Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #81, 10 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

Some combinatorial results on smooth permutations

Shoni Gilboa[1*] and Erez Lapid[2†]

[1]*Department of Mathematics and Computer Science, The Open University of Israel, Raanana, Israel*
[2]*Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel*

Abstract. We show that any smooth permutation w is characterized by the set $\mathbf{C}(w)$ of transpositions and 3-cycles that are $\leq w$ in the Bruhat order and that w is the product (in a certain order) of the transpositions in $\mathbf{C}(w)$. We also characterize the image of the map $w \mapsto \mathbf{C}(w)$. This map is closely related to the essential set (in the sense of Fulton) and gives another approach for enumerating smooth permutations and subclasses thereof. As an application, we obtain a result about the intersection of the Bruhat interval defined by a smooth permutation with a conjugate of a parabolic subgroup of the symmetric group. Finally, we relate covexillary permutations to smooth ones.

Keywords: Bruhat order, smooth permutations, pattern avoidance, Covexillary permutations

1 Introduction

This is an extended abstract to the paper [12], which contains all the proofs.

Fix an integer $n \geq 1$ and an n-dimensional vector space V over $\mathbb{C}$. Consider the (complete) *flag variety* $\mathcal{F}l_n$ consisting of all flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V, \ \ \dim V_i = i, \ i = 0, \dots, n.$$

This is a homogeneous space under the action of the general linear group $\mathrm{GL}(V)$. It can be identified with $\mathrm{GL}_n(\mathbb{C})/B_n(\mathbb{C})$ where B_n is the Borel subgroup of upper triangular matrices. By *Gauss elimination* (which is a special case of the *Bruhat decomposition*), the orbits of $B_n(\mathbb{C})$ on $\mathcal{F}l_n$ are naturally indexed by the symmetric group S_n (the Weyl group of GL_n). The *Schubert cell* Y_w pertaining to $w \in S_n$ is by definition the orbit of the permutation matrix of w. The *Schubert variety* X_w is by definition the closure of Y_w. For instance, for the identity permutation e, $Y_e = X_e$ is a singleton consisting of the standard flag (whose stabilizer is B_n), while for the longest permutation w_0, Y_{w_0} is the open cell defined by the non-vanishing of all minors in the bottom left corners and $X_{w_0} = \mathcal{F}l_n$.

*shoni@openu.ac.il.
†erez.lapid@weizmann.ac.il.

It is well known that many geometric properties of Schubert varieties can be described combinatorially. For instance, the *Bruhat order* given by

$$x \le w \iff Y_x \subseteq X_w$$

admits the following simple combinatorial description:

$$x \le w \iff r_x(i,j) \ge r_w(i,j) \text{ for all } i,j \in [n] := \{1,\dots,n\},$$

where for any permutation $y \in S_n$,

$$r_y(i,j) = \#\big(y([i]) \cap [j]\big),\ i,j \in [n].$$

(We refer to [3] for standard facts about the Bruhat order.) A more striking result due to Lakshmibai–Sandhya is that the Schubert variety X_w is *smooth* if and only if w is 3412 and 4231 avoiding [14]. (In this case we say that w is smooth.) This beautiful result opened the door to far-reaching relations between the geometry of Schubert varieties and combinatorics (in particular, pattern avoidance). We refer the reader to [1] for a recent survey.

Smooth permutations admit other (even earlier) combinatorial characterizations. For instance, by analyzing the tangent space of X_w at Y_e, Lakshmibai–Seshadri [15] proved that

$$w \text{ is smooth } \iff \#\{i<j : T_{i,j} \le w\} = \ell(w) := \{i<j : w(i) > w(j)\},$$

where $T_{i,j} \in S_n$ denotes the transposition $i \leftrightarrow j$. Another characterizing property is that the *Kazhdan–Lusztig polynomial* $P_{e,w}$ is 1 [6]. This property is important in representation theory because of the celebrated *Kazhdan–Lusztig conjecture* [13] (proved independently by Bernstein–Beilinson and Brylinski–Kashiwara). For a more recent surprising occurrence of smooth permutations in representation theory see [17]. We refer the reader to [2] for more information about singularities of Schubert varieties, excluding however more recent exciting developments in Kazhdan–Lusztig theory.

