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Order structure of shapes of predominant integral weights and cylindric Young diagrams

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Abstract. In this paper, we give a generalization of d-complete posets to "infinite" d-complete posets. Our infinite d-complete posets are realized as a subset of coroots on the Kac–Moody Lie algebra, and has two different order relations in general. One is the ordinary order on coroots. The other is called heap order. We give a sufficient condition for both orders to coincide. As an application, we give a multivariate hook formula for cylindric Young diagrams.

Résumé. Dans ce papier, nous donnons une généralisation de posets d-complet à posets d-complet "infini". De nos posets d-complets infinis se rendent compte comme un sous-ensemble de coroots sur l'algèbre du Kac–Moody Lie et ont deux relations de l'ordre différentes dans le général. On est l'ordre ordinaire sur coroots. L'autre est appelé l'ordre du tas. Nous donnons une condition suffisante pour les deux ordonne de coïncider. Comme une application, nous donnons une formule du crochet du multivariate pour cylindric Young diagrammes.

Keywords: d-complete posets, heap order

1 Introduction

Since the end of 1980's, d-complete posets, or minuscule elements equivalently, have been studied in combinatorics with the other regions in mathematics, [1, 8, 9, 4, 7] and so on. In these studies, the order structure of d-complete poset plays important rolls. In [1, 9, 4], d-complete posets are realized by certain subsets of real coroot systems for Kac–Moody Lie algebras. In such a situation, the partial order of a d-complete poset is defined two different ways: One is the ordinary coroot order (Section 2), and the other the heap order (see Section 3 for details). In [9], J. R. Stembridge proved the following theorem:

Theorem 1.1 (Stembridge [9])**.** *These two partial orders coincide with each other in a* d-*complete poset.*

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One purpose of the present paper is to give another proof of Stembridge's theorem (as Corollary 4.4).

Recently, a generalization of d-complete posets to "infinite d-complete posets" was studied in [6, 10]. In [4], the author defined the notion of *predominant* integral weights. Predominant integral weights are certain integral weights (see Section 3), which generalize minuscule elements in Weyl groups [1, 8]. Furthermore, *finite* predominant integral weights bijectively correspond to minuscule elements [4].

The main purpose of the present paper is to study the *infinite* predominant integral weights. Typical examples of infinite predominant integral weights appear if the underlying Dynkin diagram is one in the following list:

(A) affine Dynkin diagrams (see [6] for details):

$$A_n^{(1)}(n \geq 1),\ B_n^{(1)}(n \geq 3),\ C_n^{(1)}(n \geq 2),\ D_n^{(1)}(n \geq 4),\ E_6^{(1)},\ E_7^{(1)},\ D_n^{(2)}(n \geq 3),\ A_{2n-1}^{(2)}(n \geq 3).$$

(B) infinite Dynkin diagrams (see [2, 3] for details): $A_\infty, A_{+\infty}, B_\infty, C_\infty, D_\infty$.

The infinite predominant integral weights over affine Dynkin diagrams are determined in [6]. Predominant integral weights over the infinite Dynkin diagram A_∞ are closely related to combinatorics of plane partitions (e.g. MacMahon identity [5]).

In the present paper, we define the *shape* $\mathrm{D}(\lambda)^\vee$ of a predominant integral weight λ, which is a certain subset of positive real coroots. We consider two kinds of partial orders over the shape. One is the *ordinary order* of coroots (Section 2). The other is a certain order called the *heap order* (Section 3), which is a generalization of the notion defined by J. Stembridge in [9]. In the present paper, we study a relation between the ordinary order and the heap order, and give a sufficient condition for both orders to coincide (Corollary 4.2).

2 Preliminaries for root system and coroot system

Let $A = (a_{i,j})_{i,j \in I}$ be a (not necessarily symmetrizable) Cartan matrix of a Kac–Moody Lie algebra [2, 3]. We denote the set of real numbers by $\mathbb{R}$. Let $\mathfrak{h}$ be an $\mathbb{R}$ -vector space and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$ and $\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ the canonical bilinear form. We suppose the existence of linearly independent subsets $\Pi := \{ \alpha_i \mid i \in I \} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{ \alpha_i^\vee \mid i \in I \} \subset \mathfrak{h}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

The set of integral weights is denoted by P. For each $i \in I$, we define the *simple reflection* $s_i \in GL(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The group W generated by $\{ s_i \,|\, i \in I \}$ is called the *Weyl group*, which acts on $\mathfrak{h}$ by:

$$\langle w(\lambda), w(h) \rangle = \langle \lambda, h \rangle, \quad w \in W, \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

We define the *real root system* (resp. *real coroot system*) by $\Phi := W\Pi$ (resp. $\Phi^\vee := W\Pi^\vee$). The *dual* $\beta^\vee \in \Phi^\vee$ of a root $\beta \in \Phi$ is defined by

$$\beta^\vee = w(\alpha_i^\vee), \qquad (\beta = w(\alpha_i), w \in W, \alpha_i \in \Pi).$$

