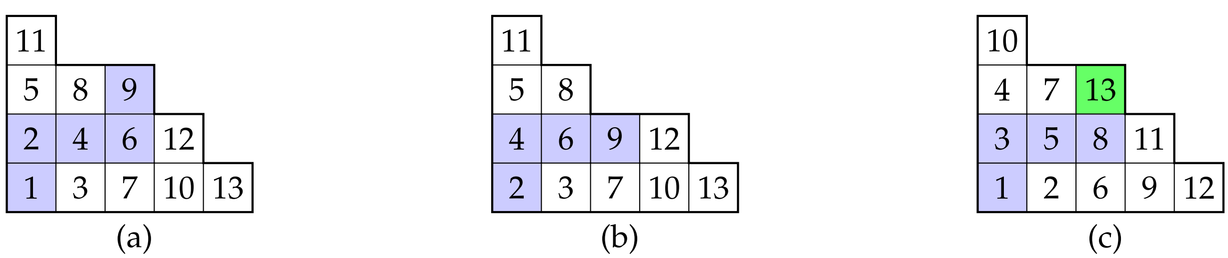
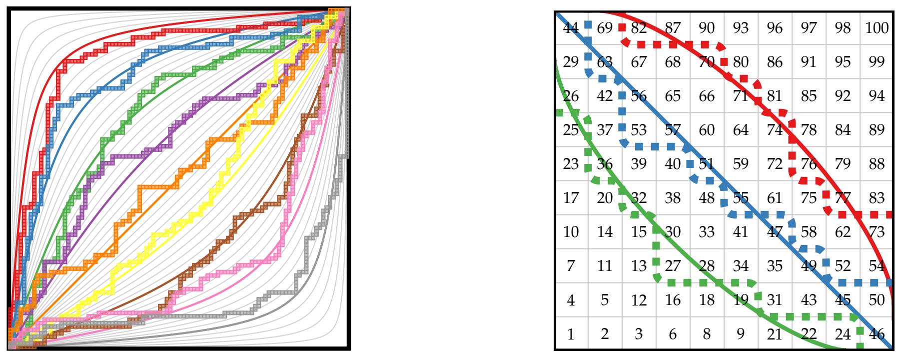



**Figure 1:** (a) Standard Young tableau $T$ of shape $\lambda$. The highlighted boxes form the *jeu de taquin path*. (b) The outcome $j(T)$ of jeu de taquin transformation. (c) A bijection on $\mathcal{T}_\lambda$, *modified jeu de taquin* $J(T)$, is defined by creating an additional box with entry $|\lambda|+1$ at the final position of the hole and then decreasing all entries by 1.




(a) (b)

**Figure 2:** (a) Zigzag lines are sample jeu de taquin paths for random square tableaux of size $N = 100$, selected so they cross the anti-diagonal near the corresponding meridians (smooth thick curves) with the longitudes $\psi \in \{1/10, 2/10, \dots, 9/10\}$. The grey lines are the meridians with the longitudes $\psi \in \{2/100, 4/100, \dots, 98/100\}$. (b) Sample random square tableau of size $N = 10$. Zigzag lines are the level curves for $\alpha \in \{1/4, 1/2, 3/4\}$. The smooth lines are the corresponding circles of latitude $g_\alpha$.

## 2 Asymptotics of evacuation paths

It is easy to change the definition of jeu de taquin so that it becomes a permutation $J \colon \mathcal{T}_\lambda \to \mathcal{T}_\lambda$ on the set of standard tableaux of prescribed shape, see Figure 1c.

We keep notations from Section 1.3 and consider the evacuation path (1.1) for $T = T_N$ and $n = N^2$. The position of each of the boxes in the evacuation path coincides with the position of a specific box in the standard Young tableau obtained by iterating $J$:

$$\operatorname{pos}_{N^2}\left(j^i(T_N)\right) = \operatorname{pos}_{N^2-i}\left(J^i(T_N)\right). \tag{2.1}$$

The latter standard tableaux $J^i(T_N)$ is a uniformly random square Young tableau.

