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On the realization space of the cube

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Abstract. We prove that the realization space of the d-dimensional cube is contractible. For this we first show that any two realizations are connected by a finite sequence of projective transformations and normal transformations. As an application we use this fact to define an analog of the connected sum construction for cubical d-polytopes, and apply this construction to certain cubical d-polytopes to conclude that the rays spanned by f-vectors of cubical d-polytopes are dense in Adin's cone. The connectivity result on cubes extends to any product of simplices, and further, it shows the respective realization spaces are contractible.

Résumé. Nous considérons l'espace de réalisation du cube en dimension d et montrons que tous les deux cubes sont liés par une combinaison de transformations projectives et de transformations normales.

Keywords: polytope, cube, realization space, f-vector, connected sum

1 Introduction

Perhaps the most natural transformations on polytopes that preserve the combinatorial type, namely the facial structure, are projective transformations and normal transformations. Loosely speaking, the former are given by perspective transformation from one hyperplane where the polytope lies to another hyperplane, while the latter are given by scaling the outer normal vectors to facets so that facets do not degenerate. While the former are connected to the projective linear group acting on vector spaces, the latter is connected to the Chow cohomology of toric varieties, and in particular inherits an algebra structure via the Minkowski sum [7]. (By *polytope* we always mean a convex polytope.)

The simplex, of any fixed dimension, is *projectively unique*, namely, any simplex can be continuously transformed to any other simplex of the same dimension by a homotopy of projective transformations. Thus, any two *simplicial* polytopes, after applying an appropriate projective transformation to one of them, can be glued along a common facet

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whose hyperplane separates them, to produce again a convex polytope. This realizes the connected sum operation geometrically.

However, the d-cube is not projectively unique for $d \geq 3$; this can be seen even by a dimension count: the realization space of the (combinatorial) d-cube has dimension larger then the dimension of the space of projective transformations. Indeed, the group of projective transformations on $\mathbb{R}^d$ is of dimension $d(d+2)$, while the realization space of the d-cube has dimension $2d^2$. In particular, we can not realize the connected sum operation geometrically for cubical d-polytopes, $d \geq 4$.

We enlarge the set of transformations by adding normal transformations to the generating set. While the first author mentioned this theorem in passing, assuming it had to be known, it was to our surprise that the following results appear to be new, even the qualitative assertion in (a).

Theorem 1.1 (Cubes are normal-projectively unique)**.** *Fix a dimension d.*

(a) For any two realizations of the d-cube, one can be obtained from the other by a composition of finitely many transformations, each is either projective or normal. In fact, $8d$ *of them suffice.*

(b) The constructed algorithm transforms cubes continuously to the standard cube. In particular, we obtain a deformation retraction to a point. Thus, the realization space of cubes is contractible.

Let us stress that we stay entirely inside the space of cubes. Every transformation takes us from one cube to another; not one of the projective transformations results in an unbounded polytope.

As a corollary of the quantitative assertion in (a), we obtain a cubical analog of the connected sum construction, at a small price.

Theorem 1.2. *(a) (Bounded towers) For any two realizations* C_1 *and* C_2 *of the* $(d-1)$*-cube, there exists a cubical d-polytope C made of m* ($m \leq 4d$) *d-cubes stacked one on top of the other, such that* C_1 *and* C_2 *are projectively equivalent to its bottom and top facets, respectively. Call C a* d-tower of m cubes.

(b) (Cubical connected sum) For any two cubical d-polytopes P_1 *and* P_2*, and facets* F_i *of* P_i*,* $i = 1, 2$*, there exists a projective transformation* ϕ *and a d-tower T of at most* $4d$ *cubes, such that* $P := P_1 \cup T \cup \phi(P_2)$ *is* convex*,* $P_1 \cap T = F_1$ *and* $\phi(P_2) \cap T = \phi(F_2)$ *are the top and bottom facets of T respectively. We call P the* C-connected sum *of* P_1 *and* P_2 *along* F_1 *and* F_2*.*

We apply this cubical connected sum operation to the cubical polytopes constructed recently in [2]; the f-vectors of the latter approach the extremal rays of Adin's cone, which is conjectured to contain all f-vectors of cubical d-polytopes [1]. The following density result extends the results of Babson, Billera and Chan (see the remark after Theorem 5.7 in [5]) from cubical spheres to cubical polytopes. Let $\square^d$ denote the d-cube and $f(P)$ denote the f-vector of the polytope P. Let $\mathcal{A}_d$ be the Adin cone (its apex is $f(\square^d)$ and its dimension is $\lfloor d/2 \rfloor$ by the cubical Dehn–Sommerville relations [1]).