This is independent from choice of $w \in W$ and $\alpha_i \in \Pi$. In general, we have

$$(w(\beta))^\vee = w(\beta^\vee), \qquad (\beta \in \Phi, w \in W).$$

The map $\Phi \ni \beta \mapsto \beta^\vee \in \Phi^\vee$ is a bijection. For $R \subseteq \Phi$, we put $R^\vee := \{ \beta^\vee \,|\, \beta \in R \} \subseteq \Phi^\vee$. For each $\beta \in \Phi$, we define $s_\beta \in W$ by:

$$s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*, \quad \text{or, equivalently, by}$$

or, equivalently, by We denote:

$$Q^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee, \qquad \text{and} \qquad Q_+^\vee := \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee,$$

where $\mathbb{N}$ is the set of non-negative integers. We have $\Phi^\vee \subseteq Q^\vee$. Put $\Phi_+^\vee := \Phi^\vee \cap Q_+^\vee$, and $\Phi_-^\vee := \Phi^\vee \cap (-Q_+^\vee)$. We have $\Phi^\vee = \Phi_+^\vee \amalg \Phi_-^\vee$. For $\beta^\vee, \gamma^\vee \in \Phi^\vee$, we denote $\beta^\vee \leq \gamma^\vee$ if

$$\gamma^\vee - \beta^\vee \in Q_+^\vee.$$

This order is called the *ordinary order* of coroots.

For each $w \in W$, we define a set $\Phi(w)$ $(\subseteq \Phi_+)$ by:

$$\Phi(w) := \{ \gamma \in \Phi_+ \,|\, w^{-1}(\gamma) < 0 \}.$$

3 Pre-dominant Integral Weights, Shapes, and λ-Paths

Definition 3.1. An integral weight λ is *pre-dominant* if

$$\langle \lambda, \beta^\vee \rangle \geq -1, \quad \beta^\vee \in \Phi_+^\vee.$$

The set of pre-dominant integral weights is denoted by $P_{\geq -1}$.

Definition 3.2. For $\lambda \in P_{\geq -1}$, we define a set $\mathrm{D}(\lambda)^\vee$ by

$$\mathrm{D}(\lambda)^\vee := \{ \beta^\vee \in \Phi_+^\vee \,|\, \langle \lambda, \beta^\vee \rangle = -1 \}$$

We call the set $\mathrm{D}(\lambda)^\vee$ the *shape of* λ. A pre-dominant integral weight λ is said to be *finite* (resp. *infinite*) if $\#\mathrm{D}(\lambda)^\vee < \infty$ (resp. $\#\mathrm{D}(\lambda)^\vee = \infty$).

Set $\mathrm{D}(\lambda) := \left\{ \beta \in \Phi_+ \,\middle|\, \langle \lambda, \beta^\vee \rangle = -1 \right\}$ as in [4]. We note that $\left(\mathrm{D}(\lambda)\right)^\vee = \mathrm{D}(\lambda)^\vee$.

Definition 3.3. Let $\beta^\vee, \gamma^\vee \in \Phi^\vee$. We denote $\beta^\vee \lhd \gamma^\vee$ if $\beta^\vee < \gamma^\vee$ and $\langle \gamma, \beta^\vee \rangle \geq 1$.

Definition 3.4. Let $\lambda \in P_{\geq -1}$. The reflective and transitive closure of the restriction of the relation $\lhd$ to $\mathrm{D}(\lambda)^\vee$ is denoted by $\preceq$. Namely, for $\beta^\vee, \gamma^\vee \in \mathrm{D}(\lambda)^\vee$, we denote $\beta^\vee \preceq \gamma^\vee$ if there exists $\beta_1^\vee, \cdots, \beta_l^\vee \in \mathrm{D}(\lambda)^\vee$ such that

$$\beta^\vee \lhd \beta_1^\vee \lhd \cdots \lhd \beta_l^\vee \lhd \gamma^\vee,$$

or $\beta^\vee = \gamma^\vee$. The partial order $\preceq$ is called the *heap order over* $\mathrm{D}(\lambda)^\vee$.

Remark 3.5. For $\beta^\vee, \gamma^\vee \in \mathrm{D}(\lambda)^\vee$, if $\beta^\vee \preceq \gamma^\vee$, then $\beta^\vee \leq \gamma^\vee$. However, in general, the ordinary order does not coincide with the heap order over a shape $\mathrm{D}(\lambda)^\vee$. We see such examples in Examples 3.11 and 3.15 later.

Lemma 3.6 ([4])**.** *Let $\lambda \in P_{\geq -1}$ and $\beta \in \mathrm{D}(\lambda)$. Then we have:*

1. $s_\beta(\lambda) \in P_{\geq -1}$.
2. $\mathrm{D}(s_\beta(\lambda)) = s_\beta\left(\mathrm{D}(\lambda) \setminus \Phi\left(s_\beta\right)\right)$.

