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Cubillages in odd dimensions

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Abstract. In this abstract we discuss novel results on fine zonotopal tilings (called "cubillages" for short) in odd-dimensional cyclic zonotopes and their relations to generalized weakly separated set-systems, triangulations of cyclic polytopes, and others.

Keywords: cyclic zonotope, cubillage, strong and weak separation, cyclic polytope, triangulation, combined tiling

1 Introduction

For positive integers $n \geq d$, by a *cyclic configuration* of size n in $\mathbb{R}^d$ we mean an ordered set Ξ of n vectors $\xi_i = (\xi_i(1), \dots, \xi_i(d)) \in \mathbb{R}^d$, $i = 1, \dots, n$, satisfying:

(1.1) (a) $\xi_i(1) = 1$ for each i; (b) any flag minor of the $d \times n$ matrix formed by $\xi_1, \dots, \xi_n$ as columns (in this order) is positive; and (c) all 0,1-combinations of these vectors are different.

(A typical sample of such configurations Ξ is generated by Veronese curve: take reals $t_1 < t_2 < \dots < t_n$ and assign $\xi_i := \xi(t_i)$, where $\xi(t) = (1, t, t^2, \dots, t^{d-1})$.)

We deal with fine zonotopal tilings related to Ξ. Recall that the (cyclic) *zonotope* $Z = Z(\Xi)$ generated by Ξ is the Minkowski sum of line segments $[0, \xi_i]$, $i = 1, \dots, n$. Then a *fine zonotopal tiling* is (the polyhedral complex determined by) a subdivision Q of Z into d-dimensional parallelotopes such that: any two intersecting ones share a common face, and each face of the boundary of Z is entirely contained in some of these parallelotopes. For brevity, we refer to these parallelotopes as *cubes*, and to Q as a *cubillage*. Note that the choice of one or another cyclic configuration Ξ (subject to (1.1)) is not important to us in essence, and we will write $Z(n, d)$ rather than $Z(\Xi)$, referring to it as the *cyclic zonotope* with parameters (n, d).

Let $[n]$ denote the set $\{1, 2, \dots, n\}$. Any point v in $Z(n, d)$ occurring as a vertex of a cubillage Q is viewed as $\sum_{i \in X} \xi_i$ for some subset $X \subseteq [n]$ and we identify such v and X. So the set $V(Q)$ of vertices of Q is identified with the corresponding collection (set-system) in $2^{[n]}$, that we call the *spectrum* of Q. It is known that

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(1.2) the size (cardinality) of $V(Q)$ is equal to $\binom{n}{\le d}$ $\left(=\binom{n}{d}+\binom{n}{d-1}+\cdots+\binom{n}{0}\right)$.

The above correspondence possesses a number of nice properties. One of them involves so-called *strongly separated* sets and set-systems. We need some definitions.

For $X,Y\subseteq[n]$, we write $X<Y$ if the maximal element $\max(X)$ of X is smaller than the minimal element $\min(Y)$ of Y, letting $\max(\emptyset):=0$ and $\min(\emptyset):=n+1$. An *interval* of $[n]$ is a subset of the form $\{a,a+1,\dots,b\}$, denoted as $[a,b]$.

Definition 1.1. For $r\in\mathbb{Z}_{\ge 0}$, sets $A,B\subseteq[n]$ are called *strongly r-separated* if there is no sequence $i_1<i_2<\cdots<i_{r+2}$ of elements of $[n]$ such that those with odd indices (namely, $i_1,i_3,\dots$) belong to one of $A-B$ or $B-A$, while those with even indices ($i_2,i_4,\dots$) belong to the other (where $A'-B'$ denotes the set difference $\{i\colon i\in A',\ i\notin B'\}$). Accordingly, a set-system $\mathcal{S}\subseteq 2^{[n]}$ is called r-separated if any two members of $\mathcal{S}$ are such.

In particular, A,B are strongly 1-separated if $\max(A-B)<\min(B-A)$ or $\max(B-A)<\min(A-B)$. This notion was introduced and studied, under the name of "strong separation", by Leclerc and Zelevinsky [7]. The case $r=2$ was studied by Galashin [5]. Extending results in [7, 5] concerning the strong 1- and 2-separation to a general r, Galashin and Postnikov [6] showed that

(1.3) The maximal size $s_{n,r}$ of a strongly r-separated collection in $2^{[n]}$ is equal to $\binom{n}{\le r+1}$; moreover (see (1.2)), for any cubillage Q on $Z(n,d)$, its spectrum $V(Q)$ constitutes a maximal by size strongly $(d-1)$-separated collection in $2^{[n]}$, and conversely, for any size-maximal strongly$(d-1)$-separated collection $\mathcal{S}\subseteq 2^{[n]}$, there exists a cubillage Q on $Z(n,d)$ with $V(Q)=\mathcal{S}$.

(As a more general version of strong r-separation, [6] considers the notion of M-separation in oriented matroids, but this is not needed to us in this paper.)

Another sort of set separation introduced by Leclerc and Zelevinsky is known under the name of *weak separation* (which appeared in [7] in connection with the problem of characterizing quasi-commuting flag minors of a quantum matrix). We generalize that notion to "higher odd dimensions" in the following way. When $A,B\subseteq[n]$ are such that $\min(A-B)<\min(B-A)$ and $\max(A-B)>\max(B-A)$, we say that A *surrounds* B. When A,B are strongly r-separated but not strongly $(r-1)$-separated, they are called $(r+1)$-*intertwined*. In other words, there are intervals $I_1<I_2<\cdots<I_{r'}$ in $[n]$ with $r'=r+1$, but not $r'=r$, such that one of $I_1\cup I_3\cup\dots$ and $I_2\cup I_4\cup\dots$ includes $A-B$, and the other $B-A$; we say that $(I_1,\dots,I_{r'})$ is an *interval cortege* for A,B. For example, $A=\{1,2,5,6,7,10,11\}$ and $B=\{1,3,4,6,9,11\}$ are 5-intertwined (with an interval cortege $(\{2\},[3,4],[5,7],\{9\},\{10\})$) and A surrounds B.