Theorem 1.3 (Ray density in Adin's cone)**.** *For any $\epsilon > 0$ and any $x \in \mathcal{A}_d$ there exists a cubical d-polytope P such that the angle $\measuredangle x f(\square^d) f(P)$ is smaller than ϵ.*

Lastly, we note that our cubical connected sum construction endows the set of f-vectors of cubical d-polytopes with the structure of an affine semigroup (see [10]).

A complete version of this extended abstract can be found in [3].

2 Preliminaries

For further background on polytopes see e.g. [11].

2.1 Two notions of equivalence of d-polytopes

Let $P = \{x \in \mathbb{R}^d \mid Ax \leq b\}$ be a d-polytope, with the origin in its **interior** P°. Denote by $r_1, \ldots, r_m$ the rows of A. By scaling we may assume these are the **facet outer normals**. The **polar polytope**

$$P^\triangle = \left\{ y \in \mathbb{R}^d \;\middle|\; \langle y, x \rangle \leq 1 \text{ for all } x \in P \right\} = \operatorname{conv}(p_1, \ldots, p_m)$$

is the d-polytope with vertices $p_1 = \frac{1}{b_1} r_1, \ldots, p_m = \frac{1}{b_m} r_m$.

A **projective transformation** is a map

$$\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

defined by

$$x \mapsto \frac{Ax + b}{c^T x + \alpha},$$

for some $A \in M_{d \times d}(\mathbb{R})$, $b, c \in \mathbb{R}^d$, and $\alpha \in \mathbb{R}$ that satisfy

$$\det \begin{pmatrix} A & b \\ c^T & \alpha \end{pmatrix} \neq 0.$$

These transformations form a group under composition.

Definition 2.1. Two d-polytopes P and Q are **projectively equivalent** if there is a projective transformation φ such that $Q = \varphi(P)$.

Recall that two d-polytopes are **combinatorially equivalent** if they have isomorphic face lattices. We will need another notion of equivalence:

Definition 2.2. Two d-polytopes P and Q are **normally equivalent** if they are combinatorially equivalent and have the same set of facet outer normals.

In this case we say $Q = \psi(P)$ for a normal transformation ψ. Thus, given a polytope P, any two polytopes normally equivalent to it differ by a normal transformation. On the dual polytopes we say $Q^{\triangle} = \psi^{\triangle}(P^{\triangle})$ for a **dual normal transformation** $\psi^{\triangle}$ (it scales the vertices along the rays from the origin while preserving the combinatorial type).

2.2 Connected sums of d-polytopes

Suppose P and Q are d-polytopes whose intersection is a common facet $F = P \cap Q$ of both. If $R = P \cup Q$ is convex then its proper faces are precisely the proper faces of either P or Q, excluding F:

$$\text{faces}(R) = (\text{faces}(P) \cup \text{faces}(Q)) \setminus \{F\}.$$

The following lemma, a proof of which can be found in [9, Lemma 3.2.4], tells us when and how the connected sum of two polytopes can be formed.

Lemma 2.3. *Let P and Q be d-polytopes that have projectively equivalent facets F_1 and F_2 respectively. Then there exists a projective transformation φ so that $P \cap \varphi(Q) = F_1 = \varphi(F_2)$ and $R = P \cup \varphi(Q)$ is convex.*

The combinatorial type of R in Lemma 2.3 is called the **connected sum** of P and Q along F_1 and F_2, denoted $P\#_{F_1 \sim F_2}Q$, or simply $P\#_F Q$ when the faces F_1, F_2 combinatorially isomorphic to F are understood.

