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Basis of totally primitive elements of **WQSym**

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Abstract. By Foissy's work, the bidendriform structure of the Word Quasisymmetric Functions Hopf algebra (WQSym) implies that it is isomorphic to its dual. However, the only known explicit isomorphism does not respect the bidendriform structure. This structure is entirely determined by so-called totally primitive elements (elements such that the two half-coproducts are 0). In this paper, we construct a basis indexed by a new combinatorial family called biplane forests in bijection with packed words. In this basis, primitive elements are indexed by biplane trees and totally primitive elements by a certain subset of trees. Thus we obtain the first explicit basis for the totally primitive elements of WQSym.

Résumé. Grâce aux travaux de Foissy, on sait que l'algèbre de Hopf WQSym est isomorphe à sa duale car bidendriforme. Cependant, le seul isomorphisme explicite connu ne respecte pas la structure bidendriforme. Cette structure est entièrement déterminée par les éléments totalement primitifs (annulés par les demi co-produits). Dans ce papier, nous construisons une base indexée par une nouvelle famille combinatoire appelée forêt biplanes, en bijection avec les mots tassés. Dans cette base, les éléments primitifs sont indexés par les arbres et les totalement primitifs par un certain sous-ensemble d'arbres. Ainsi, nous obtenons la première base explicite des éléments totalement primitifs dans WQSym.

Keywords: bidendriform Hopf algebras, word quasisymmetric functions, primitive elements, packed words

Introduction

The varied zoo of combinatorial Hopf algebras is much better understood when one considers the extra algebraic structures that each algebra can have. In this light, operad related theories are very useful. Some important examples are the Hopf algebras of non-commutative and quasi-symmetric functions related to the theory of free Lie algebras [5]. More recently, the Hopf algebra of binary trees was identified as the free dendriform algebra on one generator [7].

Closed related examples include the algebras **FQSym** of permutations of Malvenuto-Reutenauer [8] and the Hopf algebra **WQSym** of surjections or, equivalently, ordered

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set partitions [6]. These two can be seen as noncommutative versions of the algebra of quasi-symmetric functions. Though the first one is trivially self-dual, it is only by a deep theorem of Foissy [4] that one can show that the second one is too. In particular, until Vargas's work [11], no concrete isomorphism was known.

The first study of **WQSym** structure is due to Bergeron-Zabrocki [1]. They showed that it is free and co-free. Independently, Novelli-Thibon endowed it with a bidendriform bialgebra structure [9]. Recall that a dendriform algebra is an abstraction of a shuffle algebra where the product is split in two half-products. If the coproduct is also split, and certain compatibilities hold, one gets the notion of bidendriform bialgebra [4].

Building on the work of Chapoton and Ronco [10, 2], Foissy [4] showed that the structure of a bidendriform bialgebra is very rigid. In particular, he defined a specific subspace called the space of totally primitive elements, and showed that it characterizes the whole structure. This does not only re-prove the freeness and co-freeness, as well as the freeness of the primitive lie algebra, but also shows that the structure of a bidendriform bialgebra depends only on its Hilbert series (the series of dimensions of its homogeneous components). In particular, any such algebra is isomorphic to its dual. However, Foissy's isomorphism is not fully explicit and depends on a choice of a basis of the totally primitive elements. To this end, one needs an explicit basis of the totally primitive elements. Foissy described such a construction for **FQSym** [3]. Our long term goal is to effectively apply Foissy's construction to **WQSym** in order to build an explicit isomorphism respecting the bidendriform structure. With this objective in mind, we provide an explicit basis called the totally primitive elements using a bijection with certain families of trees.

We begin with a background section presenting two rigidity structure theorems that prove, among other things, the self-duality of any bidendriform bialgebra (Theorems 1.2 and 1.3). We then define the notion of packed word as well as the specific basis of **WQSym**, which will be the starting point of our combinatorial analysis ((1.7), (1.8), and (1.9)).

Section 2 is devoted to the combinatorial construction of biplane forests (Definition 2.19) which are our first key ingredient. They record a recursive decomposition of packed words according to their global descents (Lemma 2.4) and positions of the maximum letter (Lemma 2.11). We show that the cardinalities of some specific sets of biplane trees match the dimensions of primitive and totally primitive elements (Theorem 2.20).

Finally in Section 3 we construct a new basis of **WQSym** which contains a basis for the primitive and totally primitve elements (see Theorem 3.7). To do so we decompose the space of totally primitive elements as a certain direct sum which matchs the combinatorial decomposition of packed words (Lemma 3.4).