As a corollary, we get:

Corollary 3.7. *Let $\lambda \in P_{\geq -1}$ and $\alpha_i^\vee \in \mathrm{D}(\lambda)^\vee \cap \Pi^\vee$. Then we have:*

1. $s_i \lambda \in P_{\geq -1}$.
2. *The map $s_i : \mathrm{D}(\lambda)^\vee \setminus \{\alpha_i^\vee\} \ni \beta^\vee \mapsto s_i \beta^\vee \in \mathrm{D}(s_i\lambda)^\vee$ is an order isomorphism on $\preceq$.*

Definition 3.8. Let $\lambda \in P_{\geq -1}$. Let $l \in \mathbb{N}$. A sequence $\mathcal{B} = (\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_l})$ of simple roots is said to be a *simple λ-path* if

$$\alpha_{i_p}^\vee \in \mathrm{D}(s_{i_{p-1}} \cdots s_{i_1}(\lambda))^\vee, \quad 1 \leq p \leq l.$$

As mentioned above, in general, the ordinary coroot order and the heap order do not coincide. Hereafter, we consider the conditions for these two orders to coincide. For this purpose, we introduce several concepts: bad pairs, bad quartets, and bad triplets.

3.1 Bad pairs

Definition 3.9. Let $\lambda \in P_{\geq -1}$. Let $\beta^\vee, \gamma^\vee \in \mathrm{D}(\lambda)^\vee$. A pair $(\beta^\vee, \gamma^\vee)$ is called a *bad pair in* $\mathrm{D}(\lambda)^\vee$ if we have $\beta^\vee \not\preceq \gamma^\vee$ and $\beta^\vee \leq \gamma^\vee$.

We can see typical examples of bad pairs in Examples 3.11 and 3.15.

3.2 Bad quartets

Definition 3.10. Let $\lambda \in P_{\geq -1}$. Let $\delta_1^\vee, \delta_2^\vee, \beta_1^\vee, \beta_2^\vee \in \mathrm{D}(\lambda)^\vee$. A quartet $(\delta_1^\vee, \delta_2^\vee; \beta_1^\vee, \beta_2^\vee)$ is said to be a *bad quartet in* $\mathrm{D}(\lambda)^\vee$ if the following conditions hold:

1. $\delta_j^\vee \lhd \beta_{j'}^\vee \quad (j, j' = 1, 2)$,
2. $\beta_1^\vee$ and $\beta_2^\vee$ are incomparable on $\preceq$,
3. $\langle \delta_1, \delta_2^\vee \rangle = 0$.

Example 3.11. Let $\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ be a simple root system such that

$$\begin{bmatrix} \langle \alpha_0, \alpha_0^\vee \rangle & \langle \alpha_0, \alpha_1^\vee \rangle & \langle \alpha_0, \alpha_2^\vee \rangle & \langle \alpha_0, \alpha_3^\vee \rangle \\ \langle \alpha_1, \alpha_0^\vee \rangle & \langle \alpha_1, \alpha_1^\vee \rangle & \langle \alpha_1, \alpha_2^\vee \rangle & \langle \alpha_1, \alpha_3^\vee \rangle \\ \langle \alpha_2, \alpha_0^\vee \rangle & \langle \alpha_2, \alpha_1^\vee \rangle & \langle \alpha_2, \alpha_2^\vee \rangle & \langle \alpha_2, \alpha_3^\vee \rangle \\ \langle \alpha_3, \alpha_0^\vee \rangle & \langle \alpha_3, \alpha_1^\vee \rangle & \langle \alpha_3, \alpha_2^\vee \rangle & \langle \alpha_3, \alpha_3^\vee \rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

of type $A_3^{(1)}$. Put

$$\lambda := -\omega_0 + \omega_2,$$

where ω_i denotes a fundamental weight. Then $\lambda \in P_{\geq -1}$ and the shape $\mathrm{D}(\lambda)^\vee$ is depicted in Figure 1. We notice that the shape $\mathrm{D}(\lambda)^\vee$ contains a bad quartet

$$(\alpha_0^\vee + \alpha_1^\vee, \alpha_3^\vee + \alpha_0^\vee; \\ \alpha_3^\vee + \alpha_0^\vee + \alpha_1^\vee, \alpha_3^\vee + 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee).$$

Note that

$$(\alpha_3^\vee + \alpha_0^\vee + \alpha_1^\vee, \alpha_3^\vee + 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee)$$

is a bad pair.