2.3 Cubical polytopes

We give just a brief reminder of the definitions of a cubical d-polytope and its h^c-vector and g^c-vector. For more details, in particular, for the construction used in Section 5, see [2].

A d-polytope Q is **cubical** if each of its proper faces is combinatorially a cube. Its **f-polynomial** is defined by

$$f(Q,t) = \sum_{i=0}^{d-1} f_i t^i$$

where $f_i = f_i(Q)$ is the number of i-dimensional faces of Q.

We then define the **short cubical h-polynomial**:

$$h^{sc}(Q,t) = (1-t)^{d-1} f\left(Q, \frac{2t}{1-t}\right),$$

and the **cubical h-polynomial**

$$h^c(Q,t) = \sum_{i=0}^{d} h_i^c t^i = \frac{t(1-t)^{d-1}}{1+t} f\left(Q, \frac{2t}{1-t}\right) + 2^{d-1}\frac{1-(-t)^{d+1}}{1+t}.$$

Adin [1] has shown that $h^c(Q,t)$ is symmetric, that is

$$h_i^c = h_{d-i}^c \quad (0 \le i \le d).$$

These $\lceil d/2 \rceil$ equations are the **cubical Dehn–Sommerville relations**. We thus define the **cubical g-vector** $g^c(Q) = (g_0^c, \dots, g_{\lfloor d/2 \rfloor}^c)$ by

$$g_0^c = h_0^c = 2^{d-1}, \qquad\qquad g_i^c = h_i^c - h_{i-1}^c \quad \text{for } 1 \le i \le \lfloor d/2 \rfloor.$$

Adin conjectured

Conjecture 2.4 (Question 2 in [1])**.** *For a cubical d-polytope we have*

$$g_i^c \ge 0 \quad (1 \le i \le \lfloor d/2 \rfloor). \tag{2.1}$$

The cone (2.1) is the nonnegative orthant in $\mathbb{R}^{\lfloor d/2 \rfloor}$, and its image under the map transforming g^c-vectors back into f-vectors yields the Adin cone $\mathcal{A}_d$ in $\mathbb{R}^d$.

In [2], for each $1 \le i \le \lfloor d/2 \rfloor$, the authors exhibit a sequences of cubical d-polytopes whose corresponding sequence of g^c-vectors approaches the ray spanned by e_i. This translates into sequences of f-vectors approaching the extremal rays of $\mathcal{A}_d$.

3 Any two combinatorial d-cubes are related by normal and projective transformations

We will use the following lemma, which describes the effect of a projective transformation on the polar polytope.

Lemma 3.1. *Let P be a d-polytope with $0 \in P^\circ$, and $P^\triangle = \operatorname{conv}(p_1, \dots, p_m)$. Then for any $v \in P^\circ$ there exists a d-polytope Q which is projectively equivalent to P, and $Q^\triangle = \operatorname{conv}(p_1 + v, \dots, p_m + v)$.*

Proof. Consider the effect of a projective transformation $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ that takes P to Q on the polar polytopes $P^\triangle$ and $Q^\triangle$. It is easy check that the map

$$\varphi^\triangle : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

defined by

$$x \mapsto \frac{A^T x - c}{-b^T x + \alpha}$$

where $(\cdot)^T$ denotes the transpose, is a projective transformation that satisfies

$$\varphi^\triangle(Q^\triangle) = P^\triangle.$$

Denote $\varphi^{-\triangle} = (\varphi^\triangle)^{-1}$, so that

$$Q^\triangle = \varphi^{-\triangle}(P^\triangle).$$

Taking $A = I_{d\times d}$, $b = 0$, $c = v$, and $\alpha = 1$ produces a projective transformation φ for which