1 Background

1.1 Cartier-Milnor-Moore theorems for Bidendriform bialgebras

A bialgebra is a vector space over a field K, endowed with an unitary associative product $\cdot$ and a counitary coassociative coproduct Δ satisfying a compatibility relation called the Hopf relation $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$. In this paper all bialgebras are assumed to be graded and connected (*i.e.* the homogeneous component of degree 0 is K). They are therefore Hopf algebras, as the existence of the antipode is implied.

We now recall the elements of the definition of bidendriform bialgebras which are useful for the comprehension of this paper. We refer to [4] for the full list of axioms.

First of all, a *dendriform algebra* A is a K-vector space, endowed with two binary bilinear operations $\prec$, $\succ$ satisfying the following axioms, for all $a, b, c \in A$:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c), \tag{1.1}$$

$$(a \succ b) \prec c = a \succ (b \prec c), \tag{1.2}$$

$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c). \tag{1.3}$$

Adding together (1.1), (1.2), and (1.3) shows that the product $a \cdot b := a \prec b + a \succ b$ is associative. Adding a subspace of scalars, this defines a unitary algebra structure on $K \oplus A$. In this paper, all the dendriform algebras are graded and have null 0-degree component so that the associated algebra is connected.

Dualizing, one gets a notion of *co-dendriform co-algebra* which is a K-vector space with two binary co-operations (*i.e.*, linear maps $A \to A \otimes A$) denoted by $\Delta_\prec$, $\Delta_\succ$ satisfying the dual axioms of (1.1), (1.2), and (1.3). The sum of the two half coproducts $\tilde{\Delta}(a) := \Delta_\prec(a) + \Delta_\succ(a)$ is a reduced coassociative coproduct. On $K \oplus A$, setting $\Delta(a) := 1 \otimes a + a \otimes 1 + \tilde{\Delta}(a)$ defines a co-associative and co-unitary coproduct.

A *bidendriform bialgebra* is a K-vector space which is both a dendriform algebra and a co-dendriform co-algebra satisfying a set of four relations [4] relating respectively $\prec$ and $\succ$ with $\Delta_\prec$, $\Delta_\succ$. Adding those four relations shows that $\cdot$ and Δ as defined above defines a proper bi-algebra.

We recall here the relevant results of Foissy [4] on the rigidness of bidendriform bialgebras based on the works of Chapoton and Ronco [10, 2].

Let A be a bidendriform bialgebra. We define $\mathrm{Prim}(A) := \mathrm{Ker}(\tilde{\Delta})$ as the set of *primitive* elements of A. We also denote by $\mathcal{A}(z)$ and $\mathcal{P}(z)$ the Hilbert series of A and $\mathrm{Prim}(A)$ defined as $\mathcal{A}(z) := \sum_{n=1}^{+\infty} \dim(A_n) z^n$ and $\mathcal{P}(z) := \sum_{n=1}^{+\infty} \dim(\mathrm{Prim}(A_n)) z^n$. The present work is based on two analogues of the Cartier-Milnor-Moore theorems [4] which we present now. The first one is extracted from the proof of [3, Proposition 6]:

Proposition 1.1. *Let A be a bidendriform bialgebra and let $p_1 \ldots p_n \in \mathrm{Prim}(A)$. Then the map*

$$p_1 \otimes p_2 \otimes \ldots \otimes p_n \mapsto p_1 \prec (p_2 \prec (\ldots \prec p_n) \ldots). \tag{1.4}$$

is an isomorphism of co-algebras from $T^+(\mathrm{Prim}(A))$ *(the non trivial part of the tensor algebra with deconcatenation as coproduct) to* A*. As a consequence, taking a basis* $(p_i)_{i\in I}$ *of* $\mathrm{Prim}(A)$*, the family* $(p_{w_1} \prec (p_{w_2} \prec (\cdots \prec p_{w_n})\dots))_w$ *where* $w = w_1 \dots w_n$ *is a non empty word on* I *defines a basis of* A*. This implies the equality of Hilbert series* $\mathcal{A} = \mathcal{P}/(1-\mathcal{P})$*.*

One can further analyze $\mathrm{Prim}(A)$ using the so-called *totally primitive* elements of A defined as $\mathrm{TPrim}(A) = \mathrm{Ker}(\Delta_\prec) \cap \mathrm{Ker}(\Delta_\succ)$ and $\mathcal{T}(z) = \sum_{n=1}^{+\infty} \dim(\mathrm{TPrim}(A_n))z^n$. Recall that a brace algebra is a K-vector space A together with an n-multilinear operation denoted as $\langle \dots \rangle$ for all $n \geq 2$ which satisfies certain relations (see [10] for details).