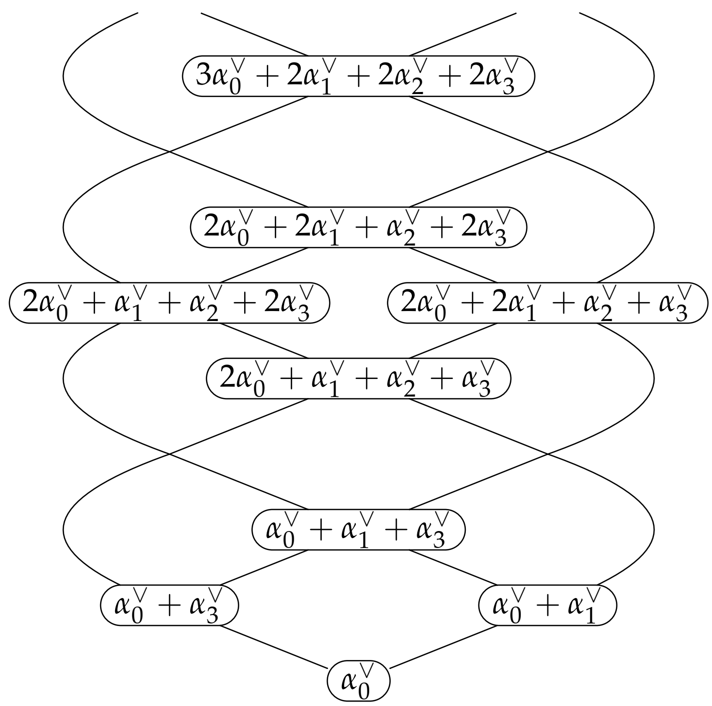


Figure 1: Hasse diagram of heap order of $\mathrm{D}(\lambda)^\vee$

Proposition 3.12. *Let $\lambda \in P_{\geq -1}$. If $\mathrm{D}(\lambda)^\vee$ contains a bad quartet, then there exists a simple λ-path $(\alpha_{i_1}, \cdots, \alpha_{i_l})$ such that $\mathrm{D}(s_{i_l} \cdots s_{i_1} \lambda)^\vee$ contains a bad pair.*

Proposition 3.13. *If the shape $\mathrm{D}(\lambda)^\vee$ contains a bad quartet, then λ is infinite.*

3.3 Bad triplets

Definition 3.14. Let $\lambda \in P_{\geq -1}$. Let $\delta^\vee, \beta_1^\vee, \beta_2^\vee \in \mathrm{D}(\lambda)^\vee$. A triplet $(\delta^\vee; \beta_1^\vee, \beta_2^\vee)$ is said to be a *bad triplet in* $\mathrm{D}(\lambda)^\vee$ if the following conditions hold:

1. $\delta^\vee \lhd \beta_{j'}^\vee \quad (j' = 1, 2)$,
2. $\beta_1^\vee$ and $\beta_2^\vee$ are incomparable on $\preceq$,
3. $\langle \delta, \beta_{j'}^\vee \rangle = 2 \quad (j' = 1, 2)$.

Example 3.15. Let $\Pi = \{\alpha_0, \alpha_1, \alpha_2\}$ be a simple root system such that

$$\begin{bmatrix} \langle \alpha_0, \alpha_0^\vee \rangle & \langle \alpha_0, \alpha_1^\vee \rangle & \langle \alpha_0, \alpha_2^\vee \rangle \\ \langle \alpha_1, \alpha_0^\vee \rangle & \langle \alpha_1, \alpha_1^\vee \rangle & \langle \alpha_1, \alpha_2^\vee \rangle \\ \langle \alpha_2, \alpha_0^\vee \rangle & \langle \alpha_2, \alpha_1^\vee \rangle & \langle \alpha_2, \alpha_2^\vee \rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

of type $D_3^{(2)}$. Put

$$\lambda := -\omega_0 + \omega_2,$$

where ω_i denotes a fundamental weight. Then $\lambda \in P_{\geq -1}$ and the shape $\mathrm{D}(\lambda)^\vee$ is depicted in Figure 2.We notice that the shape $\mathrm{D}(\lambda)^\vee$ contains the bad triplet

$$(\alpha_0^\vee + \alpha_1^\vee; \alpha_0^\vee + 2\alpha_1^\vee, 2\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee).$$

Note that

$$(\alpha_0^\vee + 2\alpha_1^\vee, 2\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee)$$

is a bad pair.

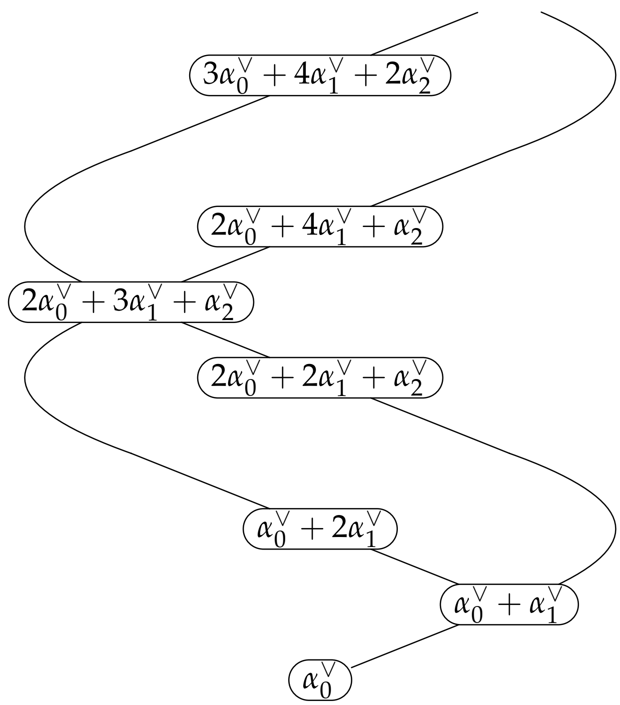


Figure 2: Hasse diagram of heap order of $\mathrm{D}(\lambda)^\vee$.