$$\varphi^{-\triangle}(x) = x + v$$

and the claim follows. $\square$

Let $Q = \left\{x \in \mathbb{R}^d \,\middle|\, Ax \leq b\right\}$ be a combinatorial d-cube, with the origin in its interior, and $r_1, \dots, r_{2d}$ the rows of A are the facet outer normals. We may assume that they are ordered by pairs of combinatorially opposite facets, that is, r_i is normal to a facet opposite to the facet normal to r_{i+1}, for $i = 1, 3, \dots, 2d-1$. The polar polytope $Q^\triangle$ is the combinatorial d-crosspolytope with vertices $p_1 = \frac{1}{b_1} r_1, \dots, p_{2d} = \frac{1}{b_{2d}} r_{2d}$, and we denote by $\ell_i = [p_{2i-1}, p_{2i}]$, for $1 \leq i \leq d$, the line segments connecting the pairs of opposite vertices. The following proposition proves Theorem 1.1(a).

Proposition 3.2. *Let Q and Q' be two combinatorial d-cubes. Then there is a sequence $\phi_1, \dots \phi_s$ $(s \leq 8d-1)$ of projective and normal transformations such that*

$$Q' = (\phi_s \circ \cdots \circ \phi_1)(Q).$$

Proof sketch. We present the sequence in terms of the polar d-crosspolytopes. For each pair of antipodal vertices of $P := Q^\triangle$ we perform a sequence of 4 transformations, alternating between projective and ray transformations, arriving at a d-crosspolytope which is ray equivalent to the standard d-crosspolytope, namely the convex hull of the standard basis elements and their negatives. We refer to the sequence of 4 transformations for the i-th pair of antipodal vertices as the *i-th iteration*. We denote the crosspolytope obtained after the i-th iteration by $P^{(i)}$, its vertices by $p_1^{(i)}, p_2^{(i)}, \dots, p_{2d}^{(i)}$ and the line segments connecting its pairs of opposite vertices $p_{2j-1}^{(i)}, p_{2j}^{(i)}$ by $\ell_j^{(i)}$ for $j = 1, 2, \dots d$.

1. Use Lemma 3.1 to translate the crosspolytope $P^{(i-1)}$ so that the origin lies on the interior of the line segment $\ell_i^{(i-1)} = [p_{2i-1}^{(i-1)}, p_{2i}^{(i-1)}]$, say on its mid point to make a canonical choice; this projective transformation produces a polytope P'.

2. For $P' = \{x|\ Ax \leq b\}$ with vertex notation as in $P^{(i-1)}$, choose $c_{2i-1} \geq \frac{1}{b_{2i-1}}$ so that there exists an affine hyperplane H_i orthogonal to ℓ_i, which strictly separates $q_{2i-1} := c_{2i-1}r_{2i-1}$ from $\mathrm{Vert}(P') \setminus \{p_{2i-1}\}$. To make a canonical choice, let c be the infimum of the possible values for such c_{2i-1}s, fix $c_{2i-1} = c+1$ and fix the H_i as above that intersects the ray spanned by q_{2i-1} at $(c+0.5)r_{2i-1}$.

   Set $P'' = \mathrm{conv}(q_{2i-1} \cup P')$. Then P'' is ray equivalent to P'.

3. Again denote the vertices of P'' by p_i, in correspondence with the vertices of $P^{(i-1)}$, so $p_{2i-1} = q_{2i-1}$. Use again Lemma 3.1 to move the origin close enough to p_{2i-1} along the segment ℓ_i, that is, so that the origin and p_{2i-1} are on the same side of the hyperplane H_i of step (2). To make a canonical choice, move the origin to $(c+0.7)r_{2i-1}$. Then

$$H_i \cap P'' \cong \mathrm{conv}(H_i \cap \mathrm{Span}(p_j) \mid j \in [2d] \setminus \{2i-1, 2i\}).$$

   (Here $\cong$ means combinatorially equivalent). The resulted polytope P''' is projectively equivalent to P''.

4. Set $q_j := H_i \cap \mathrm{Span}(p_j)$ for $j \in [2d] \setminus \{2i-1, 2i\}$. Then
   $P^{(i)} = \mathrm{conv}(q_1, \ldots, q_{2i-2}, p_{2i-1}, p_{2i}, q_{2i+1,\ldots,q_{2d}})$ is ray equivalent to P'''.