Some light on the problem of finding the right-hand side of (2.1) is provided by the work of Pittel and Romik; one of their results [5, Theorem 2] gives an explicit family of curves $(g_\alpha)$ indexed by $\alpha \in [0,1]$ which fit inside the square $[0,1]^2$, see Figure 2b. Each curve $g_\alpha$ turns out to be one of the level curves of (scaled down) random standard Young tableau of shape $\square_N$ which separates the boxes with entries $\leq \alpha N^2$ from the boxes with entries bigger than this threshold (asymptotically, as $N \to \infty$, except for an event of negligible probability). We will refer to the curves $(g_\alpha)$ as ***circles of latitude***.

It follows that **asymptotically the (scaled down) position of the box** (2.1) **in the evacuation path is very close to some point on the circle of latitude** $g_{1-t}$ **where** $t = i/N^2$**.** ***The remaining difficulty is to pinpoint a specific location of this point on the curve.*** **For this purpose we need a convenient parametrization on each circle of latitude** $g_\alpha$**.**

### 2.1 Longitude. Geographic coordinates on the square

The work of Pittel and Romik [5, Theorem 2] additionally gives the limit distribution of the (scaled down) position of the box $\lfloor \alpha N^2 \rfloor$ in a random square tableau with $N^2$ entries. This probability distribution is supported on the curve $g_\alpha$ and will be denoted by $\mu_\alpha$. We will use $\mu_\alpha$ to construct *geographical coordinate system* on the unit square $[0,1]^2$.

For any point $p \in [0,1]^2$ there is exactly one curve $g_\alpha$ on which it lies. We say that *the latitude of $p$ is equal to $\alpha$*. The ***longitude* of** $p$, denoted by $\psi(p) \in [0,1]$, is defined as the measure $\mu_\alpha$ of the set of points on the curve $g_\alpha$ which have their $u$-coordinate smaller than the $u$-coordinate of $p$. For given $\alpha, \psi \in [0,1]$ we denote by $P_{\alpha,\psi}$ the unique point of the square $[0,1]^2$ with the appropriate latitude and longitude.

### 2.2 The first main result: asymptotics of evacuation paths

For a given tableau $T_N \in \mathcal{T}_{\square_N}$ and $t \in [0,1]$ we denote by

$$X_t = X_t(T_N) = \frac{1}{N} \operatorname{pos}_{N^2}\left( j^{\lfloor tN^2 \rfloor}(T_N) \right) \in [0,1]^2$$

the scaled down point from the evacuation path, cf. (1.1).

Our first main result states that, asymptotically, *the scaled evacuation path in a random square tableau is a random meridian*, i.e. a curve which consists of points in the square $[0,1]^2$ with equal longitude $\psi$. The probability distribution of this longitude $\psi$ is the uniform distribution on the interval $[0,1]$.

**Theorem 2.1.** *For each $N \in \mathbb{N}$ there exists a map $\Psi_N \colon \mathcal{T}_{\square_N} \to [0,1]$ such that for each $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}_N \left\{ T_N \in \mathcal{T}_{\square_N} : \sup_{t\in[0,1]} \left| X_t(T_N) - P_{1-t,\Psi_N(T_N)} \right| > \varepsilon \right\} = 0. \tag{2.2}$$

*The probability distribution of the random variable $\Psi_N$ converges, as $N \to \infty$, to the uniform distribution on the unit interval $[0,1]$.*

## 3 Asymptotics of jeu de taquin paths

Let $T$ be a standard Young tableau with $n$ boxes. We define $\mathbf{q}(T) = (\mathbf{q}_1, \dots, \mathbf{q}_n) \subset \mathbb{N}^2$ to be the corresponding *jeu de taquin path in the lazy parametrization*. More specifically, $\mathbf{q}_i$ is defined as the last box along the jeu de taquin path corresponding to $T$ (cf. Figure 1a) which contains a number $\leq i$.

Our second main result states that, asymptotically, *the scaled jeu de taquin path in a random square tableau is a random meridian*.

