Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #12, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

Counting permutations by peaks, descents, and cycle type

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Abstract. We derive a general formula describing the joint distribution of two permutation statistics—the peak number and the descent number—over any set of permutations whose quasisymmetric generating function is a symmetric function. Our formula involves a certain kind of plethystic substitution on quasisymmetric generating functions. We apply this result to cyclic permutations, involutions, and derangements, and to give a generating function formula for counting permutations by peaks, descents, and cycle type. We recover as special cases results previously derived by Désarménien–Foata, Gessel–Reutenauer, Fulman, and Diaconis–Fulman–Holmes.

Keywords: permutation statistics, peaks, descents, cycle type, symmetric functions, plethysm

1 Introduction

Let $\pi = \pi(1)\pi(2)\cdots\pi(n)$ be an element of the symmetric group $\mathfrak{S}_n$ of permutations of the set $[n] := \{1, 2, \ldots, n\}$. We say that $i \in [n-1]$ is a *descent* of π if $\pi(i) > \pi(i+1)$. The *descent set* of π is the set of all descents of π. Let $\operatorname{des}(\pi)$ denote the number of descents of π (i.e., the size of the descent set) and let $\operatorname{maj}(\pi)$ denote the sum of all descents of π. The *descent number* des and *major index* maj are classical permutation statistics whose study dates back to Percy MacMahon [9].

The distribution of the descent number over $\mathfrak{S}_n$ is encoded by the nth *Eulerian polynomial*

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\operatorname{des}(\pi)+1},$$

and the joint distribution of the descent number and major index by the nth *q-Eulerian polynomial*

$$A_n(q,t) := \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{maj}(\pi)} t^{\operatorname{des}(\pi)+1}.$$

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MacMahon [9, Vol. 2, Section IX] proved a formula of which a special case is

$$\frac{A_n(q,t)}{(1-t)(1-tq)\cdots(1-tq^n)} = \sum_{k=0}^{\infty} [k]_q^n t^k, \tag{1.1}$$

where $[k]_q := 1 + q + q^2 + \cdots + q^{k-1}$. (1.1) allows one to compute the joint distribution of the descent number and major index over $\mathfrak{S}_n$, but one may also want to study the joint distribution of these statistics, as well as others, over certain interesting subsets of $\mathfrak{S}_n$.

We are concerned here with permutation statistics that are determined by the descent set; these statistics are called *descent statistics*. Examples include the descent number, major index, and also the "peak number" statistic which is defined below. In many cases, formulas for distributions of descent statistics can be extracted in a useful way from quasisymmetric functions. For example, the first author and Reutenauer proved in [6, Theorem 5.3] (see also [5, Section 4]) that if the quasisymmetric generating function $Q(\Pi)$ of $\Pi \subseteq \mathfrak{S}_n$ is a symmetric function (see Section 2.2 for relevant definitions), then

$$\frac{\sum_{\pi\in\Pi} t^{\operatorname{des}(\pi)+1} q^{\operatorname{maj}(\pi)}}{(1-t)(1-tq)\cdots(1-tq^n)} = \sum_{k=0}^{\infty} \operatorname{ps}_k(Q(\Pi))t^k \tag{1.2}$$

where $\operatorname{ps}_k(f) := f(1, q, \ldots, q^{k-1})$. This formula reduces to (1.1) when Π is taken to be $\mathfrak{S}_n$. They then use symmetric function operations to derive from (1.2) formulas for the joint distribution of des and maj over cyclic permutations, involutions, and derangements—which all have symmetric quasisymmetric generating functions. Later, Jason Fulman [3, Theorem 1] used Equation (1.2) to derive a formula for the joint distribution of des, maj, and cycle type over $\mathfrak{S}_n$. These results are notable because the descent number and major index, like all descent statistics, are statistics which encode properties of a permutation in one-line representation, and it is generally difficult to study distributions of such statistics while refining by cycle structure.

In our recent paper [7], we prove several general formulas analogous to Equation (1.2) for the distributions of various other descent statistics over any subset of $\mathfrak{S}_n$ whose quasisymmetric generating function is symmetric. These formulas involve plethysm, an operation on symmetric functions which has in recent decades been extended to more general formal power series rings and has found numerous applications within algebraic combinatorics.

This extended abstract is a summary of our results in [7] with a focus on the peak number statistic pk. Given $\pi \in \mathfrak{S}_n$, we say that $i \in \{2, \ldots, n-1\}$ is a *peak* of π if $\pi(i-1) < \pi(i) > \pi(i+1)$, and we let $\operatorname{pk}(\pi)$ denote the number of peaks of π. In Section 2, we review some relevant definitions and results from the theory of symmetric and quasisymmetric functions. We present our general formula for the joint distribution of pk and des in Section 3, followed by applications to cyclic permutations, involutions, derangements, and more generally, counting permutations by peaks, descents, and cycle type. Finally, in Section 4 we briefly discuss our general formulas for two other descent statistics: the left peak number and the number of up-down runs.

2 Preliminaries

2.1 Symmetric functions and plethysm

We assume familiarity with basic definitions from the theory of symmetric functions as described by Stanley [10, Chapter 7]. In this section we establish notation and define the plethysm operation which will be needed for our main result. (See [8] for a more comprehensive introductory reference on plethysm.)