Our purpose is to give another way of looking at smooth permutations combinatorially. Our main result is the characterization of smooth permutations in terms of their 2-3-*table*. By definition, the 2-3-table of a permutation w is the set of transpositions and the 3-cycles that are $\le w$. The 2-3-table of a smooth permutation satisfies some simple combinatorial properties and conversely, any set of transpositions and 3-cycles satisfying these conditions arises from a smooth permutation (Theorem 2.1). Moreover, we can recover a smooth permutation from its 2-3-table by taking the product of the transposition $T_{i,j} \le w$ in a suitable *compatible order*, governed by the additional data in the 2-3-table. In fact, the set of compatible orders (with respect to the 2-3-table) has a structure of a connected graph, in a way reminiscent of the graph of reduced decomposition of w under Coxeter moves (Theorem 3.1).

The result is in accordance with known enumerative results of smooth permutations (e.g., [4, 5, 18]). It also gives a bijection between smooth permutations and Dyck paths with additional data (Theorem 5.1). Another interesting consequence is yet another combinatorial characterization of smooth permutations (Theorem 4.1). This characterization is of a rather different nature than the above-mentioned. Finally, an intriguing relation between covexillary permutations and smooth ones is given (Theorem 6.1).

The 2-3-table of permutation is closely related to the notion of *essential set* conceived by Fulton in his study of degeneracy loci [10]. This notion was further studied combinatorially by Eriksson–Linusson [8]. In the case of smooth permutations the situation is particularly simple.

2 The 2-3-table of a permutation

Fix an integer $n \geq 1$. Consider the symmetric group S_n of all the permutations of the set $[n] = \{1, 2, \ldots, n\}$ with the Bruhat order $\leq$. Let $\mathcal{T} = \{T_{i,j} : 1 \leq i < j \leq n\} \subset S_n$ be the set of transpositions. For every permutation $w \in S_n$ define the *2-table* of w to be

$$\mathbf{C}_{\mathcal{T}}(w) = \{x \in \mathcal{T} : x \leq w\}.$$

For every $w \in S_n$ we have $\ell(w) \leq \#\mathbf{C}_{\mathcal{T}}(w)$ where

$$\ell(w) = \#\{i < j : w(i) > w(j)\}$$

is the number of inversions of w [15]. If $\ell(w) = \#\mathbf{C}_{\mathcal{T}}(w)$, then w is called *smooth*, a terminology that is justified by the fact that this condition also characterizes the smoothness of the Schubert variety X_w pertaining to w [ibid.]. Another well-known combinatorial characterization of the smoothness of w is that w is 4231 and 3412 avoiding [14]. We refer to [2] and the references therein for more information about singularities of Schubert varieties.

Distinct smooth permutations may have the same 2-table (for example, for $n = 3$, $\mathbf{C}_{\mathcal{T}}((231)) = \{T_{1,2}, T_{2,3}\} = \mathbf{C}_{\mathcal{T}}((312))$). However, we show that smooth permutations are distinguishable from each other at the 'next level'. More precisely, let $\mathcal{C}^{2,3} \subset S_n$ be the set of permutations consisting of a single cycle of length 2 or 3. Denote the 3-cycle permutation $i \mapsto j \mapsto k \mapsto i$ with $i < j < k$ by $R_{i,j,k}$, so that

$$\mathcal{C}^{2,3} = \mathcal{T} \cup \{R_{i,j,k}, R_{i,j,k}^{-1} : i < j < k\}.$$

We define the 2-3-*table* of a permutation $w \in S_n$ to be

$$\mathbf{C}(w) = \{x \in \mathcal{C}^{2,3} : x \leq w\}.$$

Clearly, $\mathbf{C}(w)$ is downward closed and it is easy to see that if $R_{i,j,l}, R_{i,k,l}^{-1} \in \mathbf{C}(w)$ with $i < j, k < l$, then $T_{i,l} \in \mathbf{C}(w)$.

We say that a downward closed subset A of $\mathcal{C}^{2,3}$ is *admissible* if it satisfies the following two conditions.

- If $R_{i,j,l}, R^{-1}_{i,k,l} \in A$ with $i < j, k < l$, then $T_{i,l} \in A$.
- Whenever $T_{i,j}, T_{j,k} \in A$, $i < j < k$, at least one of $R_{i,j,k}$ and $R^{-1}_{i,j,k}$ belongs to A.

Our main result is the following.