**Theorem 3.1.** *For each $N \in \mathbb{N}$ there exists a map $\widetilde{\Psi}_N \colon \mathcal{T}_{\square_N} \to [0,1]$ such that for each $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}_N \left\{ T_N \in \mathcal{T}_{\square_N} : \sup_{t\in[0,1]} \left| \frac{1}{N} \mathbf{q}_{\lceil tN^2 \rceil}(T_N) - P_{t,\widetilde{\Psi}(T_N)} \right| > \varepsilon \right\} = 0.$$

*The probability distribution of the random variable $\widetilde{\Psi}_N$ converges, as $N \to \infty$, to the uniform distribution on the unit interval $[0,1]$.*

An analogous problem was studied by Romik and Śniady [6] for random tableaux obtained by applying Robinson–Schensted correspondence to a random permutation (*"Plancherel measure"*). The proof presented there is not applicable in our context.

*Proof.* We show equivalence of Theorems 2.1 and 3.1 by comparing positions of boxes in lazy jeu de taquin path and the evacuation path in pairs of corresponding tableaux.

In the proof we use: (1.) a *Robinson-Schensted-Knuth algorithm* which is a bijection which applied to a permutation $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_n$ returns a pair $(P(\sigma), Q(\sigma))$ of tableaux of the same shape; (2.) a *Schützenberger involution* which is a map

$$\mathfrak{S}_n \ni \sigma \mapsto \varepsilon^*(\sigma) := (n+1-\sigma_n, \dots, n+1-\sigma_1);$$

and (3.) the identity [6, Section 2] in which $s(\sigma) = s(\sigma_1, \sigma_2, \dots, \sigma_n) := (\sigma_2, \dots, \sigma_n)$:

$$\big(j \circ Q\big)(\sigma) = \big(Q \circ s\big)(\sigma), \quad \sigma = (\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_n \tag{3.1}$$

((3.1) holds true up to renumbering of the boxes on the left-hand side.)

Let $\sigma \in \mathfrak{S}_{N^2}$. In the lazy jeu de taquin path $\mathbf{q}(T) := \mathbf{q}(Q(\sigma))$ we have by (3.1):

$$\mathbf{q}_n(T) = Q(\sigma_1, \sigma_2 \dots, \sigma_n) \setminus Q(\sigma_2, \dots, \sigma_n).$$

On the other hand, using (3.1) many times, in the evacuation path (1.1) in the tableau $T^* := Q(\varepsilon^*(\sigma))$, the position of the box with entry $N^2$ in $j^{N^2-p}(T^*)$ is

$$Q\left(N^2+1-\sigma_p, \dots, N^2+1-\sigma_2, N^2+1-\sigma_1\right) \setminus Q\left(N^2+1-\sigma_p, \dots, N^2+1-\sigma_2\right).$$

By Greene theorem, this position is equal to $Q\left(\sigma_1, \sigma_2 \dots, \sigma_p\right) \setminus Q\left(\sigma_2, \dots, \sigma_p\right) = \mathbf{q}_p(T)$. □

# 4 Sketch of proof of Theorem 2.1

*Proof of Theorem 2.1.* In order to investigate the point $X_t$ we will pass to its geographic coordinates $\alpha(X_t)$ and $\psi(X_t)$; our goal is to show that for each $c > 0$ and $\varepsilon > 0$

$$\sup_{t\in[0,1]} \Big| \alpha(X_t) - (1-t) \Big| < \varepsilon \qquad \text{and} \qquad \sup_{t\in[c,1-c]} \Big| \psi(X_t) - \Psi_N(T_N) \Big| < \varepsilon \tag{4.1}$$

hold true except for $T_N$ in a set which has asymptotically negligible probability $o(1)$.

The discussion from the beginning of Section 2 shows that the first statement in (4.1) indeed holds true (except for an event of negligible probability, for $N \to \infty$).