Proposition 3.16. *Let $\lambda \in P_{\geq -1}$. If $\mathrm{D}(\lambda)^\vee$ contains a bad triplet, then there exists a simple λ-path $(\alpha_{i_1}, \cdots, \alpha_{i_l})$ such that $\mathrm{D}(s_{i_l} \cdots s_{i_1} \lambda)^\vee$ contains a bad pair.*

Proposition 3.17. *If the shape $\mathrm{D}(\lambda)^\vee$ contains a bad triplet, then λ is infinite.*

4 Main theorem

Theorem 4.1. *Let $\lambda \in P_{\geq -1}$. Then the following two conditions are equivalent:*

1. *There exists a simple λ-path $(\alpha_{i_1}, \cdots, \alpha_{i_l})$ $(l \geq 0)$ such that $\mathrm{D}(s_{i_l} \cdots s_{i_1}\lambda)^\vee$ contains a bad pair.*

2. *$\mathrm{D}(\lambda)^\vee$ contains a bad quartet or a bad triplet.*

We note that we cannot replace condition (1) in Theorem 4.1 with the simpler condition (3) below:

3. the shape $\mathrm{D}(\lambda)^\vee$ contains a bad pair.

Take $\lambda \in P_{\geq -1}$ in Example 3.11. The ordinary coroot order of the shape $\mathrm{D}(\lambda)^\vee$ is depicted in Figure 3. Note that the pair $(\alpha_3^\vee + \alpha_0^\vee + \alpha_1^\vee, \alpha_3^\vee + 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee)$ is a bad pair. The poset $(\mathrm{D}(\lambda)^\vee; \leq)$ is not order-isomorphic to the poset $(\mathrm{D}(\lambda)^\vee; \preceq)$. By $\alpha_0^\vee \in \mathrm{D}(\lambda)^\vee$, we have $s_0\lambda \in P_{\geq -1}$. The ordinary coroot order of the shape $\mathrm{D}(s_0\lambda)^\vee$ is depicted in Figure 4. The shape $\mathrm{D}(s_0\lambda)^\vee$ contains no bad pairs. In general, for $\alpha_i^\vee \in \mathrm{D}(\lambda)^\vee$, the poset $(\mathrm{D}(\lambda)^\vee \setminus \{\alpha_i^\vee\}; \leq)$ is not order-isomorphic to the poset $(\mathrm{D}(s_i\lambda)^\vee; \leq)$, in contrast to Corollary 3.7. However, the shape $\mathrm{D}(s_i\lambda)^\vee$ still contains bad quartets. Hence, we cannot replace the condition 1 in Theorem 4.1 with the condition 3.

As a corollary of Theorem 4.1, we get:

Corollary 4.2. *Let $\lambda \in P_{\geq -1}$. If the shape $\mathrm{D}(\lambda)^\vee$ contains no bad quartets and no bad triplets, then the ordinary order coincides with the heap order over the shape $\mathrm{D}(\lambda)^\vee$.*

By Propositions 3.13 and 3.17, we get:

Proposition 4.3. *If the shape $\mathrm{D}(\lambda)^\vee$ contains a bad quartet or a bad triplet, then we have* $\#\mathrm{D}(\lambda)^\vee = \infty$.

Immediately we get:

Corollary 4.4. *If $\lambda \in P_{\geq -1}$ is finite, then the ordinary order coincides with the heap order over the shape $\mathrm{D}(\lambda)^\vee$.*

Furthermore, we can get:

Corollary 4.5. *Suppose that the underlying Dynkin diagram Γ is of affine type, listed as (A) in the introduction. Denote by W the Weyl group. Let λ be an infinite predominant integral weight. Then the following conditions are equivalent to each other:*

1. *the shape $\mathrm{D}(\lambda)^\vee$ contains no bad quartets and no bad triplets.*