Theorem 2.1. *The map $w \mapsto \mathbf{C}(w)$ defines a bijection between the smooth permutations of S_n and the admissible sets. The inverse bijection $A \mapsto \pi(A)$ is given by*

$$\pi(A) = \max\{x \in S_n : \mathbf{C}(x) = A\} = \max\{x \in S_n : \mathbf{C}_{\mathcal{T}}(x) = A_{\mathcal{T}}, \mathbf{C}(x) \subseteq A\},$$

where max *denotes the greatest element with respect to the Bruhat order.*

3 Compatible orders

We give an alternative, more constructive definition of $\pi(A)$ for an admissible set $A \subseteq \mathcal{C}^{2,3}$. We say that a total order $\prec$ on $A_{\mathcal{T}} = A \cap \mathcal{T}$ is *compatible* (with A) if whenever $T_{i,j}, T_{j,k} \in A$, $i < j < k$, the following hold:

1. If $T_{i,k} \in A$, then either $T_{i,j} \prec T_{i,k} \prec T_{j,k}$ or $T_{j,k} \prec T_{i,k} \prec T_{i,j}$.
2. If $T_{i,k} \notin A$, then $R_{i,j,k} \in A \iff T_{i,j} \prec T_{j,k}$.

Note that the first condition also occurs in the notion of *reflection order* (cf. [7], [3, Section 5.2]) except that we do not consider a total order on the whole of $\mathcal{T}$.

Theorem 3.1. *Let A be an admissible subset of $\mathcal{C}^{2,3}$. Then, a compatible order on $A_{\mathcal{T}}$ always exists and $\pi(A)$ is equal to the product of the elements of $A_{\mathcal{T}}$ taken with respect to a compatible order $\prec$. (In particular, the product depends only on A.) Consequently, every smooth permutation may be written as the product, in an appropriate order, of the transpositions in its 2-table (each appearing exactly once).*

More precisely, we define a graph $\mathcal{G}_A$ whose vertices are the compatible orders on $A_{\mathcal{T}}$ and whose edges connect two compatible orders that can be obtained from one another by one of the following elementary operations.

1. Interchanging the order of two adjacent commuting transpositions, or
2. Switching the order of consecutive $T_{i,j}$, $T_{i,k}$, $T_{j,k}$ to $T_{j,k}$, $T_{i,k}$, $T_{i,j}$, or vice versa.

These operations do not change the product of the elements of $A_{\mathcal{T}}$, taken in the respective orders. We show that $\mathcal{G}_A$ is connected (and in particular, non-empty). In other words, every two compatible orders are obtained from one another by a sequence of elementary operations. The situation is reminiscent of the case of reduced decompositions of a permutation w, which form the vertices of a connected graph $G(w)$ whose edges are given by basic Coxeter relations. In fact, for $A = \mathcal{C}^{2,3}$ itself, there is a natural isomorphism between $\mathcal{G}_A$ and $G(w_0)$ where w_0 is the longest permutation [22]. However, for a general smooth permutation w, the number of compatible orders on $\mathbf{C}_{\mathcal{T}}(w)$ with respect to $\mathbf{C}(w)$ does not agree with the number of reduced decompositions of w, which is given by a well-known formula of Stanley [20].

4 Intersection of Bruhat intervals with conjugates of parabolic subgroups

As an application of Theorem 2.1, consider an arbitrary partition X (i.e., an equivalence relation) of $[n]$ and the subgroup S_X of S_n preserving all subsets of X. The group S_X is isomorphic to the direct product of $S_{\#y}$ over $y \in X$. However, the product order on S_X (which we denote by $\leq_X$) is in general stronger than the one induced from S_n. We say that an element of S_X is *X-smooth* if all its coordinates in $S_{\#y}$, $y \in X$ are smooth. This condition is weaker than smoothness in S_n. For instance, if X is the partition $\{\{1,3\},\{2,4\}\}$ then the permutation (3412) is X-smooth but not smooth.

Theorem 4.1. *$w \in S_n$ is smooth if and only if for every partition X of $[n]$, the set*

$$\{x \in S_X : x \leq w\}$$

admits a maximum w_X with respect to $\leq_X$. Moreover, in this case w_X is X-smooth.

5 Relation to Dyck paths

We may also interpret the bijection of Theorem 2.1 in terms of more familiar combinatorial objects, namely Dyck paths. We may view a Dyck path as a weakly increasing function $f : [n] \to [n]$ such that $f(i) \geq i$ for all i. Suppose that in addition to f, we are given a function $g : [n] \to \{0,1\}$ such that

1. $g(i) = 0$ whenever $f(f(i)) = f(i)$.

2. $g(i) = g(i+1)$ whenever $i < n$ and $f(i+1) < f(f(i))$.