Theorem 1.2 ([3, Theorem 5])**.** *Let* A *be a bidendriform bialgebra. Then* $\mathrm{Prim}(A)$ *is freely generated as a brace algebra by* $\mathrm{TPrim}(A)$ *with brackets given by*

$$\langle p_1, \dots, p_{n-1}; p_n \rangle := \sum_{i=0}^{n-1} (-1)^{n-1-i} \\ (p_1 \prec (p_2 \prec (\cdots \prec p_i)\cdots)) \succ p_n \prec ((\cdots (p_{i+1} \succ p_{i+2}) \succ \cdots) \succ p_{n-1}).$$

A basis of $\mathrm{Prim}(A)$ is described by ordered trees that are decorated with elements of $\mathrm{TPrim}(A)$ where p_n is the root and $p_1, \dots, p_{n-1}$ are the children (see [10, 2, 3]). This is reflected on their Hilbert series as [4, Corollary 37]: $\mathcal{T} = \mathcal{A}/(1+\mathcal{A})^2$ or equivalently $\mathcal{P} = \mathcal{T}(1+\mathcal{A})$.

Using Proposition 1.1 and Theorem 1.2 together with a dimension argument, one can show the following theorem:

Theorem 1.3 ([3, Theorem 2])**.** *Let* A *be a bidendriform bialgebra. Then* A *is freely generated as a dendriform algebra by* $\mathrm{TPrim}(A)$*.*

1.2 The Hopf algebra of word-quasisymmetric functions WQSym

The algebra **WQSym** is a Hopf algebra whose bases are indexed by ordered set partitions or equivalently surjections or packed words. In this paper, we use the latter which we define now.

Definition 1.4. A word over the alphabet $\mathbb{N}_{>0}$ is *packed* if all the letters from 1 to its maximum m appears at least once. By convention, the empty word ϵ is packed. For $n \in \mathbb{N}$, we denote by $\mathbf{PW}_n$ the set of all packed words of length (also called size) n and $\mathbf{PW} = \bigsqcup_{n\in\mathbb{N}} \mathbf{PW}_n$ the set of all packed words.

Definition 1.5. The packed word $u := pack(w)$ associated with a word over the alphabet $\mathbb{N}_{>0}$ is obtained by the following process: if $b_1 < b_2 < \cdots < b_r$ are the letters occurring in w, then u is the image of w by the homomorphism $b_i \mapsto i$.

A word u is packed if and only if $pack(u) = u$.

Example 1.6. The word 4152142 is not packed because the letter 3 does not appear while the maximum letter is $5 > 3$. Meanwhile $pack(4152142) = 3142132$ is a packed word. Here are all packed words of size 1, 2 and 3 in lexicographic order:

1, 11 12 21, 111 112 121 122 123 132 211 212 213 221 231 312 321

We will use the following notations and operations on words over the alphabet $\mathbb{N}_{>0}$: First, $\max(w)$ is the maximum letter of the word w with the convention that $\max(\epsilon) = 0$. Then $|w|$ is the length (or size) of the word w. The concatenation of the two words u and v is denoted as $u \cdot v$. Moreover, u/v (resp. $u\backslash v$) is the left-shifted (resp. right-shifted) concatenation of the two words where all the letters of the left (resp. right) word are shifted by the maximum of the right (resp. left) word: $1121/3112 = 44543112$ and $1121\backslash 3112 = 11215334$.

Finally $u \sqcup\!\sqcup v$ is the shuffle product of the two words. It is recursively defined as $u \sqcup\!\sqcup \epsilon = \epsilon \sqcup\!\sqcup u = u$ and

$$ua \sqcup\!\sqcup vb = (u \sqcup\!\sqcup vb) \cdot a + (ua \sqcup\!\sqcup v) \cdot b \tag{1.5}$$

where u and v are words and a and b are letters. Analogously to the shifted concatenation, one can define the right shifted-shuffle $u \,\overline{\sqcup\!\sqcup}\, v$ where all the letters of the right word v are shifted by the maximum of the left word u.

Example 1.7. $12 \,\overline{\sqcup\!\sqcup}\, 11 = 12 \sqcup\!\sqcup 33 = 1233 + 1323 + 1332 + 3123 + 3132 + 3312$.

Bidendriform bialgebra structure Novelli-Thibon [9] proved that **WQSym** is a bidendriform bialgebra. Their products and coproducts involve overlapping-shuffle. However it will be easier for us to chose, among the various bases known in the literature [6, 1, 9, 11] a basis where the shuffle are non-overlapping. Therefore, we take the dual basis denoted $(\mathbb{R}_w)_{w\in\mathbf{PW}}$ of [1, Equation 23], using the classical bijection between ordered partitions and packed words and redefine the bidendriform structure. The Hopf algebra product and reduced coproduct are respectively recovered as the sum of the half products (see (1.7)) and half coproducts (see (1.8) and (1.9)).