If the second statement in (4.1) instead of a supremum would involve only a fixed value of $t_0 \in (0,1)$, we could simply define $\Psi_N(T_N) := \psi(X_{t_0})$ to be longitude of $X_{t_0}$. Unfortunately, we need to justify that this choice of $\Psi_N(T_N)$ is also good for other values of $t \neq t_0$. This kind of result is provided by Proposition 4.1. □

**Proposition 4.1.** *Assume that* $0 < t_1 < t_2 < 1$. *Then for each* $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}_N \left\{ T_N \in \mathcal{T}_{\square_N} : \left| \psi(X_{t_1}) - \psi(X_{t_2}) \right| > \varepsilon \right\} = 0.$$

# 5 Sketch of proof of Proposition 4.1. Surfers on the sink

Proposition 4.1 is equivalent to the conjuction of the following two statements for $\varepsilon > 0$:

$$\lim_{N\to\infty} \mathbb{P}_N \left\{ T_N \in \mathcal{T}_{\square_N} : \psi(X_{t_1}) - \psi(X_{t_2}) > \varepsilon \right\} = 0, \tag{5.1}$$

$$\lim_{N\to\infty} \mathbb{P}_N \left\{ T_N \in \mathcal{T}_{\square_N} : \psi(X_{t_1}) - \psi(X_{t_2}) < -\varepsilon \right\} = 0. \tag{5.2}$$

We present the proof of (5.1); the other claim (5.2) can be proved in an analogous way.

## 5.1 Single surfer scenario

We present the story of the evacuation path in a different light. We will speak about a *square pool of side* $N$ (=the square Young diagram $\square_N$) filled with $N^2 - 1$ *particles of water* (=the Young tableau $T_N$ with the largest entry removed), a passive *surfer* (=the box with the biggest entry) and its behaviour when the pool is drained (=jeu de taquin). Our goal in Theorem 2.1 is to show that, when pool is big enough, the surfer has some typical paths along which he/she moves.

**In our proof of Proposition 4.1 we start our analysis at time $t_1$ when jeu de taquin was already applied $m = \lfloor t_1 N^2 \rfloor$ times.** Our starting point is therefore the tableau

$$T'_N := j^m(T_N)$$

with $N^2 - m$ boxes. Let us renumber the entries of this tableau so it becomes standard. We denote $w = N^2 - m - 1$. In this way the boxes with numbers $1, \dots, w$ correspond to the *water* and the box with the number $w + 1$ to the *surfer*. By removing the box with the surfer

$$W'_N := T'_N \setminus \{w + 1\}$$

we get a standard Young tableau which encodes *the configuration of the water*.

## 5.2 Pieri tableaux. Multisurfer scenario

Let $k = \lfloor \sqrt[4]{N} \rfloor$. Let $M$ be a tableau in which the $k$ largest entries are numbered by consecutive integers $l+1, \dots, l+k$. We say that $M$ is a *Pieri tableau* if these $k$ largest boxes are in *an increasing order* from north-west to south-east:

$$u^M_{l+1} < \cdots < u^M_{l+k}.$$

By $\widetilde{\mathcal{T}}_{\square_N}$ we denote the set of standard Young tableaux of shape $\square_N$ which are Pieri, and by $\widetilde{\mathbb{P}}_N$ the uniform distribution on the set $\widetilde{\mathcal{T}}_{\square_N}$.

It is easy to check that if $M$ has at least $k+1$ boxes then $M$ is a Pieri tableau if and only if $j(M)$ is a Pieri tableau.

To pinpoint the position of the surfer in random water, we will need a point of reference. We introduce **the multisurfer scenario** in which we consider the square pool filled with $N^2 - k$ *particles of water* on which are surfing $k$ *surfers* (=$k$ boxes with the biggest entries). We assume that the multisurfers are in an increasing order; it follows that this scenario corresponds to a Pieri tableau $M_N \in \widetilde{\mathcal{T}}_{\square_N}$ We assume that $M_N$ is a random tableau sampled with the probability distribution $\widetilde{\mathbb{P}}_N$.