Definition 1.2. Let r be *odd*. Sets $A,B\subseteq[n]$ are called *weakly r-separated* if they are either strongly r-separated, or they are $r+2$-intertwined, and in the latter case, if A surrounds B then $|A|\le|B|$, while if B surrounds A then $|B|\le|A|$. Accordingly, a set-system $\mathcal{W}\subseteq 2^{[n]}$ is called weakly r-separated if any two members of $\mathcal{W}$ are such.

In case $r = 1$, this turns into the weak separation of [7].

Using a machinery of cubillages in cyclic zonotopes of odd dimensions, we generalize, to an arbitrary odd $r \geq 1$, two well-known results on weakly separated collections obtained in [7] and develop a method of constructing a representable class of size-maximal weakly r-separated set-systems. One of those results [7] says that

(1.4) the maximal sizes of strongly and weakly separated collections in $2^{[n]}$ are the same (and equal to $\frac{1}{2}n(n+1)+1 = \binom{n}{2}+\binom{n}{1}+\binom{n}{0}$).

Let $w_{n,r}$ denote the maximal possible size of a weakly r-separated collection in $2^{[n]}$. We generalize (1.4) as follows.

Theorem 1.3. *Let r be odd. Then $w_{n,r} = s_{n,r}$.*

(Note that for even $r > 0$, at present we see no way of defining the notion of weak r-separation ensuring that the maximal size of such collections in $2^{[r+1]}$ does not exceed $s_{n,r}$. So the odd and even cases behave differently. Note also that for an odd $r \geq 3$, a maximal by inclusion weakly r-separated collection need not be maximal by size.)

Another impressive result in [7] says that a weakly separated collection can be transformed into another one by making a *flip* (a sort of mutation) "in the presence of four witnesses". This relies on the following property ([7, Theorem 7.1]):

(1.5) Let $\mathcal{W} \subset 2^{[n]}$ be weakly separated, and suppose that there are elements $i < j < k$ of $[n]$ and a set $X \subseteq [n] - \{i,j,k\}$ such that $\mathcal{W}$ contains four sets ("witnesses") Xi, Xk, Xij, Xjk and a set $U \in \{Xj, Xik\}$; then the collection obtained from $\mathcal{W}$ by replacing U by the other member of $\{Xj, Xik\}$ is again weakly separated.

Hereinafter for disjoint sets A and $\{a,\dots,b\}$, we write $Aa\dots b$ for $A \cup \{a,\dots,b\}$. Also for $a \in A$, we abbreviate $A - \{a\}$ as $A - a$. We generalize (1.5) as follows.

Theorem 1.4. *For an odd r, let $r' := (r+1)/2$. Let $P = \{p_1,\dots,p_{r'}\}$ and $Q = \{q_0,\dots,q_{r'}\}$ consist of elements of $[n]$ such that $q_0 < p_1 < q_1 < p_2 < \dots < p_{r'} < q_{r'}$, and let $X \subseteq [n] - (P \cup Q)$. Define the sets of "upper" and "lower"* neighbors *(or "witnesses") of P, Q to be*

$$\mathcal{N}^{\uparrow}(P,Q) := \{Pq\colon q \in Q\} \cup \{(P-p)q\colon p \in P,\ q \in Q\}; \quad \textit{and} \tag{1.6}$$

$$\mathcal{N}^{\downarrow}(P,Q) := \{Q-q\colon q \in Q\} \cup \{(Q-q)p\colon p \in P,\ q \in Q\}. \tag{1.7}$$

Suppose that a weakly r-separated collection $\mathcal{W} \subset 2^{[n]}$ contains the set $X \cup P$ (resp. $X \cup Q$) and the sets $X \cup S$ for all $S \in \mathcal{N}^{\downarrow}(P,Q)$ (resp. $S \in \mathcal{N}^{\uparrow}(P,Q)$). Then the collection obtained from $\mathcal{W}$ by replacing $X \cup P$ by $X \cup Q$ (resp. $X \cup Q$ by $X \cup P$) is weakly r-separated as well.

The above theorems give rise to an important construction. More precisely, for a cubillage Q in $Z(n,d)$, we introduce a natural *fragmentation* $Q^{\equiv}$ of Q, by cutting each "cube" C of Q by the "horizontal" hyperplanes through the vertices of C, and define a class of $(d-1)$-dimensional subcomplexes M of $Q^{\equiv}$, called *weak membranes*. These membranes form a distributive lattice. Based on Theorem 1.4, we show that if $r := d-2$ is odd, then the vertex set of M has size exactly $w_{n,r}$ and constitutes a weakly r-separated collection in $2^{[n]}$. This gives a plenty of size-maximal weakly r-separated collections associated with Q, and any two collections among these are linked by a sequence of (lowering or raising) "elementary" *flips*.

In this abstract, Section 2 contains additional definitions and reviews some basic facts. Section 3 outlines a proof of Theorem 1.3. The construction of max-size weakly r-separated collections via weak membranes in cubillages is described in Section 4. The concluding Section 5 discusses issues related to the problem of extending a triangulation in a cyclic polytope to a cubillage and raises some conjectures.