The recursive definition of the shuffle product (see (1.5)) contains two summands. We define them respectively as $\prec$ and $\succ$:

$$ua \prec vb := (u \sqcup\!\sqcup vb) \cdot a, \qquad \text{and} \qquad ua \succ vb := (ua \sqcup\!\sqcup v) \cdot b. \tag{1.6}$$

We define $\prec$, $\succ$, $\Delta_\prec$ and $\Delta_\succ$ on $(\mathbf{WQSym})_+ = Vect(\mathbb{R}_u \mid u \in \mathbf{PW}_n, n \geq 1)$ in the following way: for all $u = u_1 \cdots u_n \in \mathbf{PW}_{n\geq 1}$ and $v \in \mathbf{PW}_{m\geq 1}$,

$$\mathbb{R}_u \prec \mathbb{R}_v := \sum_{w \in u \prec v} \mathbb{R}_w, \qquad \text{and} \qquad \mathbb{R}_u \succ \mathbb{R}_v := \sum_{w \in u \succ v} \mathbb{R}_w. \tag{1.7}$$

$$\Delta_{\prec}(\mathbb{R}_u) := \sum_{\substack{i=k \\ \{u_1,\dots,u_i\}\cap\{u_{i+1},\dots,u_n\}=\emptyset \\ u_k=\max(u)}}^{n-1} \mathbb{R}_{pack(u_1\cdots u_i)} \otimes \mathbb{R}_{pack(u_{i+1}\cdots u_n)}, \tag{1.8}$$

$$\Delta_{\succ}(\mathbb{R}_u) := \sum_{\substack{i=1 \\ \{u_1,\dots,u_i\}\cap\{u_{i+1},\dots,u_n\}=\emptyset \\ u_k=\max(u)}}^{k-1} \mathbb{R}_{pack(u_1\cdots u_i)} \otimes \mathbb{R}_{pack(u_{i+1}\cdots u_n)}. \tag{1.9}$$

Example 1.8.

$$\begin{aligned}
\mathbb{R}_{211} \prec \mathbb{R}_{12} &= \mathbb{R}_{21341} + \mathbb{R}_{23141} + \mathbb{R}_{23411} + \mathbb{R}_{32141} + \mathbb{R}_{32411} + \mathbb{R}_{34211},\\
\mathbb{R}_{221} \succ \mathbb{R}_{12} &= \mathbb{R}_{21134} + \mathbb{R}_{21314} + \mathbb{R}_{23114} + \mathbb{R}_{32114},\\
\Delta_{\prec}(\mathbb{R}_{2125334}) &= \mathbb{R}_{2123} \otimes \mathbb{R}_{112} + \mathbb{R}_{212433} \otimes \mathbb{R}_{1},\\
\Delta_{\succ}(\mathbb{R}_{2125334}) &= \mathbb{R}_{212} \otimes \mathbb{R}_{3112}.
\end{aligned}$$

Theorem 1.9. *[9, Theorem 2.5]* $(\mathbf{WQSym})_+, \prec, \succ, \Delta_\prec, \Delta_\succ)$ *is a bidendriform bialgebra.*

From now on Prim($\mathbf{WQSym}$) and TPrim($\mathbf{WQSym}$) are respectively abbreviated to Prim and TPrim. Moreover, we denote homogeneous components using indices as in Prim_n. We give the first several values of the dimensions $a_n := \dim(\mathbf{WQSym}_n)$, $p_n := \dim(\mathrm{Prim}_n)$ and $t_n := \dim(\mathrm{TPrim}_n)$:

n	1	2	3	4	5	6	7	8	9	OEIS
a_n	1	3	13	75	541	4 683	47 293	545 835	7 087 261	A000670
p_n	1	2	8	48	368	3 376	35 824	430 512	5 773 936	A095989
t_n	1	1	4	28	240	2 384	26 832	337 168	4 680 272	

2 Decorated forests

In this section we will generalize the construction of [3]. We will start by defining some forests called biplane that are labeled by certain lists of integers. We will construct a bijection between these packed forests and packed words thanks to a decomposition of packed words through global descents and removal of maximums. The recursive structure of forests can then be understood as a chaining of operations generating the elements of $\mathbf{WQSym}$. This will allow us to construct a basis of TPrim by characterizing a subfamily of biplane trees.

2.1 Decompositions of packed words

Definition 2.1. A **global descent** of a packed word w is a position c such that all the letters before or at position c are greater than all letters after position c.