**We start our analysis when jeu de taquin was already applied $m+1-k$ times.** Our starting point is therefore the tableau

$$M'_N := j^{m+1-k}(M_N)$$

with $N^2 + k - m - 1 = w + k$ boxes. Let us renumber the entries of this tableau so it becomes standard. In this way, just as in the single surfer scenario, the boxes with numbers $1, \dots, w$ correspond to the *water*. On the other hand, the boxes with the numbers $w+1, \dots, w+k$ correspond to the *multisurfers*. By removing the multisurfers

$$\widetilde{W}'_N := M'_N \setminus \{w+1, \dots, w+k\} = J^{m+1-k}(M_n) \setminus \{w+1, \dots, N^2\} \tag{5.3}$$

we get a standard Young tableau which encodes the configuration of the water.

## 5.3 Counting multisurfers gives longitude

For $0 \le i \le N^2 - k$ we consider the situation after jeu de taquin was applied $i$ times in the multisurfer scenario, i.e. $M'_N = j^i(M_N)$, and for $u \in [-1, 1]$ we define the random variable $\overline{\Psi}_i(u)$ to be the fraction of the number of multisurfers which have scaled $u$-coordinates smaller than $u$, i.e.:

$$\overline{\Psi}_i(u) := \frac{1}{k} \max \left\{ p \in \{1, \dots, k\} : \frac{1}{N} u^{M'_N}_{w+p} \le u \right\}. \tag{5.4}$$

The following result states that $\overline{\Psi}_i$ gives a good approximation of the longitude.

**Proposition 5.1.** *Let $\alpha \in (0,1)$ be fixed. For a given $\psi \in [0,1]$ we set $(x,y) = P_{\alpha,\psi}$ to be the Cartesian coordinates of the point with geographic coordinates $\alpha, \psi$ and define $U_{\alpha,\psi} := x - y$.*

*We set $i := \lfloor (1-\alpha)N^2 \rfloor$. The random variable $\overline{\Psi}_i\left(U_{\alpha,\psi}\right)$ converges in probability to $\psi$ and this convergence is uniform over $\psi$; in other words for each $\varepsilon > 0$*

$$\widetilde{\mathbb{P}}_N\left\{ M_N \in \widetilde{\mathcal{T}}_{\square_N} : \sup_{\psi\in[0,1]} \left|\psi - \overline{\Psi}_i\left(U_{\alpha,\psi}\right)\right| > \varepsilon \right\} = O\left(\frac{1}{\sqrt{N}}\right).$$

*Proof.* The scaled $u$-coordinates of the multisurfers may be encoded by a random probability measure on $\mathbb{R}$

$$\mu := \frac{1}{k} \sum_{1\leq p\leq k} \delta\left(\frac{1}{N}\, u^{M'_N}_{w+p}\right),$$

where $\delta(x)$ denotes the point measure concentrated in $x$. Our goal is to prove that the cumulative probability function of $\mu$ converges uniformly to some explicit limit; the convergence should hold true in probability.

In order to achieve this goal we investigate the *moments* of the measure $\mu$:

$$\mathcal{M}_r = \int_{\mathbb{R}} z^r \, \mathrm{d}\mu = \frac{1}{k} \sum_{1\leq p\leq k} \left(\frac{1}{N}\, u^{M'_N}_{w+p}\right)^r \qquad \text{for } r \in \{1,2,\dots\}.$$

By applying Lemma 5.2 for some specific symmetric polynomials $W$ we are able to find explicit values for the mean value and the variance of the random variable $\mathcal{M}_r$. It turns out that, asymptotically for $N \to \infty$, the mean value $\mathbb{E}\mathcal{M}_r$ converges to the 'right' value and the variance $\operatorname{Var}\mathcal{M}_r$ tends to zero which implies that the convergence of measures *in the sense of moments* indeed holds true in probability.

Since the limit measure is compatly supported, the convergence *in the sense of moments* implies also convergence of measures *in the weak topology of probability measures*, as required. □

Lemma 5.2 **provides a link between the statistical properties of the multisurfers and the representation theory of the symmetric groups.** The problem of understanding (the $u$-coordinates of) the positions of the multisurfers after $i$ applications of jeu de taquin is equivalent to finding (the $u$-coordinates of) the boxes with numbers from the set $M := \{a, \dots, b\} := \{N^2+1-k-i, \dots, N^2-i\}$ in the random tableau $S := J^i(M_N)$.