The abstract is based on abridged versions of parts of [4] and [2], and some results are also reflected in the survey [3].

2 Preliminaries

This section contains additional definitions, notation and conventions. Also we review some known properties of cubillages. For details, see [4, 3].

- Let π denote the projection $\mathbb{R}^d \to \mathbb{R}^{d-1}$ given by $(x(1),\dots,x(d)) \mapsto (x(1),\dots,x(d-1))$. Due to (1.1)(b), the vectors $\pi(\xi_1),\dots,\pi(\xi_n)$ form a cyclic configuration as well, and we may say that π projects $Z(n,d)$ to the zonotope $Z(n,d-1)$.
- The 0-, 1-, and $(d-1)$-dimensional faces of a cubillage Q in $Z(n,d)$ are called *vertices*, *edges*, and *facets*, respectively. While each vertex is identified with a subset of $[n]$, each edge e is a parallel translation of some segment $[0,\xi_i]$; we say that e has *color* i.
- When a cell (face) C of Q has the lowest point $X \subseteq [n]$ and when $T \subseteq [n]$ is the set of colors of edges in C, we say that C has the *root* X and *type* T, and may write $C = (X \mid T)$. One easily shows that $X \cap T = \emptyset$.
- For a closed subset U of points in $Z = Z(n,d)$, let $U^{\rm fr}$ ($U^{\rm rear}$) be the subset of U "seen" in the direction of the last, d-th, coordinate vector e_d (resp. $-e_d$), i.e., formed by the points $x \in \pi^{-1}(x') \cap U$ with $x(d)$ minimum (resp. maximum) for all $x' \in \pi(U)$. It is called the *front* (resp. *rear*) *side* of U.

In particular, $Z^{\rm fr}$ and $Z^{\rm rear}$ denote the front and rear sides, respectively, of the zonotope Z. We call $Z^{\rm rim} := Z^{\rm fr} \cap Z^{\rm rear}$ the *rim* of Z.

- When a set $X \subseteq [n]$ is the union of k intervals and k is as small as possible, we say that X is a *k-interval*. Then its complementary set $[n]-X$ is a k'-interval with $k' \in \{k-1,k,k+1\}$. We will use the following known characterization of the sets of vertices in the front and rear sides of a zonotope of an odd dimension.

(2.1) Let d be odd. Then for $Z = Z(n,d)$,

(i) $V(Z^{\rm fr})$ is formed by all k-intervals of $[n]$ with $k \le (d-1)/2$; and

(ii) $V(Z^{\rm rear})$ is formed by the subsets of $[n]$ complementary to those in $V(Z^{\rm fr})$; so it consists of all k-intervals with $k < (d-1)/2$, all $(d-1)/2$-intervals containing at least one of the elements 1 and n and all $(d+1)/2$-intervals with both 1 and n.

This implies that the set of *inner* vertices in $Z^{\rm fr}$, i.e., $V(Z^{\rm fr}) - V(Z^{\rm rim})$, consists of the $(d-1)/2$-intervals containing none of 1 and n, whereas $V(Z^{\rm rear}) - V(Z^{\rm rim})$ consists of the $(d+1)/2$-intervals containing both 1 and n.

The rest of this section describes an important class of subcomplexes in a cubillage Q and associate with Q a certain path structure (used in the next section).

Definition 2.1. Let Q be a cubillage in $Z(n,d)$. A *strong membrane*, or, briefly, an *s-membrane*, in Q is a subcomplex M of Q such that M (regarded as a subset of $\mathbb{R}^d$) is *bijectively* projected by π onto $Z(n,d-1)$.

Then each facet of Q occurring in M is projected to a cube of dimension $d-1$ in $Z(n,d-1)$ and these cubes constitute a cubillage in $Z(n,d-1)$, denoted as $\pi(M)$. In view of (1.3) and (1.2) (applied to $\pi(Q)$),

(2.2) all s-membranes M in a cubillage Q in $Z(n,d)$ have $s_{n,d-2}$ vertices, and the vertex set of M (regarded as a collection in $2^{[n]}$) is strongly $(d-2)$-separated.

Two s-membranes are of a particular interest. These are the front side $Z^{\rm fr}$ and the rear side $Z^{\rm rear}$ of $Z = Z(n,d)$. Following terminology in [2, 3], their projections $\pi(Z^{\rm fr})$ and $\pi(Z^{\rm rear})$ are called the *standard* and *anti-standard* cubillages in $Z(n,d-1)$, respectively.

Next we distinguish certain vertices in cubes. When $n = d$, the zonotope turns into the cube $C = (\emptyset|[d])$, and there holds:

(2.3) the front side $C^{\rm fr}$ (rear side $C^{\rm rear}$) of $C = (\emptyset|[d])$ has a unique inner vertex, namely, $t_C := \{i \in [n] \colon d-i \text{ odd}\}$ (resp. $h_C := \{i \in [n] \colon d-i \text{ even}\}$.

When n is arbitrary and Q is a cubillage in $Z = Z(n,d)$, we distinguish vertices t_C and h_C of a cube $C(X\,|\,T)$ with $T = (p_1 < \ldots < p_d)$ in Q in a similar way; namely,

(2.4) $\quad t_C = X \cup \{p_i \colon d-i \text{ odd}\}$ and $h_C = X \cup \{p_i \colon d-i \text{ even}\}$.

Note that for each vertex v of Q, unless v is in $Z^{\rm rear}$, there is a unique cube $C \in Q$ such that $t_C = v$, and symmetrically, unless v is in $Z^{\rm fr}$, there is a unique cube $C \in Q$ such that $h_C = v$ (to see this, consider the line going through v and parallel to e_d).