Example 2.2. The global descents of $w = 54664312$ are the positions 5 and 6. Indeed, all letters of 54664 are greater than the letters of 312 and this is also true for 546643 and 12.

Definition 2.3. A packed word w is **irreducible** if it has no global descent.

Lemma 2.4. *Each word w admits a unique factorization as $w = w_1/w_2/\dots/w_k$ such that w_i is irreducible for all i.*

Example 2.5. The global descent decomposition of 54664312 is 21331/1/12. The word $n \cdot n-1 \cdot \dots \cdot 1$ has 1/1/.../1 as global descent decomposition.

Definition 2.6. Fix $n \in \mathbb{N}$ and $w \in \mathbf{PW}_n$. We write $m := \max(w) + 1$. For any $p > 0$ and any subset $I \subseteq [1, \dots, n+p]$ of cardinality p, $\phi_I(w) = u_1 \dots u_{n+p}$ is the packed word of length $n+p$ obtained by inserting p occurences of the letter m in w so that they end up in positions $i \in I$. In other words $u_i = m$ if $i \in I$ and w is obtained from $\phi_I(w)$ by removing all occurrences of m.

Example 2.7. $\phi_{2,4,7}(1232) = 1424324$ and $\phi_{1,2,3}(\epsilon) = 111$.

Let $\mathbf{PW}_n^I$ denote the set of packed words of size n whose maximums are in positions $i \in I$. This way $\phi_I(w) \in \mathbf{PW}_{n+p}^I$. The following lemma is immediate.

Lemma 2.8. *Let $n \in \mathbb{N}$ and $p > 0$, for any $I \subseteq [1, \dots, n+p]$, ϕ_I is a bijection from $\mathbf{PW}_n$ to the $\mathbf{PW}_{n+p}^I$.*

Moreover, for any $W \in \mathbf{PW}_\ell$ where $\ell > 0$ there exists a unique pair (I, w) where $I \subseteq [1 \dots \ell]$ and w is packed, such that $W = \phi_I(w)$.

Definition 2.9. Let $u, v \in \mathbf{PW}$ with $v \neq \epsilon$. By Lemma 2.8, there is a unique pair (I, v') such that $v = \phi_I(v')$. We denote by I' the set obtained by adding $|u|$ to the elements of I. We define $u \blacktriangleright v = \phi_{I'}(u/v')$. In other words, we remove the maximum letter of the right word, perform a left shifted concatenation and reinsert the removed letters as new maximums.

Example 2.10. $2123 \blacktriangleright 322312 = \phi_{1+4,4+4}(43452212) = 4345622612$.

Lemma 2.11. *Let w be an irreducible packed word. There exists a unique factorization of the form $w = u \blacktriangleright \phi_I(v)$ which maximizes the size of u. In this factorization*

- *either $v = \epsilon$ and $I = [1, \dots, p]$ for some p;*
- *or the factorization $v = v_1/\dots/v_r$ of v into irreducibles satisfies the inequalities $i_1 \leq |v_1|$, and $(\sum |v_i| + |I|) + 1 - i_p \leq |v_r|$.*

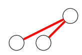
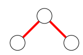
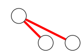
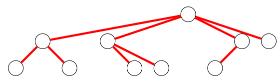
Example 2.12. Here are some decompositions according to Lemma 2.11:

$$21331 = 1 \blacktriangleright \phi_{2,3}(11) \qquad 1231 = \epsilon \blacktriangleright \phi_3(121)$$
$$1233 = 12 \blacktriangleright \phi_{1,2}(\epsilon) \qquad 111 = \epsilon \blacktriangleright \phi_{1,2,3}(\epsilon)$$
$$543462161 = (1/212) \blacktriangleright \phi_{1,4}(1/11) = (3212) \blacktriangleright \phi_{1,4}(211)$$

2.2 Forests from decomposed packed words

We now apply recursively the decomposition of the former section to construct a bijection between packed words and a certain kind of trees that we now define.

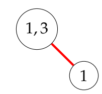
Definition 2.13. An unlabeled **biplane tree** is an ordered tree (sometimes also called a planar) whose children are organized in a pair of two (possibly empty) ordered forests, which we call the left and right forests.

Example 2.14. The biplan trees ,  and  are different. Indeed in the first case, the left forest contains two trees and the right forest is empty, in the second case both forests contain exactly one tree while in the third case we have the opposite of the first case. Here is an example of a bigger biplane tree where the root has two trees in both left and right forests .

Remark 2.15. These biplan trees are counted by the sequence A006013 in OEIS.

In our construction we will deal with labeled biplane trees where the labels are sorted lists of positive integer. For a labeled biplane tree, we denote by $\mathrm{Node}(x, f_\ell, f_r)$ the tree whose root is labeled by x and whose left (resp. right) forest is given by f_ℓ (resp. f_r). We also denote by $[t_1, \dots t_k]$ a forest of k trees.