We will view the symmetric group $\mathfrak{S}_k \subset \mathfrak{S}_{N^2}$ as the group of permutations of the set $M$ and define an element of the symmetric group algebra $p_{\mathfrak{S}_k} = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \sigma \in \mathbb{C}\mathfrak{S}_{N^2}$. We also consider Jucys–Murphy elements $Z_s := \sum_{i=1}^{s-1}(i,s) \in \mathbb{C}\mathfrak{S}_{N^2}$ (see [3]).

**Lemma 5.2.** *Let $W(x_1, \dots, x_k)$ be a symmetric polynomial in $k$ variables. Let $S$ be a random element (sampled with the uniform distribution) of the set of tableaux $T \in \mathcal{T}_{\square_N}$ with the additional property that*

$$u^T_a < \cdots < u^T_b.$$

*Then*

$$\mathbb{E}\, W\left(u_a^S,\, \dots,\, u_b^S\right) = \frac{\chi^{\square_N}\Big(p_{\mathfrak{S}_k} \cdot W\left(Z_a,\, \dots,\, Z_b\right)\Big)}{\chi^{\square_N}\left(p_{\mathfrak{S}_k}\right)}. \tag{5.5}$$

*Proof.* The proof has the following ingredients.

Firstly, the observation that the vector space $V^{\square_N}$ has a linear basis $(e_T)$ indexed by standard Young tableaux $T \in \mathcal{T}_{\square_N}$ in which the action of Jucys–Murphy elements is diagonal, with the eigenvalue equal to the $u$-coordinate of the appropriate box:

$$Z_s\, e_T = u_s^T\, e_T.$$

Secondly, with the help of Littlewood–Richardson coefficients, we investigate the decomposition of the restricted representation $V^{\square_N}\big\downarrow^{\mathfrak{S}_{N^2}}_{\mathfrak{S}_{N^2-i-k}\times\mathfrak{S}_k\times\mathfrak{S}_i}$ into irreducible components □

## 5.4 Presence of multisurfers does not influence the water

We will show that the probability distribution of water in the single surfer scenario is very close to its multisurfer counterpart.

**Lemma 5.3.** *For any standard Young tableau $S$ with shape $\lambda := \operatorname{sh} S$ and $w$ boxes*

$$\frac{\mathbb{P}_N\Big\{T_N \in \mathcal{T}_{\square_N} : W'_N = S\Big\}}{\widetilde{\mathbb{P}}_N\left\{M_N \in \widetilde{\mathcal{T}}_{\square_N} : \widetilde{W}'_N = S\right\}} = \frac{\mathbb{P}_N\Big\{T_N \in \mathcal{T}_{\square_N} : \operatorname{sh} W'_N = \lambda\Big\}}{\widetilde{\mathbb{P}}_N\left\{M_N \in \widetilde{\mathcal{T}}_{\square_N} : \operatorname{sh} \widetilde{W}'_N = \lambda\right\}} = 1 + O\left(\frac{1}{\sqrt{N}}\right). \tag{5.6}$$

*Proof.* The first equality follows from the observation that the conditional probability distribution of $W'_N$ under the event that $\operatorname{sh} W'_N = \lambda$ is the uniform measure on $\mathcal{T}_\lambda$ and the analogous result for $\widetilde{W}'_N$.

Since $\widetilde{\mathbb{P}}_N(\cdot) = \mathbb{P}_N(\cdot|\widetilde{\mathcal{T}}_{\square_N})$ is a conditional probability, Bayes rule can be applied and the middle part of (5.6) is equal to

$$\frac{\mathbb{P}_N\left(\widetilde{\mathcal{T}}_{\square_N}\right)}{\mathbb{P}_N\left(\widetilde{\mathcal{T}}_{\square_N}\middle|\operatorname{sh} W'_N = \lambda\right)}. \tag{5.7}$$

We will show the following estimate for the denominator:

$$\mathbb{P}_N\left(\widetilde{\mathcal{T}}_{\square_N}\middle|\operatorname{sh} W'_N = \lambda\right) = \frac{1 + O\left(\frac{1}{\sqrt{N}}\right)}{k!}. \tag{5.8}$$