Therefore, by drawing for each cube $C \in Q$, the edge-arrow from t_C to h_C, we obtain a directed graph whose connected components are directed paths going from $Z^{\rm fr} - Z^{\rm rim}$ to $Z^{\rm rear} - Z^{\rm rim}$. We call these paths *bead-threads* in Q. It is convenient to add to this graph the elements of $V(Z^{\rm rim})$ as isolated vertices, forming *degenerate* bead-threads, each going from a vertex to itself. Let B_Q be the resulting directed graph. Then

(2.5) B_Q contains all vertices of Q, and each component of B_Q is a bead-thread going from $Z^{\rm fr}$ to $Z^{\rm rear}$.

Note that the heights $|X|$ of vertices X along a bead-thread are monotone increasing when d is odd (whereas they are constant when d is even).

3 Proof of Theorem 1.3

Let r be odd and $n > r$. We have to show that

(3.1) if $\mathcal{W}$ is a weakly r-separated collection in $2^{[n]}$, then $|\mathcal{W}| \le \binom{n}{\le r+1}$.

This is valid when $r = 1$ (see (1.4)) and is trivial when $n = r+1$. So one may assume that $3 \le r \le n-2$. We prove (3.1) by induction, assuming that the corresponding inequality holds for $\mathcal{W}', n', r'$ when $n' \le n$, $r' \le r$, and $(n', r') \ne (n, r)$. Define the following subcollections in $\mathcal{W}$:

$$\begin{array}{rcl} \mathcal{W}^- & := & \{A \subseteq [n-1] : \{A, An\} \cap \mathcal{W} \ne \emptyset\}, \quad \text{and} \\ \mathcal{T} & := & \{A \subseteq [n-1] : \{A, An\} \subseteq \mathcal{W}\}, \end{array}$$

One easily shows that $\mathcal{W}^-$ is weakly r-separated. Then by induction, $|\mathcal{W}^-| \le \binom{n-1}{\le r+1}$. Also $|\mathcal{W}| = |\mathcal{W}^-| + |\mathcal{T}|$. Therefore, in view of the identity $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$ for any $j \le n-1$, it suffices to show that

$$|\mathcal{T}| \le \binom{n-1}{\le r}. \tag{3.2}$$

For $i = 0, 1, \ldots n-1$, define $\mathcal{T}^i := \{A \in \mathcal{T} : |A| = i\}$. We rely on two claims.

Claim 1 *For each i, the collection $\mathcal{T}^i$ is strongly $(r-1)$-separated; moreover, $\mathcal{T}^i$ is weakly $(r-2)$-separated.*

Proof. Let $A, B \in \mathcal{T}^i$. Take an interval cortege $(I_1, \ldots, I_{r'})$ for A, B, and we may assume that $I_{r'} \cap (A - B) \ne \emptyset$. Then $(I_1, \ldots, I_{r'}, I_{r'+1} := \{n\})$ is an interval cortege for A and $B' := Bn$. Since $|A| < |B'|$ and $\max(A - B') < \max(B' - A) = n$, and since A, B' are weakly r-separated, $r' + 1 < r + 2$. Then $r' \le r$, implying that A, B are $(r-1)$-separated. Since $|A| = |B|$ and r is odd, we also can conclude that A, B are weakly $(r-2)$-separated. □

Now consider the zonotope $Z = Z(n-1, r)$. For $j = 0, 1, \ldots, n-1$, define $\mathcal{S}^j$ ($\mathcal{A}^j$) to be the set of vertices X of $Z^{\rm fr}$ (resp. $Z^{\rm rear}$) with $|X| = j$. We extend each collection $\mathcal{T}^i$ to

$$\mathcal{D}^i := \mathcal{T}^i \cup (\mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1}) \cup (\mathcal{A}^0 \cup \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^{i-1}). \tag{3.3}$$

Claim 2 *$\mathcal{D}^i$ is weakly $(r-2)$-separated.*

Proof. The vertex sets of Z^{fr} and $\pi(Z^{\text{fr}})$ are essentially the same (regarding a vertex as a subset of $[n-1]$), and similarly for Z^{rear} and $\pi(Z^{\text{rear}})$. Since $\pi(Z^{\text{fr}})$ and $\pi(Z^{\text{rear}})$ are cubillages on $Z(n-1,r-1)$ (the so-called "standard" and "anti-standard" ones), (1.3) implies that both collections $V(Z^{\text{fr}}) = \mathcal{S}^0 \cup \ldots \cup \mathcal{S}^{n-1}$ and $V(Z^{\text{rear}}) = \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{n-1}$ are $(r-2)$-separated, and therefore, they are weakly $(r-2)$-separated as well.

Next, by (2.1)(i), each vertex X of Z^{fr} is a k-interval with $k \le (r-1)/2$. Such an X and any subset $Y \subseteq [n-1]$ are k'-intertwined with $k' \le 2k+1$. Then $k' \le r$ and this holds with equality when X and Y are r-intertwined and Y surrounds X. It follows that X is weakly $(r-2)$-separated from any $Y \subseteq [n-1]$ with $|Y| \le |X|$ (in particular, if $X \in \mathcal{S}^j$ and $j \ge i$, then X is weakly $(r-2)$-separated from each member of $\mathcal{T}^i \cup \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{i-1}$).

Symmetrically, by (2.1)(ii), each vertex X of Z^{rear} is the complement to $[n-1]$ of a k-interval with $k \le (r-1)/2$. We can conclude that such an X is weakly $(r-2)$-separated from any $Y \subseteq [n-1]$ with $|Y| \ge |X|$.