Example 2.16. $\mathrm{Node}((1),\ [], []) =$ 

, and $\mathrm{Node}((1,3),\ [], [\mathrm{Node}((1),\ [], [])]) =$ 

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We now apply recursively the decompositions of Lemmas 2.4 and 2.11 to get an algorithm which takes a packed word and returns a biplane forest where nodes are decorated by lists of integers:

Definition 2.17. The forest $F(w)$ (resp. tree $T(w)$) associated to a packed word (resp. a non empty irreducible packed word) w are defined in a mutual recursive way as follows:

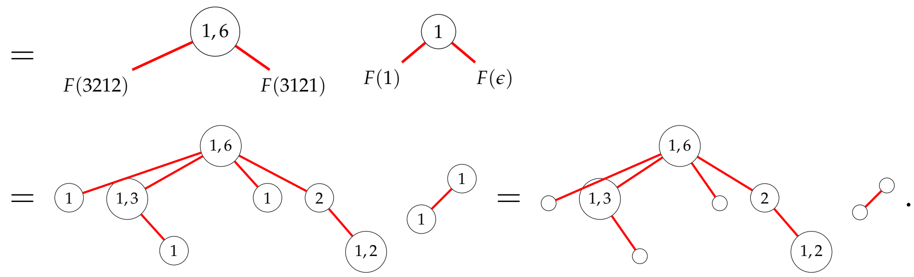
- $F(\epsilon) = []$ (empty forest),
- for any packed word w, let $w_1/w_2/\dots/w_k$ be the global descent decomposition of w, then $F(w) := [T(w_1), T(w_2), \dots, T(w_k)]$.
- for any non empty irreducible packed word w, define $T(w) := \mathrm{Node}(I, F(u), F(v))$ where $w = u \blacktriangleright \phi_I(v)$ and u is of maximal length.

Example 2.18. Let $w = 876795343912$, the decomposition of Lemma 2.4 gives $w = w_1/w_2$ with $w_1 = 6545731217$ and $w_2 = 12$. Now, we decompose w_1 and w_2 using Lemma 2.11 as

$$w_1 = 3212 \blacktriangleright \phi_{1,6}(3121) = (1/212) \blacktriangleright \phi_{1,6}(1/121), \quad \text{and} \quad w_2 = 1 \blacktriangleright \phi_1(\epsilon).$$

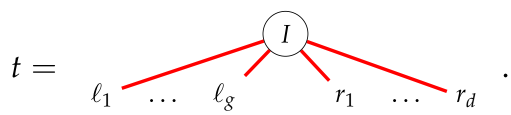
It gives the following forest:

$$F(876795343912) = [T(6545731217),\ T(12)]$$



As we can see in this example, in order to simplify the notation, we do not write the label when it is the list 1. We now characterize the trees obtained this way:

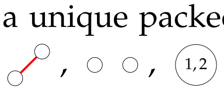
Definition 2.19. Let t be a labeled biplane tree. We write $t = \mathrm{Node}(I, f_\ell, f_r)$ where $I = [i_1, \ldots, i_p]$, $f_\ell = [\ell_1, \ldots, \ell_g]$ and $f_r = [r_1, \ldots, r_d]$, which is depicted as follows:

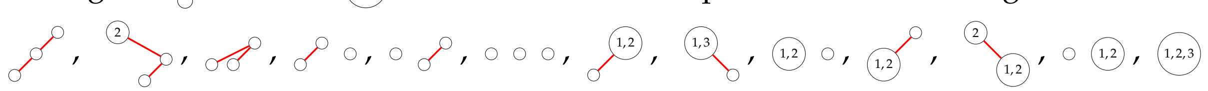


The *weight* of t is recursively defined by $\omega(t) = p + \sum_{i=0}^{g} \omega(\ell_i) + \sum_{j=0}^{d} \omega(r_j)$. In particular, if t is a leaf then $\omega(t) = p$. The *right-weight* of t is defined by $\omega_r(t) = p + \sum_{j=0}^{d} \omega(r_j)$. We say that t is a **packed tree** if it satisfies:

- If there are no right children ($d = 0$) then $i_k = k$ for all k.
- Otherwise:
$$\begin{cases} 1 \leq & i_1 & \leq \omega(r_1), \\ 1 \leq & \omega_r(t) + 1 - i_p & \leq \omega(r_d). \end{cases} \tag{2.1}$$
- f_ℓ and f_r are packed forests (*i.e.* lists of packed trees).