2. *we have either*

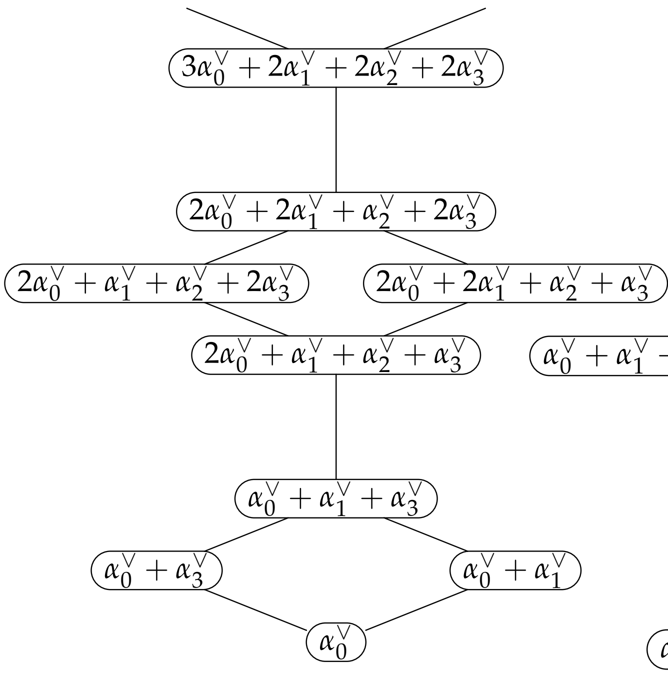


Figure 3: Hasse diagram of ordinary coroot order of $\mathrm{D}(\lambda)^\vee$

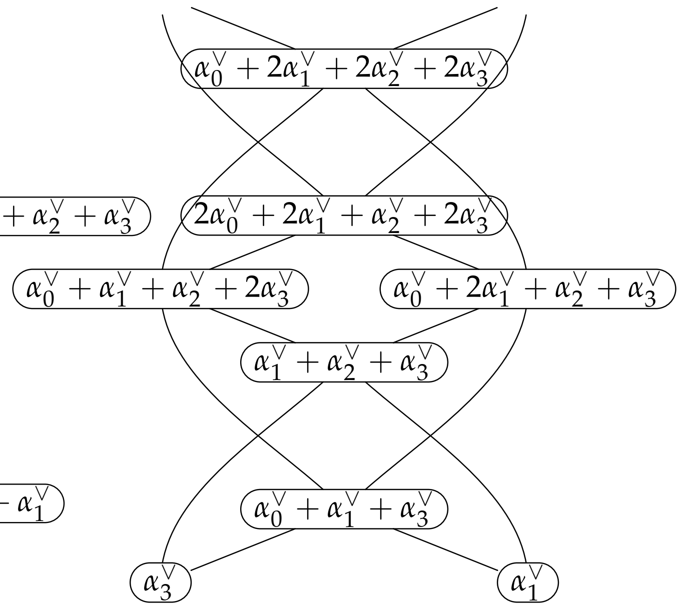


Figure 4: Hasse diagram of ordinary coroot order of $\mathrm{D}(s_0\lambda)^\vee$

(a) *Γ is of type $A_{l-1}^{(1)}$ for some $l \geq 2$ and there exists $\mu \in P_{\geq -1}$ such that $\lambda \in W\mu$ and*

$$\langle \mu, \alpha_i^\vee \rangle = \begin{cases} 1 & \text{if } i = 0 \\ -1 & \text{if } i = 1 \ (\text{resp.}\, l-1) \\ 0 & \text{if } 2 \leq i \leq l-1 \ (\text{resp.}\, 1 \leq i \leq l-2) \end{cases} \quad ; \quad \textit{or}$$

(b) *Γ is of type $C_l^{(1)}$ for some $l \geq 2$.*

Corollary 4.6. *Suppose that the underlying Dynkin diagram Γ is of infinite type, listed as (B) in the introduction. Let λ be an infinite predominant integral weight. Then the shape $\mathrm{D}(\lambda)^\vee$ contains no bad quartets and no bad triplets.*

4.1 Case of type $A_n^{(1)}$

In this subsection, we assume Theorem 4.1 and suppose that the underlying Dynkin diagram is of type $A_n^{(1)}$. The purpose of this subsection is to prove the following theorem:

Theorem 4.7. *Denote by W the Weyl group. Let λ be an infinite predominant integral weight. Then the following conditions are equivalent to each other:*

1. *$\lambda \in W(\omega_1 - \omega_0)$ or $\lambda \in W(\omega_n - \omega_0)$.*

2. *for any $\mu \in W\lambda$, the ordinary order coincides with the heap order over the shape* $\mathrm{D}(\mu)^\vee$.

First, we review several results in [6]. Let the index set I be $\{0, 1, \cdots, n\}$ and set the cartan matrix $A = (a_{ij})$ be

$$a_{ij} = \begin{cases} 2 & i = j \\ -1 & i - j \equiv 1, n \pmod{n+1} \\ 0 & i - j \not\equiv 0, 1, n \pmod{n+1}. \end{cases}$$

Denote by P the set of integral weights. For $i \in I$, set

$$P(i, 0; I) := \left\{ \lambda \in P \,\middle|\, \langle \lambda, \alpha_k^\vee \rangle = \delta_{i,k} - \delta_{0,k}, \text{ for each } k \in I \right\}$$

Denote by P_{sig} the set of integral weight λ such that

$$\forall \beta \in \Phi; \langle \lambda, \beta^\vee \rangle = 1, 0, -1.$$

Theorem 4.8 ([6, Theorem 5.15])**.**

$$P_{\mathrm{sig}} = \coprod_{i=0}^{n} W \cdot P(i, 0; I)$$

Denote by $P_{\geq -1}^{\mathrm{inf}}$ the set of infinite predominant integral weights.