Now the result is provided by the inequalities $|X| > |A| > |X'|$ for any $A \in \mathcal{T}^i$, $X \in \mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1}$, and $X' \in \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{i-1}$. □

By induction, $|\mathcal{D}^i| \le \binom{n-1}{\le r-1}$. Then, using (2.2) (for $n-1$ and $r-2$), we have

$$|\mathcal{D}^i| \le \tbinom{n-1}{\le r-1} = s_{n-1,r-2} = |V(Z^{\text{fr}})|. \tag{3.4}$$

Let $\mathcal{S}' := \mathcal{S}^0 \cup \mathcal{S}^1 \cup \ldots \cup \mathcal{S}^i$ and $\mathcal{A}' := \mathcal{A}^0 \cup \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^{i-1}$. Since $\mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1} = V(Z^{\text{fr}}) - \mathcal{S}'$, we obtain from (3.3) and (3.4) that

$$|\mathcal{T}^i| = |\mathcal{D}^i| - (|V(Z^{\text{fr}}) - \mathcal{S}'|) - |\mathcal{A}'| \le |\mathcal{S}'| - |\mathcal{A}'|. \tag{3.5}$$

We now finish the proof by using a bead-thread technique (see Section 2). Fix an arbitrary cubillage Q in $Z = Z(n-1,r)$. Let $\mathcal{R}^i$ be the set of vertices X of Q with $|X| = i$, and let $\mathcal{B}$ be the set of paths in the graph B_Q beginning at Z^{fr} and ending at Z^{rear}. Since r is odd, each edge (X,Y) of B_Q is "ascending" (satisfies $|Y| > |X|$). This implies that each path $P \in \mathcal{P}$ beginning at $\mathcal{S}'$ must meet either $\mathcal{R}^i$ or $\mathcal{A}'$, and conversely, each path meeting $\mathcal{R}^i \cup \mathcal{A}'$ begins at $\mathcal{S}'$. This and (3.5) imply $|\mathcal{T}^i| \le |\mathcal{R}^i|$. Summing up these inequalities for $i = 0, 1, \ldots, n-1$, we have

$$|\mathcal{T}| = \textstyle\sum_i |\mathcal{T}^i| \le \sum_i |\mathcal{R}^i| = |V_Q| = s_{n-1,r-1} = \binom{n-1}{\le r},$$

yielding (3.2) and completing the proof of Theorem 1.3.

4 Weakly r-separated collections generated by cubillages

We have seen an interrelation between strongly $*$-separated collections on the one hand, and cubillages and s-membranes on the other hand (see (1.3) and (2.2)). This section is devoted to geometric aspects of the weak r-separation when r is odd. Being motivated by geometric constructions for maximal weakly 1-separated collections elaborated in [1, 2], we explain how to construct maximal by size weakly r-separated collections by use of *weak membranes*, which are analogs of s-membranes in *fragmentations* of cubillages.

4.1 Fragmentation and weak membranes.

Let Q be a cubillage in $Z(n,d)$. The *fragmentation* of Q is the complex $Q^{\equiv}$ obtained by cutting Q by the "horizontal" hyperplanes $H_\ell := \{x \in \mathbb{R}^d : x(1) = \ell\}$, $\ell = 1, \dots, n-1$.

Such hyperplanes subdivide each cube $C = (X \,|\, T)$ of Q into pieces $C_1^{\equiv}, \dots, C_d^{\equiv}$, where $C_h^{\equiv}$ is the portion of C between $H_{|X|+h-1}$ and $H_{|X|+h}$, called a *fragment* of C (and of $Q^{\equiv}$). Let $S_h(C)$ denote h-th horizontal section $C \cap H_{|X|+h}$ of C; this is the convex hull of the set of vertices $(X \,|\, \binom{T}{h}) := \{X \cup A : A \subset T, |A| = h\}$ (forming a *hyper-simplex* and turning into a simplex when $h = 1$ or $d-1$). We call $S_{h-1}(C)$ and $S_h(C)$ the *lower* and *upper* (horizontal) facets of the fragment $C_h^{\equiv}$, respectively. (Here $S_0(C)$ and $S_d(C)$ degenerate to the single points X and $X \cup T$, respectively.) The other facets of $C_h^{\equiv}$ are conditionally called *vertical* ones.

Note that the horizontal facets are "not fully seen" under the projection π. To make all facets of fragments of $Q^{\equiv}$ visible, we look at them as though "from the front and slightly from below", i.e., by using the projection $\pi^\epsilon : \mathbb{R}^d \to \mathbb{R}^{d-1}$ defined by

$$x = (x(1), \dots, x(d)) \mapsto (x(1) - \epsilon x(d), x(2), \dots, x(d-1)) =: \pi^\epsilon(x) \tag{4.1}$$

for a sufficiently small $\epsilon > 0$. (Compare π^ϵ with π.)

This projection makes slanting front and rear sides of objects in $Q^{\equiv}$. More precisely, for a closed set U of points in $Z = Z(n,d)$, let $U^{\epsilon,\mathrm{fr}}$ ($U^{\epsilon,\mathrm{rear}}$) be the subset of U formed by the points $x \in (\pi^\epsilon)^{-1}(x') \cap U$ with $x(d)$ minimum (resp. maximum) for all $x' \in \pi^\epsilon(U)$. We call it the *ϵ-front* (resp. *ϵ-rear*) *side* of U.