In this case we say that (f,g) is a *good pair*. Write $g^{-1}(0) = \{i_1,\dots,i_k\}$ and $g^{-1}(1) = \{j_1,\dots,j_l\}$ with $i_1 < \dots < i_k$ and $j_1 < \dots < j_l$.

For every $1 \le i < j \le n$, let $C_{i \to j} \in S_n$ be the cycle permutation $i \to i+1 \to \cdots \to j \to i$ and let $C_{i \leftarrow j} = C_{i \to j}^{-1}$.

Theorem 5.1. *The map*

$$(f, g) \to w(f, g) = C_{j_1 \leftarrow f(j_1)} \cdots C_{j_l \leftarrow f(j_l)} C_{i_k \to f(i_k)} \cdots C_{i_1 \to f(i_1)} \tag{5.1}$$

is a bijection between good pairs and the smooth permutations in S_n. The inverse is given by $w \mapsto (f, g)$, where for every $i \in [n]$,

$$f(i) = \max\Big(\{i\} \cup \{j > i : T_{i,j} \in \mathbf{C}(w)\}\Big)$$

$$g(i) = \begin{cases} 1 & \text{if } i < f(i) \text{ and } R_{i,f(i),f(i)+1} \in \mathbf{C}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the expression on the right-hand side of (5.1) is reduced. Finally,

$$\begin{aligned} \mathbf{C}(w(f,g)) = & \{T_{i,j} : i < j \le f(i)\} \cup \{R_{i,j,k}, R_{i,j,k}^{-1} : i < j < k \le f(i)\} \\ & \bigcup \{R_{i,j,k} : i < j \le f(i) < k \le f(j),\ g(i) = 1\} \\ & \bigcup \{R_{i,j,k}^{-1} : i < j \le f(i) < k \le f(j),\ g(i) = 0\}. \end{aligned}$$

Theorem 5.1 is in the spirit of Skandera's factorization of smooth permutation [19]. Using Theorem 5.1, we can recover several known enumerative results concerning smooth permutations [4, 5, 9, 18, 21].

6 Relation to covexillary permutations

Using Theorem 2.1, we can also give an interesting relation between smooth permutations and *covexillary* ones. Recall that a permutation is called covexillary if it avoids the pattern 3412.

Theorem 6.1. *For any covexillary $x \in S_n$, $\mathbf{C}(x)$ is admissible. Therefore, the map $x \mapsto \pi(\mathbf{C}(x))$ is an idempotent function from the set of covexillary permutations onto the subset of smooth permutations. Moreover, this map is order preserving and for any covexillary $x \in S_n$,*

$$\pi(\mathbf{C}(x)) = \min\{w \in S_n \text{ smooth} : w \ge x\}.$$

7 Relation to coessential set

In [10] Fulton introduced the notion of the *essential set* of a permutation $w \in S_n$. For our purpose it is more convenient to use the following slight variant:

$$\mathcal{E}(w) = \{(i,j) \in [n-1] \times [n-1] : w(i) \le j < w(i+1) \text{ and } w^{-1}(j) \le i < w^{-1}(j+1)\}.$$

For any $w \in S_n$ we have

$$\forall x \in S_n,\ x \le w \iff r_x(i,j) \ge r_w(i,j) \text{ for all } (i,j) \in \mathcal{E}(w).$$

Moreover, the set $\mathcal{E}(w)$ is minimal with respect to this property.

In particular, w is defined by the set $\mathcal{E}(w)$ and the restriction of r_w to $\mathcal{E}(w)$. The image of the injective map

$$w \in S_n \mapsto (\mathcal{E}(w), r_w|_{\mathcal{E}(w)})$$

was described in [8], extending Fulton's result in the covexillary case.

We say that w is *defined by inclusion* if $r_w(i,j) = \min(i,j)$ (i.e., if $w([i]) \subseteq [j]$ or $[j] \subseteq w([i])$) for all $(i,j) \in \mathcal{E}(w)$. It was proved by Gasharov–Reiner that w is defined by inclusion if and only if w is 4231, 35142, 42513 and 351624 avoiding [11]. In particular, w is smooth if and only if w is covexillary and defined by inclusions.

In general, consider the subset

$$\mathcal{E}^\circ(w) = \{(i,j) \in \mathcal{E}(w) : w([i]) \subseteq [j] \text{ or } [j] \subseteq w([i])\}.$$

Thus, w is defined by inclusion if and only if $\mathcal{E}(w) = \mathcal{E}^\circ(w)$, in which case w is determined by the set $\mathcal{E}(w)$. In particular, this is the case if w is smooth.