Proposition 4.9 ([6, Proposition 5.8])**.**

$$P_{\mathrm{sig}} = P(0, 0; I) \sqcup P_{\geq -1}^{\mathrm{inf}}.$$

Since the set $P(0, 0; I)$ is closed under the action of the Weyl group W, we get

Corollary 4.10.

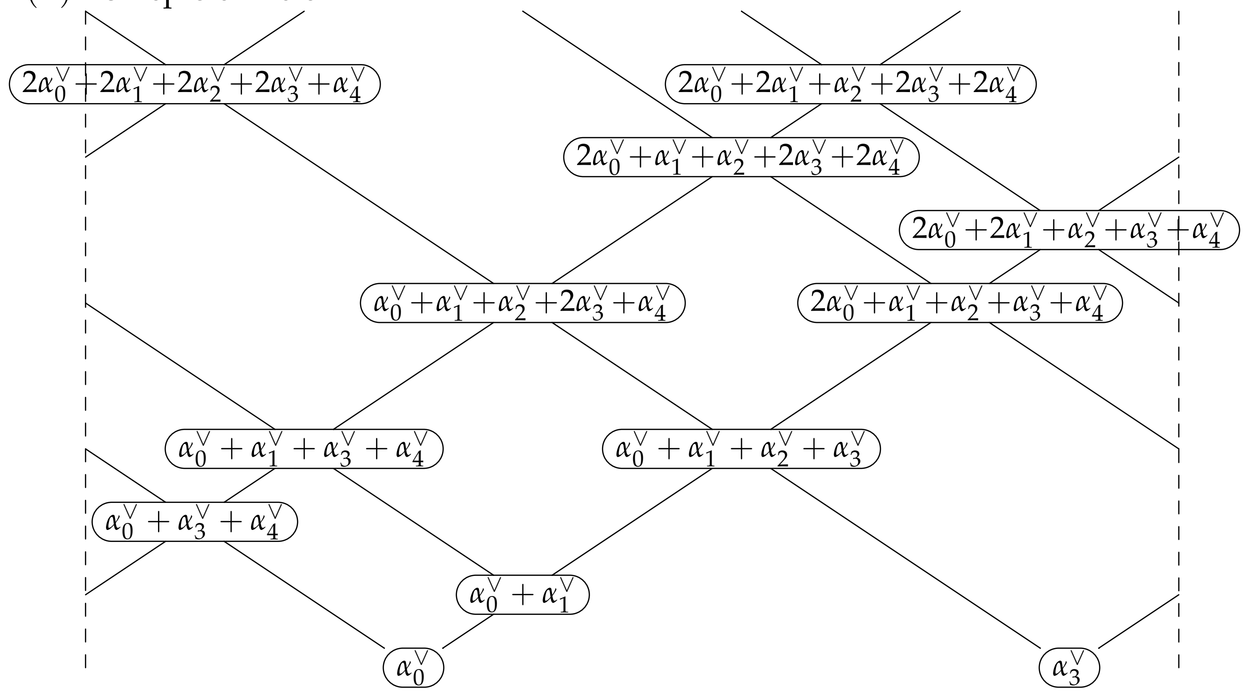
$$P_{\geq -1}^{\mathrm{inf}} = \coprod_{i=1}^{n} W \cdot P(i, 0; I).$$

For simplicity, we denote $W \cdot P(i, 0; I) = W(\omega_i - \omega_0)$.
Now, we give a proof of Theorem 4.7.

5 Application of main theorem

We give an application of main theorem to the hook length formula. In this section, we suppose that the underlying Dynkin diagram is of type $A_{l-1}^{(1)}$. Let $\lambda \in P_{\geq -1}$.

Example 5.1. Let $l = 5$ and $\lambda = -\omega_0 + \omega_2 - \omega_3 + \omega_4$. Then the heap order of the shape $D(\lambda)^\vee$ is depicted below:



The west end and the east end are connected cylindrically. This shape is order isomorphic to the cylindric Young diagram depicted below:

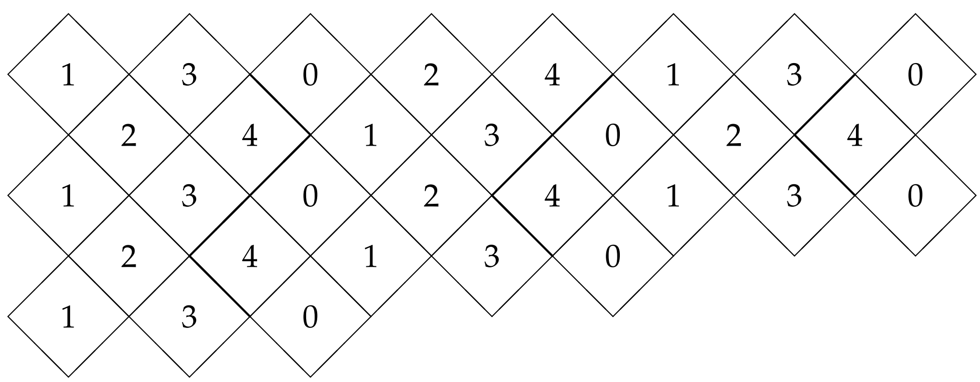


Figure 5: corresponding cylindric Young diagram

The entries of cells are contents.