Obviously, $Z^{\epsilon,\mathrm{fr}} = Z^{\mathrm{fr}}$ and $Z^{\epsilon,\mathrm{rear}} = Z^{\mathrm{rear}}$, and similarly for any cube $C = (X|T)$ in Z. As to fragments of C, their ϵ-front and ϵ-rear sides are viewed as follows:

(4.2) for $h = 1, \dots, d$, $C_h^{\epsilon,\mathrm{fr}} = C_h^{\mathrm{fr}} \cup S_{h-1}(C)$ and $C_h^{\epsilon,\mathrm{rear}} = C_h^{\mathrm{rear}} \cup S_h(C)$.

So $C_h^{\epsilon,\mathrm{fr}} \cup C_h^{\epsilon,\mathrm{rear}}$ is just the boundary of $C_h^{\equiv}$.

Next we explain the notion of weak membranes. They represent certain $(d-1)$-dimensional subcomplexes of the fragmentation $Q^{\equiv}$ of Q and use the projection π^ϵ (in contrast to strong membranes which deal with Q and π).

To introduce them, we slightly modify cyclic zonotopes in $\mathbb{R}^{d-1}$. Specifically, given a cyclic configuration $\Xi = (\xi_1, \dots, \xi_n)$ as in (1.1), define $\psi_i^\epsilon := \pi^\epsilon(\xi_i)$, $i = 1, \dots, n$. When ϵ is small enough, $\Psi^\epsilon = (\psi_1^\epsilon, \dots, \psi_n^\epsilon)$ obeys the condition (1.1)(b), though slightly violates (1.1)(a). Yet we keep the term "cyclic configuration" for Ψ^ϵ as well, and consider the zonotope in $\mathbb{R}^{d-1}$ generated by Ψ^ϵ, denoted as $Z^\epsilon(n, d-1)$.

Definition 4.1. A *weak membrane*, or, briefly, a *w-membrane*, of a cubillage Q in $Z(n,d)$ is a subcomplex M of the fragmentation $Q^{\equiv}$ such that M (regarded as a subset of $\mathbb{R}^d$) is bijectively projected by π^ϵ to $Z^\epsilon(n,d-1)$.

A w-membrane M uses facets of fragments in $Q^{\equiv}$ which are of two sorts, namely, "horizontal" and "vertical" ones as mentioned above. The set $\mathcal{M}^{\rm w}(Q)$ of w-membranes of Q is rich and forms a distributive lattice. To see this, for fragments $\Delta = C_i^{\equiv}$ and $\Delta' = (C')_j^{\equiv}$ of $Q^{\equiv}$, let us say that Δ *immediately precedes* Δ' if the ϵ-rear side of Δ and the ϵ-front side of Δ' share a facet. In other words, either $C \ne C'$ and $\Delta^{\rm rear} \cap (\Delta')^{\rm fr}$ is a vertical facet, or $C = C'$ and $j = i+1$. A nice property of this relation is that the directed graph whose vertices are the fragments in $Q^{\equiv}$ and whose edges are the pairs (Δ, Δ') of fragments such that Δ immediately precedes Δ' is acyclic (see [4, 3]).

It follows that the transitive closure of this relation forms a partial order on the fragments of $Q^{\equiv}$; denote it as $(Q^{\equiv}, \prec)$. To see that it is a lattice, associate with each w-membrane M the set $Q^{\equiv}(M)$ of fragments in $Q^{\equiv}$ lying in the region of $Z(n,d)$ between $Z^{\rm fr}$ and M. One easily shows that for fragments Δ, Δ' of $Q^{\equiv}$, if Δ immediately precedes Δ' and if $\Delta' \in Q^{\equiv}(M)$, then $\Delta \in Q^{\equiv}(M)$ as well. This implies a similar property for fragments Δ, Δ' with $\Delta \prec \Delta'$. So $Q^{\equiv}(M)$ is an ideal of $(Q^{\equiv}, \prec)$. A converse property is true as well. Thus,

(4.3) $\mathcal{M}^{\rm w}(Q)$ is a distributive lattice in which for $M, M' \in \mathcal{M}^{\rm w}(Q)$, the w-membranes $M \wedge M'$ and $M \vee M'$ satisfy $Q^{\equiv}(M \wedge M') = Q^{\equiv}(M) \cap Q^{\equiv}(M')$ and $Q^{\equiv}(M \vee M') = Q^{\equiv}(M) \cup Q^{\equiv}(M')$; the minimal and maximal elements of this lattice are the s-membranes $Z^{\rm fr}$ and $Z^{\rm rear}$, respectively.

Next, if $M \in \mathcal{M}^{\rm w}(Q)$ is different from $Z^{\rm fr}$, then $Q^{\equiv}(M) \ne \emptyset$. Take a maximal (w.r.t. $\prec$) fragment Δ in $Q^{\equiv}(M)$. Then $\Delta^{\epsilon,\rm rear}$ is entirely contained in M and the set $Q^{\equiv}(M) - \{\Delta\}$ is again an ideal of $(Q^{\equiv}, \prec)$; so it is expressed as $Q^{\equiv}(M')$ for a w-membrane M'. Moreover, M' is obtained from M by replacing the disk $\Delta^{\epsilon,\rm rear}$ by $\Delta^{\epsilon,\rm fr}$. We call the transformation $M \mapsto M'$ the *lowering flip* in M *using* Δ, and call the reverse transformation $M' \mapsto M$ the *raising flip* in M' *using* Δ. As a result, we obtain that

(4.4) for any $M \in \mathcal{M}^{\rm w}(Q)$, there exists a sequence of w-membranes $M_0, M_1, \dots, M_k \in \mathcal{M}^{\rm w}(Q)$ such that $M_0 = Z^{\rm fr}$, $M_k = M$, and for $i = 1, \dots, k$, M_i is obtained from M_{i-1} by the raising flip using some fragment in $Q^{\equiv}$.