Note that for any $w \in S_n$, the 2-3-table $\mathbf{C}(w)$ is determined by the set $\mathcal{E}^\circ(w)$. More precisely, we have

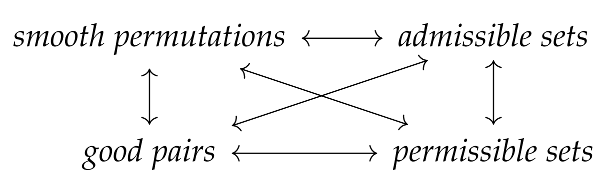
$$\begin{aligned}
T_{i,j} \in \mathbf{C}(w) &\iff \mathcal{E}^\circ(w) \cap \big([i,j) \times [i,j)\big) = \emptyset,\\
R_{i,j,k} \in \mathbf{C}(w) &\iff \mathcal{E}^\circ(w) \cap \big([i,j) \times [i,j)\big) = \mathcal{E}^\circ(w) \cap \big([j,k) \times [i,k)\big) = \emptyset,\\
R^{-1}_{i,j,k} \in \mathbf{C}(w) &\iff \mathcal{E}^\circ(w) \cap \big([i,j) \times [i,k)\big) = \mathcal{E}^\circ(w) \cap \big([j,k) \times [j,k)\big) = \emptyset.
\end{aligned}$$

We say that a subset E of $[n-1] \times [n-1]$ is *permissible* if for every two distinct points (i_1, j_1) and (i_2, j_2) in E such that $\min(i_2, j_2) \ge \min(i_1, j_1)$ we have

$$i_2 \ge i_1,\ \ j_2 \ge j_1,\ \ \max(i_2, j_2) > \max(i_1, j_1) \text{ and } \min(i_2, j_2) > \min(i_1, j_1).$$

It is easy to see that $\mathcal{E}^\circ(w)$ is permissible for every covexillary $w \in S_n$.

Theorem 7.1. *We have a commutative diagram of bijections*



that is compatible with those of Theorems 2.1 and 5.1. The good pair corresponding to a permissible set E is given by

$$f(k) = \min\Big(\{n\} \cup \{\max(i,j) : (i,j) \in E,\ i,j \ge k\}\Big),$$
$$g(k) = \begin{cases} 1 & \text{if } j < f(j) \text{ and } (j, f(j)) \in E, \text{ where } j = \max f^{-1}(f(k)), \\ 0 & \text{otherwise.} \end{cases}$$

The permissible set corresponding to a good pair (f,g) is

$$\{(i, f(i)) : i \in [n-1],\ f(i+1) > f(i) \text{ and } g(i) = 1\}$$
$$\bigcup \{(f(i), i) : i \in [n-1],\ f(i+1) > f(i) \text{ and } g(i) = 0\}.$$

The admissible set corresponding to a permissible set E is

$$\{T_{i,j} : E \cap \big([i,j) \times [i,j)\big) = \emptyset\}$$
$$\bigcup \{R_{i,j,k} : E \cap \big([i,j) \times [i,j)\big) = E \cap \big([j,k) \times [i,k)\big) = \emptyset\}$$
$$\bigcup \{R^{-1}_{i,j,k} : E \cap \big([i,j) \times [i,k)\big) = E \cap \big([j,k) \times [j,k)\big) = \emptyset\}.$$

The permissible set corresponding to an admissible set A is

$$\{(i,i) : i < n,\ T_{i,i+1} \notin A\}$$
$$\bigcup \{(i,j) : i < j < n,\ T_{i,j}, T_{i+1,j+1} \in A,\ T_{i,j+1} \notin A,\ R_{i,j,j+1} \in A\}$$
$$\bigcup \{(i,j) : j < i < n,\ T_{j,i}, T_{j+1,i+1} \in A,\ T_{j,i+1} \notin A,\ R^{-1}_{j,i,i+1} \in A\}.$$

The permissible set corresponding to a smooth permutation w is $\mathcal{E}^\circ(w)$.

Finally, we can relate Theorems 6.1 and 7.1 as follows.

Theorem 7.2. *For any covexillary $x \in S_n$ we have $\mathcal{E}(\pi(\mathbf{C}(x))) = \mathcal{E}^\circ(x)$.*

8 Odds and ends

Theorem 4.1 was the original motivation of this work. It came up in studying a related problem, which is discussed in [16]. The result of [ibid.] is relevant for a certain representation-theoretic context. We hope that the same will be true for Theorem 4.1 and its variants, although we will not discuss these possible applications here.