According to [4], we define the hook at $\beta^\vee \in D(\lambda)^\vee$:

Definition 5.2. For $\beta^\vee \in D(\lambda)^\vee$, we put

$$H_\lambda\left(\beta^\vee\right) := D(\lambda)^\vee \cap \Phi\left(s_\beta\right)^\vee.$$

The set $H_\lambda(\beta^\vee)$ is called the *hook at* $\beta^\vee \in D(\lambda)^\vee$. See also Example 5.3.

Then we can define a unique coloring $c : \mathrm{D}(\lambda)^\vee \to \{0, 1, \cdots, l-1\}$ with

$$\sum_{\gamma^\vee \in \mathrm{H}_\lambda(\beta^\vee)} \alpha_{c(\gamma^\vee)} = \beta$$

for any $\beta^\vee \in \mathrm{D}(\lambda)^\vee$. See also Example 5.3.

Example 5.3. In the shape of Example 5.1, the hook $\mathrm{H}_\lambda(\beta^\vee)$ at the coroot

$$\beta^\vee = 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee$$

is depeicted as the set of boxes with shadow below.

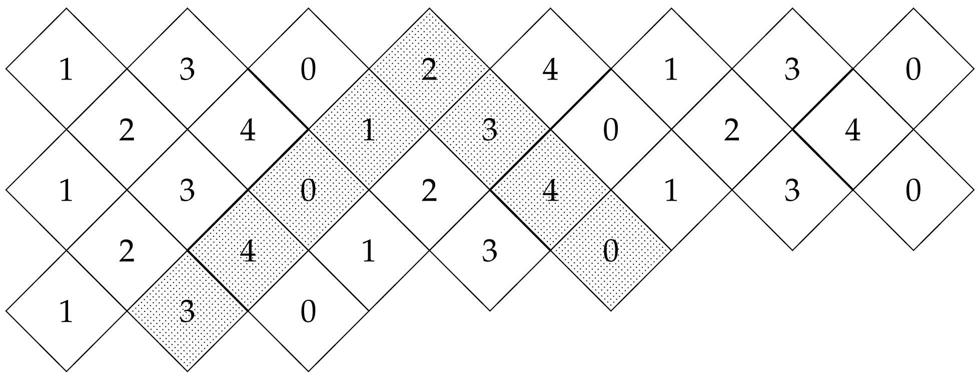


Figure 6: corresponding cylindric Young diagram

Furthermore, we notice that

$$\sum_{\gamma^\vee \in \mathrm{H}_\lambda(\beta^\vee)} \alpha_{c(\gamma^\vee)} = \alpha_3 + \alpha_4 + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_0 = \beta.$$

According to R. P. Stanley, we define $\mathrm{D}(\lambda)^\vee$-partitions:

Definition 5.4. A map $\pi : \mathrm{D}(\lambda)^\vee \to \mathbb{N} = \{0, 1, 2, \cdots\}$ is said to be a $\mathrm{D}(\lambda)^\vee$*-partition* if the following two conditions hold:

1. if $\beta^\vee, \gamma^\vee \in \mathrm{D}(\lambda)^\vee$ satisfy $\beta^\vee \triangleleft \gamma^\vee$, then we have $\pi(\beta^\vee) \geq \pi(\gamma^\vee)$.
2. the number of elements $\beta^\vee \in \mathrm{D}(\lambda)^\vee$ with $\pi(\beta^\vee) > 0$ is finite.

The set of $\mathrm{D}(\lambda)^\vee$-partitions is denoted by $\mathsf{A}(\mathrm{D}(\lambda)^\vee)$. Let $x_0, x_1, \cdots, x_{l-1}$ be indeterminates. For $\pi \in \mathsf{A}(\mathrm{D}(\lambda)^\vee)$, we define a monomial x^π by

$$x^\pi := \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} x_{c(\beta^\vee)}^{\pi(\beta^\vee)}.$$

For $\beta = \sum_{i \in I} c_i \alpha_i \in Q_+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$, we define a monomial x^β by

$$x^\beta := \prod_{i \in I} x_i^{c_i} = x_0^{c_0} x_1^{c_1} \cdots x_{l-1}^{c_{l-1}}.$$

Then, we get the following result:

Theorem 5.5. *Let λ be an infinite predominant integral weight over the root system of type $A_{l-1}^{(1)}$. Then the generating function of $\mathrm{D}(\lambda)^{\vee}$-partitions is decomposed as:*

$$\sum_{\pi \in \mathsf{A}(\mathrm{D}(\lambda)^{\vee})} x^{\pi} = \prod_{n=1}^{\infty} \frac{1}{1 - x^{n\delta}} \cdot \prod_{\beta \in \mathrm{D}(\lambda)} \frac{1}{1 - x^{\beta}},$$

where δ denotes the null root: $\delta = \sum_{i=0}^{l-1} \alpha_i$,

The proof of the above theorem is given by induction on heap order.

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