4.2 Weakly r-separated collections via w-membranes.

Based on Theorem 1.4 (see [4, Section 5] for the proof), we establish the following

Theorem 4.2. *Let r be odd and $d = r+2$. For each w-membrane M of a cubillage Q in $Z = Z(n,d)$, its spectrum $V(M)$ has size $w_{n,r}$ and constitutes a maximal by size weakly r-separated collection in $2^{[n]}$.*

Proof. For $M \in \mathcal{M}^{\rm w}(Q)$, consider a sequence $Z^{\rm fr} = M_0, M_1, \dots, M_k = M$ as in (4.4). Let M_i ($i > 0$) be obtained from M_{i-1} by the raising flip using a fragment Δ_i of $Q^{\equiv}$. Since $V(Z^{\rm fr})$ is strongly r-separated and $V(Z^{\rm fr}) = s_{n,r} = w_{n,r}$ (see (2.2)), it suffices to show that if $V(M_{i-1})$ has size $w_{n,r}$ and is weakly r-separated, then so is $V(M_i)$.

To show this, let $\Delta := \Delta_i = C_h^{\equiv}$ for a cube $C = (X \,|\, T = (p(1) < \dots < p(d)))$ and $h \in [d]$. Then $V(C^{\rm fr}) = V(C^{\rm rim}) \cup \{t_C\}$ and $V(C^{\rm rear}) = V(C^{\rm rim}) \cup \{h_C\}$, where $t_C = Xp(2)p(4)\dots p(d-1)$ and $h_C = Xp(1)p(3)\dots p(d)$ (see (2.3)). Let R be the set of vertices in $C^{\rm rim} \cap \Delta$, and let $r' := (d-1)/2$. Then r' is an integer, t_C lies in the section $S_{r'}(C)$, and h_C lies in $S_{r'+1}(C)$. Three cases are possible.

Case 1: $h \le r'$. Since the vertices of Δ are formed by the sections $S_{h-1}(C)$ and $S_h(C)$,

$$V(\Delta) = (X \,|\, \tbinom{T}{h-1}) \cup (X \,|\, \tbinom{T}{h}) \quad \text{and} \quad R \subseteq V(\Delta^{\rm fr}) \cup V(\Delta^{\rm rear}).$$

Also $V(\Delta^{\rm fr}) \subseteq V(\Delta^{\epsilon,\rm fr})$ and $V(\Delta^{\rm rear}) \subseteq V(\Delta^{\epsilon,\rm rear})$. When $h < r'$, all vertices of Δ belong to $C^{\rm rim}$, implying $V(\Delta^{\epsilon,\rm fr}) = R = V(\Delta^{\epsilon,\rm rear})$. And when $h = r'$, the only vertex of Δ not in R is t_C. Since $t_C \in V(C^{\rm fr})$, t_C belongs to $\Delta^{\epsilon,\rm fr}$. But t_C also lies in the upper facet $S_{r'}(C)$, and this facet is included in $\Delta^{\epsilon,\rm rear}$. Hence $t_C \in \Delta^{\epsilon,\rm fr} \cap \Delta^{\epsilon,\rm rear}$, implying $V(\Delta^{\epsilon,\rm fr}) = V(\Delta^{\epsilon,\rm rear})$.

Case 2: $h \ge r' + 2$. This is "symmetric" to the previous case.

Thus, in both cases the raising flip $M \mapsto M'$ using Δ gives $V(M) = V(M')$.

Case 3: $h = r' + 1$. This case is most important. Here the lower facet $S_{h-1=r'}(C)$ of Δ contains t_C, and the upper facet $S_{h=r'+1}(C)$ contains h_C. Hence $t_C \in V(\Delta^{\epsilon,\rm fr})$ and $h_C \in V(\Delta^{\epsilon,\rm rear})$. On the other hand, neither t_C belongs to $\Delta^{\epsilon,\rm rear}$ $(= \Delta^{\rm rear} \cup S_{r'+1}(C))$, nor h_C belongs to $\Delta^{\epsilon,\rm fr}$ $(= \Delta^{\rm fr} \cup S_{r'}(C))$.

It follows that $V(\Delta^{\epsilon,\rm rear}) = (V(\Delta^{\epsilon,\rm fr}) - \{t_C\}) \cup \{h_C\}$. Hence the raising flip $M \mapsto M'$ using Δ replaces t_C by h_C, while preserving the other vertices of the w-membrane. Also the vertices of Δ different from t_C, h_C form just the collection of sets XS such that S runs over $\mathcal{N}^{\downarrow}(\widetilde{P}, \widetilde{Q})$, the set of lower neighbors of $\widetilde{P} := p(2)p(4)\dots p(d-1)$ and $\widetilde{Q} := p(1)p(3)\dots p(d)$. Now applying Theorem 1.4 to $\mathcal{W} := V(M)$, $X, \widetilde{P}, \widetilde{Q}$, we conclude that $V(M')$ is weakly r-separated, as required. □

Note that the case $r = 1$ of Theorem 4.2 is obtained in [2, Corollary 6.5].