For the rest of the article we denote the set of packed forests of weight n by $\mathfrak{F}_n$ and the set of packed trees of weight n by $\mathfrak{T}_n$. We also denote by $\mathfrak{F}_n^I$ the set of packed forests that satisfy (2.1) for $I = (i_1, \ldots, i_p)$ with $0 < i_1 < \ldots < i_p$. Finally we denote the set of packed trees of weight n with an empty left forest by $\mathfrak{P}_n$.

There is a unique packed forest of weight 1, namely , here are the packed forests of weight 2: . We show below the packed forests of weight 3:



The following theorem is a generalisation of the construction of [3] for **FQSym** and permutations to **WQSym** and packed words.

Theorem 2.20. *For all $n \in \mathbb{N}$ we have the three following equalities :*

$$\dim(\mathbf{WQSym}_n) = \#\mathfrak{F}_n \quad \text{and} \quad \dim(\mathrm{Prim}_n) = \#\mathfrak{T}_n \quad \text{and} \quad \dim(\mathrm{TPrim}_n) = \#\mathfrak{P}_n$$

Proof. The construction of Definition 2.17 defines a bijection from packed words to packed forests which restrict to a bijection from irreducible packed words to packed trees. Thanks to Proposition 1.1 this proves the first two equalities. Recall that a basis of primitive elements is given by Theorem 1.2 as ordered trees decorated by totally primitive elements. If we consider the label I together with the right forest of each node as decoration, we get that packed trees are in bijection with planar trees decorated by I and an element of $\mathfrak{F}^I$ that is an element of $\mathfrak{P}$. □

3 A basis for totally primitive elements

In this section we construct a basis of primitive and totally primitive elements of **WQSym**. Thanks to Theorem 2.20 we now have the combinatorial objects to index those basis. To have the linear independency, we need to show that the decomposition through maximum is compatible with the algebraic structure.

3.1 Decomposition through maximums and totally primitive elements

Definition 3.1. Let $I = (i_1, \ldots, i_p)$ with $0 < i_1 < \ldots < i_p$. We define the linear map $\Phi_I : \mathbf{WQSym} \to \mathbf{WQSym}$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\Phi_I(\mathbb{R}_w) := \begin{cases} \mathbb{R}_{\phi_I(w)} & \text{if } i_p \le n+p, \\ 0 & \text{if } i_p > n+p\,. \end{cases} \tag{3.1}$$

Definition 3.2. Let $I = (i_1, \ldots, i_p)$ with $0 < i_1 < \ldots < i_p$. We define the projector $\tau_I : \mathbf{WQSym} \to \mathbf{WQSym}$ as follows: for all $n \in \mathbb{N}$ and $w = w_1 \cdot w_2 \cdots w_n \in \mathbf{PW}_n$,

$$\tau_I(\mathbb{R}_w) := \begin{cases} \mathbb{R}_w & \text{if } w_i = \max(w) \text{ if and only if } i \in I, \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

These are orthogonal projectors in the sense that $\tau_I^2 = \tau_I$ and $\tau_I \circ \tau_J = 0$ ($I \neq J$).

Lemma 3.3. *For any I, we have $Im(\Phi_I) = Im(\tau_I)$ where $Im(f)$ denotes the image of f.*

Lemma 3.4. *For any I, the projection by τ_I of a totally primitive element is still a totally primitive element, so that $\tau_I(\mathrm{TPrim}) = Im(\tau_I) \bigcap \mathrm{TPrim}$. Moreover,*

$$\mathrm{TPrim} = \bigoplus_I Im(\tau_I) \cap \mathrm{TPrim}\,. \tag{3.3}$$

Proof. Let w a packed word. We have $\Delta_\prec(\tau_I(\mathbb{R}_w)) = (\tau_I \otimes Id) \circ \Delta_\prec(\mathbb{R}_w)$ by definition of τ and $\Delta_\prec$. By linearity, for all $p \in \mathrm{TPrim}$, we have $\Delta_\prec(\tau_I(p)) = (\tau_I \otimes Id) \circ \Delta_\prec(p) = 0$. The same argument works on the right so that $\tau_I(p) \in \mathrm{TPrim}$. Morevover τ_I are orthogonal projectors so $\mathrm{TPrim} = \bigoplus_I \tau_I(\mathrm{TPrim}) = \bigoplus_I Im(\tau_I) \cap \mathrm{TPrim}$. □

3.2 The new basis $\mathbb{P}$

Definition 3.5. Let $t_1, \ldots, t_k \in \mathfrak{T}$ and $f_l = [\ell_1, \ldots, \ell_g], f_r \in \mathfrak{F}$,