After performing this process for every pair of antipodal vertices we obtain a combinatorial d-crosspolytope, with segments $\ell_1, \ldots, \ell_d$, such that, for each $1 \leq i \leq d$, there exists an affine hyperplane H_i, which is orthogonal to ℓ_i, and contains all other segments ℓ_j, $j \neq i$. It follows that the segments $\ell_1, \ldots, \ell_d$ all intersect in a point, and are pairwise orthogonal.

We perform the same procedure for $Q'^{\triangle}$ to get a combinatorial d-crosspolytope which is normally equivalent to the standard d-crosspolytope. To finish, we do a final normal transformation to concatenate the two sequences of transformations performed on Q and on Q'. In fact, the resulted 3 normal transformations in a row can be replaced by a single one. This algorithm gives $s = 8d - 1$. □

To conclude Theorem 1.1(b), that is, that the realization space R is contractible, we note that the (arbitrarily) canonical choices in each of the 4 steps of our algorithm, produce a continuous path from any point $x \in R$ to the point $p \in R$ corresponding to the standard cube. See [3] for the details, and a figure depicting an iteration of the algorithm for an octahedron.

4 A cubical connector d-polytope and the C-connected sum

Definition 4.1. A **d-tower** of s cubes is a cubical stacked d-polytope T obtained by stacking on the facet opposite to the facet stacked on in the previous step.

Explicitly, for $s = 1$ it is just a d-cube. Mark some two opposite facets as bottom and top. For $s > 1$, a d-tower of s cubes is obtained from a d-tower of $s-1$ cubes with bottom facet F and top facet F' by stacking a d-cube onto F'. Then the polytope T has a unique **bottom** facet and a unique **top** facet.

Given two combinatorial $(d-1)$-cubes Q_1 and Q_2, we use Proposition 3.2 to construct a d-tower having bottom facet Q'_1 and top facet Q'_2, with Q'_i projectively equivalent to Q_i, $i = 1, 2$. The following lemma shows how to translate each normal transformation from Proposition 3.2 into a d-tower of 1 cube.

Lemma 4.2. *Let Q_1 and Q_2 be two combinatorial $(d-1)$-cubes which are normally equivalent. Then there exists a d-cube Q in which Q_1 and Q_2 (both realized in $\mathbb{R}^d$) are opposite facets.*

Proof. Assume that both Q_1 and Q_2 are realized in $\mathbb{R}^d$ on the last coordinate $= 0$ hyperplane. Lift the vertices of Q_2 (say to height 1), and take the convex hull, denote it by Q.

Here is an explicit description of Q: Let $A_1, A_2 \in \mathbb{R}^{(2d-2)\times(d-1)}$ and $b_1, b_2 \in \mathbb{R}^{2d-2}$ be such that

$$Q_1 = \left\{ x \in \mathbb{R}^{d-1} \,\middle|\, A_1 x \le b_1 \right\}, \qquad Q_2 = \left\{ x \in \mathbb{R}^{d-1} \,\middle|\, A_2 x \le b_2 \right\}.$$

The fact that Q_1 and Q_2 are normally equivalent means that $A_1 = A_2$. We define

$$A = \left(\begin{array}{ccc|c} & & & | \\ & A_1 & & b_1 - b_2 \\ & & & | \\ \hline 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & -1 \end{array} \right), \qquad b = \left(\begin{array}{c} | \\ b_1 \\ | \\ \hline 1 \\ 0 \end{array} \right), \tag{4.1}$$

and

$$Q = \left\{ x \in \mathbb{R}^d \,\middle|\, Ax \le b \right\}. \quad \square$$

Applying Lemma 4.2 to each of the normal transformations in Proposition 3.2, and Lemma 2.3 to glue each such new d-cube to the previously constructed polytope so that the result is again a convex polytope, we conclude Theorem 1.2:

Corollary 4.3. *Let Q and Q' be two combinatorial $(d-1)$-cubes. Then there is a d-tower of $4d$ cubes with bottom facet projectively equivalent to Q and top facet projectively equivalent to Q'. We call this tower a* **cubical connector** *and denote it $C(Q, Q')$.*

Definition 4.4. Let Q_1 and Q_2 be cubical d-polytopes. Let F_1 be a facet of Q_1, F_2 a facet of Q_2, and $C = C(F_1, F_2)$ the appropriate cubical connector (a tower of $4d$ cubes). The **C-connected sum** $Q = Q_1 \# Q_2$ is the cubical d-polytope obtained by taking the connected sum $Q_1 \#_{F_1} C \#_{F_2} Q_2$.