A natural question is whether any two size-maximal weakly separated collections in $2^{[n]}$ can be connected by a sequence of flips. This is strengthened in the following conjecture (which was proved for $r = 1$ in [2, Theorem 7.1]):

Conjecture 4.3. *for r odd, any size-maximal weakly r-separated collection in $2^{[n]}$ is representable as the spectrum of a weak membrane of some cubillage Q in $Z(n, r+2)$.*

5 Triangulations, hyper-combies, and cubillages

Consider the polytope $P = P(n, d-1)$ that is the section of the zonotope $Z(n, d)$ by the hyperplane $H_1 = \{x \in \mathbb{R}^d : x(1) = 1\}$, called the *cyclic polytope* with n vertices of dimension $d-1$. Let $\mathcal{T}(P)$ be the set of *triangulations* of P that are subdivisions of P into $(d-1)$-dimensional simplexes whose vertices are vertices of P (i.e., occur in Ξ as in (1.1)). It has been known (see [8] for details) that

(5.1) for any $\tau \in \mathcal{T}(P(n, d-1))$, there exists a cubillage Q in $Z(n, d)$ whose section by H_1 (formed by the simplexes $C \cap H_1$ for cubes C with the root $\varnothing$ in Q) is τ.

To define more general objects, consider the projection π^ϵ and the modified zonotope $Z^\epsilon(n, d-1)$ as in Section 4. Let $\mathcal{F}(n, d)$ be the set of facets in fragments $C_h^{\equiv}$ of all (abstract) cubes $C = (X \mid T)$ in $Z(n, d)$ (running $X, T \subset [n]$ with $|T| = d$ and $X \cap T = \varnothing$).

Definition 5.1. A *hyper-combi* K is a subdivision of $Z^\epsilon(n, d-1)$ into $(d-1)$-dimensional polytopes of the form $\pi^\epsilon(F)$, where $F \in \mathcal{F}(n, d)$.

In particular, any w-membrane M of a cubillage in $Z(n, d)$ generates the hyper-combi $\pi^\epsilon(M)$. An important special case arises when M is a *principal* w-membrane in level $\ell \in [1, n-1]$. This means that M is the section by $H_\ell = \{x \in \mathbb{R}^d : x(1) = \ell\}$ of some cubillage in $Z = Z(n, d)$ to which the boundary parts

$$Z^{\rm fr}_{\ell\uparrow} := Z^{\rm fr} \cap \{x \in \mathbb{R}^d : x(1) \ge \ell\} \quad \text{and} \quad Z^{\rm rear}_{\ell\downarrow} := Z^{\rm rear} \cap \{x \in \mathbb{R}^d : x(1) \le \ell\}$$

are added, where $Z^{\rm fr}$ and $Z^{\rm rear}$ are the (properly fragmented) front and rear sides of Z. Then the essential ("horizontal") part of a principal w-membrane in level 1 is just a triangulation in $\mathcal{T}(P(n, d-1))$ (while for an arbitrary ℓ it is known as "hypersimplicial subdivision" of the corresponding section of the zonotope, see [8]).

Conjecture 5.2. *For any hyper-combi K in $Z^\epsilon(n, d-1)$ with d odd, there exists a cubillage Q in $Z(n, d)$ and a w-membrane M in (the fragmentation) of Q such that $\pi^\epsilon(M) = K$.*

The validity of Conjecture 5.2 for $d = 3$ is proved in [2, Section 7] (where the desired Q and M are explicitly constructed for an arbitrary (properly triangulated) combi K in $Z^\epsilon(n, 2)$); also we are able to prove this for $d = 5$.

Next, Oppermann and Thomas [9] revealed a nice property of triangulations of a cyclic polytope $P = P(n, 2r)$ having an even dimension $2r = d-1$. More precisely, identify each r-dimensional face in a triangulation of τ (regarded as a complex) with the corresponding increasing $(r+1)$-tuple in $[n]$. Let $e(\tau)$ denote the set of sparse r-faces in τ, where a face (tuple) is called *sparse* if it has no pair $i, i+1$. For increasing tuples $A = (a_0, \ldots, a_r)$ and $B = (b_0, \ldots, b_r)$, one says that A *intertwines* B if $a_0 < b_0 < a_1 < b_1 < \cdots < a_r < b_r$, and a collection $\mathcal{A}$ of $(r+1)$-tuples is called *non-intertwining* if no two tuples in $\mathcal{A}$ intertwine. In other words, $\mathcal{A}$ is weakly $(2r-1 = d-2)$-separated (since all elements of $\mathcal{A}$ have the same size). By [9, Theorems 2.4 and 2.5], (5.2) (a) For $P = P(n, 2r)$ and $\tau \in \mathcal{T}(P)$, the collection $e(\tau)$ has cardinality $\binom{n-r-1}{r}$ and is non-intertwining. (b) Conversely, any non-intertwining collection $\mathcal{A}$ of $\binom{n-r-1}{r}$ sparse $(r+1)$-tuples in $[n]$ represents $e(\tau)$ for a unique $\tau \in \mathcal{T}(P)$.

We can use this as follows. Consider $\mathcal{A}$ and τ as in (5.2)(b). By (5.1), there exists a cubillage Q in $Z = Z(n, d)$ such that τ is the section of Q by H_1. Then each element $A \in \mathcal{A} = e(\tau)$ labels a vertex of Q contained in level r. This vertex is not in Z^{fr}, which follows from (2.1) and the fact that A is sparse. Let M be the principal w-membrane for Q in level r. Then $|V(Z^{\text{fr}}_{r\uparrow})| + |\mathcal{A}| + |V(Z^{\text{rear}}_{(r-1)\downarrow})| \le |V(M)| = w_{n,d-2}$ (in view of Theorem 4.2). Moreover, the inequality here holds with equality (which is seen by directly counting the first and third summands and using $|\mathcal{A}| = \binom{n-r-1}{r}$).

As a consequence, (5.1) implies a weakened version of Conjecture 4.3: for d odd, any size-maximal collection of weakly $(d-2)$-separated subsets $A \subset [n]$ with $|A| = (d-1)/2$ is contained in the spectrum of a w-membrane of some cubillage in $Z(n, d)$.

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