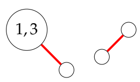
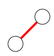
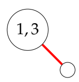
$$\mathbb{P}_{\circ} := \mathbb{R}_1, \tag{3.4}$$

$$\mathbb{P}_{t_1,\ldots,t_k} := \mathbb{P}_{t_k} \prec (\mathbb{P}_{t_{k-1}} \prec (\ldots \prec \mathbb{P}_{t_1})\ldots), \tag{3.5}$$

$$\mathbb{P}_{\mathrm{Node}(I,[],f_r)} := \Phi_I(\mathbb{P}_{f_r}), \tag{3.6}$$

$$\mathbb{P}_{\mathrm{Node}(I,f_l=[\ell_1,\ldots,\ell_g],f_r)} := \langle \mathbb{P}_{l_1}, \mathbb{P}_{l_2}, \ldots, \mathbb{P}_{l_g}; \Phi_I(\mathbb{P}_{f_r})\rangle. \tag{3.7}$$

Example 3.6.

$\mathbb{P}$  $= \mathbb{P}$  $\prec \mathbb{P}$  $= (\mathbb{P}_{\circ} \prec \mathbb{P}_{\circ} - \mathbb{P}_{\circ} \succ \mathbb{P}_{\circ}) \prec \Phi_{1,3}(\mathbb{P}_{\circ})$

$$= \mathbb{R}_{14342} + \mathbb{R}_{41342} + \mathbb{R}_{43142} + \mathbb{R}_{43412} - \mathbb{R}_{24341} - \mathbb{R}_{42341} - \mathbb{R}_{43241} - \mathbb{R}_{43421}$$

Theorem 3.7. *For all $n \in \mathbb{N}_{>0}$*

1. $(\mathbb{P}_f)_{f\in\mathfrak{F}_n}$ *is a basis of* $\mathbf{WQSym}_n$,
2. $(\mathbb{P}_t)_{t\in\mathfrak{T}_n}$ *is a basis of* Prim_n,
3. $(\mathbb{P}_t)_{t\in\mathfrak{P}_n}$ *is a basis of* TPrim_n.

Proof. As $\dim(\mathbf{WQSym}_1) = \dim(\mathrm{Prim}_1) = \dim(\mathrm{TPrim}_1) = 1$ the base case is trivial. We argue in a mutually recursive way: by Proposition 1.1, Item 2 up to degree n implies Item 1 up to degree n. Similarly, Theorem 1.2 shows that Item 3 up to degree n implies Item 2 up to degree n. By induction it is sufficient to show that Items 1 and 2 up to degree $n-1$ implies Item 3 for n.

For all $k \in \mathbb{N}$, let π_k be the projector on the homogeneous component of degree k of **WQSym**. We define $\pi_{<k} := \sum_{i=0}^{k-1} \pi_i$. Fix I of length p. In the coproduct $\Delta_\prec(\Phi_I(\mathbb{R}_u))$ all the maximums must be in the left tensor factor, which therefore must be at least of degree i_p. By linearity, this can be used to prove that $\Delta_\prec(\Phi_I(\mathbb{P}_f)) = 0$. A similar reasoning applies to $\Delta_\succ(\Phi_I(x))$, so that thanks to the conditions of (2.1), $\mathbb{P}_t$ with $t \in \mathfrak{P}_n$ is totally primitive.

In other words, if we define $\mathrm{Prim}_n(i,j) := \mathrm{Ker}(\pi_{<i} \otimes \pi_{<j}) \circ \tilde{\Delta}$ we have proved that the image of the restriction to $\mathrm{Prim}_n(i_1, n+1-i_p)$ of Φ_I is included in $Im(\tau_I) \cap \mathrm{TPrim}_n$. By Proposition 1.1, $\{\mathbb{P}_f \mid f \in \mathfrak{F}^I_{n-p}\}$ is a basis of $\mathrm{Prim}_{n-p}(i_1, n+1-i_p)$. Since Φ_I is injective on $\mathbf{WQSym}_{n-p}$ then $\{\Phi_I(\mathbb{P}_f) \mid f \in \mathfrak{F}^I_{n-p}\}$ are linearly independent. Then by Lemma 3.4 $\{\mathbb{P}_t \mid t \in \mathfrak{P}_n\}$ are linearly independent. By Theorem 2.20 it is a basis of TPrim_n. □

Conclusion

Our next obvious step is to adapt the decomposition through maximum of Definitions 3.1 and 3.2 to the dual in order to get an explicit bidendriform isomorphism from **WQSym** to its dual. Another generalization which should be easy is to do the same for the Hopf algebra **PQSym** of parking functions. Indeed **PQSym** is a bidendriform bialgebra [9], and thus self-dual, but no explicit isomorphism with the dual is known.

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