5 Filling the g^c-cone

We apply the connected sum construction to appropriate AKN-polytopes (see [2]) thus obtaining sequences of cubical d-polytopes with corresponding g^c-vector sequences approaching any ray in the nonnegative orthant of $\mathbb{R}^{\lfloor d/2 \rfloor}$.

Lemma 5.1. *Let $Q_1 \# Q_2$ be a C-connected sum then*

$$\begin{aligned} g_1^c(Q_1 \# Q_2) &= g_1^c(Q_1) + g_1^c(Q_2) + (4d+1)2^{d-1}, \\ g_i^c(Q_1 \# Q_2) &= g_i^c(Q_1) + g_i^c(Q_2) \qquad (2 \leq i \leq \lfloor d/2 \rfloor). \end{aligned} \tag{5.1}$$

Proof sketch. Let us first observe that for the (usual) connected sum $Q \#_F Q'$, when Q and Q' are cubical d-polytopes we have

$$f(Q \#_F Q', t) = f(Q, t) + f(Q', t) - f(\square^{d-1}, t) - t^{d-1}.$$

Then by a straightforward computation the claim on the cubical g-vectors follows; we omit the details here. $\square$

The following proves Theorem 1.3:

Proposition 5.2. *Let r be any ray in the nonnegative orthant in $\mathbb{R}^{\lfloor d/2 \rfloor}$. Then there exists a sequence $\{Q_n\}_{n=1}^{\infty}$ of cubical d-polytopes with the sequence $\{g^c(Q_n)\}_{n=1}^{\infty}$ approaching r.*

Proof sketch. To construct the sequence Q_n approaching r, the ray spanned by $(s_1, \ldots, s_{\lfloor d/2 \rfloor})$, we start by constructing a sequence having the correct ratio between the $\lfloor d/2 \rfloor$-th coordinate and the $(\lfloor d/2 \rfloor - 1)$-th coordinate. Take the sequences from [2] approaching the $\lfloor d/2 \rfloor$-th coordinate and the $(\lfloor d/2 \rfloor - 1)$-th coordinate:

$$Q_m = Q(\lfloor d/2 \rfloor, d, m), \quad m \to \infty \qquad \text{and} \qquad Q'_l = Q(\lfloor d/2 \rfloor - 1, d, l), \quad l \to \infty,$$

Let $c = \frac{s_{\lfloor d/2 \rfloor - 1}}{s_{\lfloor d/2 \rfloor}}$. For each $m \geq d$, define $l = \lceil \log c + m + \log m \rceil$, take the corresponding subsequence of Q'_l's (we abuse notation and denote it again by Q'_l), and construct their C-connected sum:

$$Q_n = Q_m \# Q'_l, \quad n \to \infty.$$

Analyzing the g^c-vector of Q_m and Q'_l, and using Lemma 5.1 we obtain

$$\lim_{n \to \infty} \frac{g^c_{\lfloor d/2 \rfloor - 1}(Q_n)}{g^c_{\lfloor d/2 \rfloor}(Q_n)} = \frac{s_{\lfloor d/2 \rfloor - 1}}{s_{\lfloor d/2 \rfloor}}.$$

Do the same with the new sequence and an AKN-sequence approaching the $(\lfloor d/2 \rfloor - 2)$-th ray, etc. Note that proceeding in this way (from the last coordinate backwards) does not influence the ratios already taken care of, because the g^c-entries are 0 after the dominating coordinate in the AKN construction. For the last ratio a slight adaptation is needed according to the formula for g_1^c in Lemma 5.1. □

6 Concluding remarks: Generalizations and open questions

Let us start off by remarking that the bound on the number of iterated projections and normal transformations may not be optimal, and the reason for this may lie in the fact that we are not allowing the full action by projective transformations and normal transformations, as we want to stay in the world of polytopes. Indeed, purely from a naive dimension count for the realization space of the d-cube $(2d^2)$ compared to the projective linear group $(d(d+2))$ it might be possible that only a constant number of these operations suffice (namely 3, projective followed by normal followed by projective). We leave this as an open problem.