Equation (5.3) shows that under condition $\operatorname{sh} W'_N = \lambda$ the conditional probability distribution of $U := J^{m+1-k}(M_N) \setminus \{1, \dots, w\}$ coincides with the uniformly random tableau with the skew shape $\square_N \setminus \lambda$. The conditional probability (5.8) is thus equal to the probability that the smallest $k$ boxes of $U$ (or, equivalently, the smallest $k$ boxes of the rectified tableau $V := \operatorname{rect} U$) are ordered from north-west to south-east *á la Pieri*. We denote by $\mu$ the shape of $V$. It follows that

$$\mathbb{P}_N \left( \widetilde{\mathcal{T}}_{\square_N} \middle| \operatorname{sh} W'_N = \lambda \right) = \sum_{\mu \in \mathbb{Y}_{N^2-w} \text{ such that } \mu \subset \square_N} \mathbb{P}_N \left\{ T_N \in \mathcal{T}_{\square_N} : V = \mu \right\} \cdot P_\mu, \tag{5.9}$$

where $P_\mu$ denotes the probability that $V$ (viewed as a uniformly random tableau sampled from $\mathcal{T}_\mu$) has entries $1, \dots, k$ ordered from north-west to south-east or, equivalently, the shape of the restricted tableau $\operatorname{sh}\left(V|_{1,\dots,k}\right) = (k)$ is the one-row Young diagram with $k$ boxes.

The link between standard Young tableaux and combinatorics of irreducible representations of the symmetric groups implies that

$$P_\mu = \frac{\text{multiplicity of } V^{(k)} \text{ in } \left( V^\mu \big\downarrow^{\mathfrak{S}_{N^2-w}}_{\mathfrak{S}_k} \right)}{\dim V^\mu} = \frac{\left\langle \chi^\mu \big\downarrow^{\mathfrak{S}_{N^2-w}}_{\mathfrak{S}_k}, \chi^{(k)} \right\rangle}{\dim V^\mu} = \frac{1}{k!} \left( 1 + \sum_{\pi \in \mathfrak{S}_k \text{ and } \pi \neq \mathrm{id}} \frac{\chi^\mu(\pi)}{\dim V^\mu} \right), \tag{5.10}$$

where the second equality follows from the orthogonality of irreducible characters.

Let $C > 0$ be fixed. We say that a Young diagram $\mu$ is $C$-balanced if both its number of rows and its number of columns are $\leq C\sqrt{|\lambda|}$. Each tableau $\mu$ which contributes to (5.9) is $C$-balanced for $C = \frac{N}{\sqrt{t_1 N^2}} = \frac{1}{\sqrt{t_1}}$. The result of Feráy and Śniady [1, Theorem 1] gives a good upper bound for the character ratios on the right-hand side of (5.10) for $\mu$ in the class of $C$-balanced diagrams which implies that

$$P_\mu = \frac{1 + O\left(\frac{1}{\sqrt{N}}\right)}{k!}.$$

The latter estimate combined with (5.9) implies that (5.8) indeed holds true.

An analogous (but simpler) reasoning shows that the numerator from (5.7) fulfills

$$\mathbb{P}_N \left( \widetilde{\mathcal{T}}_{\square_N} \right) = \frac{1 + O\left(\frac{1}{\sqrt{N}}\right)}{k!}. \tag{5.11}$$

The estimates (5.8) and (5.11) for the terms in (5.7) complete the proof. □

## 5.5 Single- and multisurfer scenario on the same water

**In Sections 5.1 and 5.2 we considered two random tableaux: $T'_N$ and $M'_N$ defined on two different probability spaces. Our goal is to define variants of these two random tableaux $\mathbf{T}'_N$ and $\mathbf{M}'_N$ on *the same* probability space in such a way that the corresponding configurations of water would coincide: $\mathbf{W}'_N := W'_N = \widetilde{W}'_N$.**