Likewise, it would be interesting to find a geometric context for Theorems 2.1 and 4.1.

It is natural to ask whether Theorem 3.1 admits an analogue for other Weyl groups W. In particular, one may ask whether any smooth element w of W can be written as the product (in a suitable order) of the reflections that are smaller than or equal to w in the Bruhat order (each reflection occurring exactly once).

Acknowledgements

We are very grateful to the anonymous referee for making several important suggestions. In particular, the possible relation to the essential set was suggested to us by her.

References

[1] H. Abe and S. Billey. "Consequences of the Lakshmibai-Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry". Adv. Stud. Pure Math. **71** (2016), pp. 1–52.

[2] S. Billey and V. Lakshmibai. *Singular loci of Schubert varieties*. Vol. 182. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2000, pp. xii+251. Link.

[3] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. New York: Springer, 2005, pp. xiv+363.

[4] M. Bóna. "The permutation classes equinumerous to the smooth class". *Electron. J. Combin.* **5** (1998), Research Paper 31, 12. Link.

[5] M. Bousquet-Mélou and S. Butler. "Forest-like permutations". *Ann. Comb.* **11**.3-4 (2007), pp. 335–354. Link.

[6] V. V. Deodhar. "Local Poincaré duality and nonsingularity of Schubert varieties". *Comm. Algebra* **13**.6 (1985), pp. 1379–1388. Link.

[7] M. J. Dyer. "Hecke algebras and shellings of Bruhat intervals". *Compositio Math.* **89**.1 (1993), pp. 91–115. Link.

[8] K. Eriksson and S. Linusson. "Combinatorics of Fulton's essential set". *Duke Math. J.* **85**.1 (1996), pp. 61–76. Link.

[9] C. K. Fan. "Schubert varieties and short braidedness". *Transform. Groups* **3**.1 (1998), pp. 51–56. Link.

[10] W. Fulton. "Flags, Schubert polynomials, degeneracy loci, and determinantal formulas". *Duke Math. J.* **65**.3 (1992), pp. 381–420. Link.

[11] V. Gasharov and V. Reiner. "Cohomology of smooth Schubert varieties in partial flag manifolds". *J. London Math. Soc. (2)* **66**.3 (2002), pp. 550–562. Link.

[12] S. Gilboa and E. Lapid. "Some combinatorial results on smooth permutations". Preprint, `arXiv:1912.04725`. J. Combin. (to appear).

[13] D. Kazhdan and G. Lusztig. "Representations of Coxeter groups and Hecke algebras". *Invent. Math.* **53**.2 (1979), pp. 165–184. Link.

[14] V. Lakshmibai and B. Sandhya. "Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$". *Proc. Indian Acad. Sci. Math. Sci.* **100**.1 (1990), pp. 45–52. Link.

[15] V. Lakshmibai and C. S. Seshadri. "Singular locus of a Schubert variety". *Bull. Amer. Math. Soc. (N.S.)* **11**.2 (1984), pp. 363–366. Link.

[16] E. Lapid. "A tightness property of relatively smooth permutations". *J. Combin. Theory Ser. A* **163** (2019), pp. 59–84. Link.

[17] E. Lapid and A. Mínguez. "Geometric conditions for □-irreducibility of certain representations of the general linear group over a non-archimedean local field". *Adv. Math.* **339** (2018), pp. 113–190. Link.

[18] E. Richmond and W. Slofstra. "Staircase diagrams and enumeration of smooth Schubert varieties". *J. Combin. Theory Ser. A* **150** (2017), pp. 328–376. Link.

[19] M. Skandera. "On the dual canonical and Kazhdan-Lusztig bases and 3412-, 4231-avoiding permutations". *J. Pure Appl. Algebra* **212**.5 (2008), pp. 1086–1104. Link.

[20] R. P. Stanley. "On the number of reduced decompositions of elements of Coxeter groups". *European J. Combin.* **5**.4 (1984), pp. 359–372. Link.

[21] J. West. "Generating trees and forbidden subsequences". *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*. Vol. 157. 1-3. 1996, pp. 363–374. Link.

[22] D. P. Zhelobenko. "Extremal cocycles on Weyl groups". *Funktsional. Anal. i Prilozhen.* **21**.3 (1987), pp. 11–21, 95.