Problem 6.1. *How many normal and projective transformations are needed to transform any combinatorial cube into the standard one?*

Second is the natural question of more classes of combinatorial types of polytopes that are connected by normal and projective transformations. Let us call those polytopes PN-unique. Dually, let us call polytopes weakly-PR-unique if they are related by projective transformation, and a movement of its vertices along the rays they generate within the same combinatorial type. But in the dual, this permits moving some facet hyperplanes to infinity. If we want the dual to PN-uniqueness, then we add the condition that the origin has to be in the interior of the polytope at all times; we call such polytopes PR-unique. Then the PR-unique polytopes are precisely dual to the PN-unique polytopes.

We note the following simple fact about simplicial stacking (connected sum with a simplex S) on PR-unique polytopes:

Proposition 6.2. *If P is* PR*-unique and F a simplex facet of P, then $P\#_F S$ is* PR*-unique.*

Proof. Do PR-transformations so that the P part has the correct shape, then get the new vertex v to its desired position u with transformations that do not effect the P part: this can always be done with a sequence of 3 PR-transformations. For example, scale v by ϵ so that ϵv is close enough to F, namely so that the line through u and ϵv intersects the interior of F, say at w. Then move the origin to w, then scale ϵv to u. □

We immediately conclude:

Corollary 6.3. *Every polygon, and more generally every stacked polytope, is* PR*-unique. In particular, in every dimension $d \geq 2$, there are infinitely many combinatorial types of* PR*-unique polytopes.*

This is in contrast to projectively unique polytopes, which are only finitely many in dimension 2 and 3. (However, in sufficiently large fixed dimension d there exist projectively unique d-polytopes with arbitrarily many vertices – this was proved for $d \geq 69$ by Adiprasito and Ziegler [4], answering a question of Perles and Shephard [8].)

Let us briefly recall two polytopal constructions from [6], and then give two results related to these constructions:

- The **free join** of two polytopes P and Q is the $(\dim P + \dim Q + 1)$-polytope obtained by taking $\operatorname{conv}(P \cup Q)$ when P and Q are realized in skew affine spaces.
- The **subdirect sum** of two polytopes P and Q is the $(\dim P + \dim Q)$-polytope obtained by taking $\operatorname{conv}(P \cup Q)$ when P and Q are realized so that their affine hulls intersect in a single point, which is a relatively interior point of the face F of P and the face G of Q. The dual construction is called the **subdirect product**.

Proposition 6.4. *The free join of two polytopes P and Q is weakly-*PR*-unique if and only if both components are.*

This follows easily, as we may act on each component separately. The same is not true for PR-uniqueness, and therefore PN-uniqueness. A counterexample is the cone over the crosspolytope. Indeed, it follows from the following observation, that is straightforward from the definitions:

Lemma 6.5. *If P is* PR*-unique then every facet of P is projectively unique.*

The next result holds for PN-uniqueness, by following the proof of Proposition 3.2 for the cube case.

Theorem 6.6. *The subdirect sum of a* PR*-unique polytope with a simplex is* PR*-unique, and vice versa. Dually, the subdirect product with a simplex is* PN*-unique if and only if the original polytope is.*

This is especially interesting if one considers only those polytopes that are obtained as products of simplices. These are PN-unique by the above theorem (and include the cube). Moreover, the algorithm described in Proposition 3.2 goes through verbatim, and is continuously dependent on the starting geometry. Hence, we once again obtain that the realization space of such polytopes is contractible (a fact not known for general PN-unique or PR-unique polytopes). We end with a question:

Problem 6.7. *Is the dodecahedron* PN*-unique?*

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