In order to achieve this goal we start by randomly sampling the tableau $\mathbf{W}'_N$ (=*water*) with the probability distribution of $W'_N$ from Section 5.1. Once $\mathbf{W}'_N$ is selected we randomly choose the tableaux $\mathbf{T}'_N$ (=*single surfer*) and the Pieri tableau $\mathbf{M}'_N$ (=*multisurfers*) according to the conditional probability distributions (conditioned by choosing $\mathbf{W}'_N$)

$$\mathbb{P}(\mathbf{T}'_N = S) := \mathbb{P}_N \left( T'_N = S | W'_N = \mathbf{W}'_N \right),$$
$$\mathbb{P}(\mathbf{M}'_N = S) := \widetilde{\mathbb{P}}_N \left( M'_N = S | \widetilde{W}'_N = \mathbf{W}'_N \right)$$

for an arbitrary standard tableau $S$.

The probability distribution of $\mathbf{T}'_N$ coincides with the distribution of $T'_N$ from Section 5.1. On the other hand Lemma 5.3 shows that the total variation distance between the distribution of $\mathbf{M}'_N$ and the distribution of $M'_N$ from Section 5.2 is small, of order $O\left(\frac{1}{\sqrt{N}}\right)$.

## 5.6 Ghosts of multisurfers do not overtake the surfer from right to left

Let us fix a pair of tableaux $\mathbf{T}'_N$ and $\mathbf{M}'_N$ given by the above construction. For any $0 \leq q \leq w$ we will iteratively apply jeu de taquin $q$ times to both tableaux and compare the outcomes. (Informally speaking, with the notations of Section 5.3, this means that jeu de taquin was applied $i = \lfloor t_1 N^2 \rfloor + q$ times to the tableau $T_N$.)

A simple inductive argument shows that after $q$ steps the configurations of water in $j^q(\mathbf{T}'_N)$ and $j^q(\mathbf{M}'_N)$ coincide:

$$j^q(\mathbf{T}'_N) \setminus \{w+1\} = j^q(\mathbf{M}'_N) \setminus \{w+1, \dots, w+k\}.$$

**In other words, we are dealing with two alternative universes in which the history of water is the same, and differ only by the number of surfers on top. Since the dynamics of the surfer and the multisurfers depends only on the water, there is no interaction between the surfer and the multisurfers (they are like ghosts).**

We define $\widetilde{\psi}_q$ as the fraction of the multisurfers which are to the left of the surfer:

$$\widetilde{\psi}_q := \frac{1}{k} \max \left\{ p \in \{1, \dots, k\} : u_{w+p}^{j^q(\mathbf{M}'_N)} \leq u_{w+1}^{j^q(\mathbf{T}'_N)} \right\}.$$

This quantity is a modification of (5.4) in which the role of $j^i(M_N)$ is played by $j^q(\mathbf{M}'_N)$ evaluated at $u = \frac{1}{N} u_{w+1}^{j^q(\mathbf{T}'_N)}$. The discussion from Section 5.5 shows that for the conclusion of Proposition 5.1 remains valid also for this modification. It follows that both at time $t_1$ as well as at time $t_2$ we have

$$\left|\psi(X_{t_1}) - \widetilde{\psi}_0\right| < \varepsilon/2, \tag{5.12}$$

$$\left|\psi(X_{t_2}) - \widetilde{\psi}_{\lfloor t_2 N^2 \rfloor - \lfloor t_1 N^2 \rfloor}\right| < \varepsilon/2. \tag{5.13}$$

hold true, except for an event of negligible probability.

By comparing the action of jeu de taquin on $j^q(\mathbf{M}'_N)$ with its action on $j^q(\mathbf{T}'_N)$ it follows that the sequence $(\widetilde{\psi}_q)$ is weakly decreasing:

$$\widetilde{\psi}_0 \geq \cdots \geq \widetilde{\psi}_w; \tag{5.14}$$

in other words, **if we compare the *relative* positions of the surfer and the ghosts of the multisurfers in the alternative universe, the multisurfers can only move from the left of the surfer to the right, but not in the opposite direction**.

By combining (5.12), (5.13) and (5.14) it follows that (5.1) indeed holds true, as we claimed. This concludes the proof of Proposition 4.1.

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