We use the notation $|\lambda| = n$ to indicate that λ is a partition of n, and we let $l(\lambda)$ denote the number of parts of λ. We write $\lambda = (1^{m_1}2^{m_2}\cdots)$ to mean that λ has m_1 parts of size 1, m_2 parts of size 2, and so on; alternatively, we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ to mean that λ has parts $\lambda_1, \lambda_2, \ldots, \lambda_r$. Given a partition $\lambda = (1^{m_1}2^{m_2}\cdots)$, let $z_\lambda := 1^{m_1}m_1!\,2^{m_2}m_2!\cdots$.

Let Λ denote the $\mathbb{Q}$-algebra of symmetric functions in the variables $x_1, x_2, \ldots$. In particular, we will need the complete symmetric functions h_n, the elementary symmetric functions e_n, and the power sum symmetric functions p_λ. We will often work in the ring $\hat{\Lambda}$ of symmetric functions of unbounded degree (with coefficients in some polynomial ring). In the ring $\hat{\Lambda}$ we define $H(x) := \sum_{n=0}^{\infty} h_n x^n$ and $E(x) := \sum_{n=0}^{\infty} e_n x^n$ to be the ordinary generating functions for the h_n and the e_n, respectively. We also adopt the notation $H := H(1) = \sum_{n=0}^{\infty} h_n$.

Let $\langle \cdot, \cdot \rangle \colon \Lambda \times \Lambda \to \mathbb{Q}$ denote the usual scalar product on symmetric functions which may be defined by

$$\langle p_\lambda, p_\tau \rangle := \begin{cases} z_\lambda, & \text{if } \lambda = \tau, \\ 0, & \text{otherwise} \end{cases}$$

for all λ and τ, and then extending bilinearly.

Let A be a $\mathbb{Q}$-algebra of formal power series in some set of variables (possibly containing Λ). We define an operation $\Lambda \times A \to A$, where the image of $(f, a) \in \Lambda \times A$ is denoted $f[a]$, by these two properties:

1. For any $i \geq 1$, $p_i[a]$ is the result of replacing each variable in a with its ith power.

2. For any fixed $a \in A$, the map $f \mapsto f[a]$ is a $\mathbb{Q}$-algebra homomorphism from Λ to A.

For a symmetric function $f \in \Lambda$, this means that $p_i[f(x_1, x_2, \ldots)] = f(x_1^i, x_2^i, \ldots)$. If f contains other variables than the x_i then they are raised to the ith power as well. For example, if q and t are variables then $p_i[qt^2p_m] = q^i t^{2i} p_{im}$. If f is expressed in terms of the power sums, $p_i[f]$ can be obtained from f by replacing each p_m with p_{mi}. The map $(f, a) \mapsto f[a]$ is called *plethysm*. It is important to note that plethysm does not commute with evaluation of variables. If α is a variable then $p_n[\alpha] = \alpha^n$, but if y is a rational number then $p_n[y] = y$; thus in this case $p_n[\alpha]|_{\alpha=y} \neq p_n[y]$.

We extend plethysm to a partial map $\hat{\Lambda} \times A \to A$ in the obvious way: if each f_i is in Λ then $\left(\sum_{k=0}^{\infty} f_i\right)[a] := \sum_{k=0}^{\infty} f_i[a]$ as long as both infinite sums converge as formal power series.

2.2 Descent compositions, cycle type, and quasisymmetric generating functions

We use the notation $L \vDash n$ to indicate that L is a composition of n. Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences— or equivalently, maximal consecutive subsequences containing no descents—which we call *increasing runs*. The *descent composition* of π, denoted $\mathrm{Comp}(\pi)$, is the composition whose parts are the lengths of the increasing runs of π in the order that they appear. For example, the increasing runs of $\pi = 85712643$ are 8, 57, 126, 4, and 3, so the descent composition of π is $\mathrm{Comp}(\pi) = (1, 2, 3, 1, 1)$.

For a composition $L = (L_1, L_2, \ldots, L_k)$, let $\mathrm{Des}(L) := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}$. It is easy to see that if L is the descent composition of π, then $\mathrm{Des}(L)$ is the descent set of π. Recall that the fundamental quasisymmetric function F_L is defined by

$$F_L := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}(L)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Given a set Π of permutations, its *quasisymmetric generating function* $Q(\Pi)$ is defined by

$$Q(\Pi) := \sum_{\pi \in \Pi} F_{\mathrm{Comp}(\pi)}.$$

Moreover, given a composition $L = (L_1, \ldots, L_k)$, let r_L denote the skew Schur function of ribbon shape L. That is,

$$r_L := \sum_{i_1, \ldots i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

where the sum is over all $i_1, \ldots, i_n$ satisfying

$$\underbrace{i_1 \le \cdots \le i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \le \cdots \le i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \le \cdots \le i_n}_{L_k}.$$

The following is [5, Corollary 4].

Theorem 2.1. *Suppose that $Q(\Pi)$ is symmetric. Then the number of permutations in Π with descent composition L is equal to $\langle Q(\Pi), r_L \rangle$.*

We say that a permutation π has *cycle type* $\lambda = (1^{m_1}2^{m_2}\cdots)$ if π has exactly m_1 cycles of size 1, m_2 cycles of size 2, and so on. Henceforth, cycles of size i are called *i-cycles*, 1-cycles in particular are called *fixed points*, and the number of fixed points of a permutation π is denoted $\mathrm{fix}(\pi)$. For $n \in \mathbb{P}$, define the symmetric function L_n by

$$L_n := \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d} \tag{2.1}$$

where μ is the number-theoretic Möbius function. Given a partition $\lambda = (1^{m_1}2^{m_2}\cdots)$, define L_λ by

$$L_\lambda := h_{m_1}[L_1]h_{m_2}[L_2]\cdots .$$

The symmetric functions L_λ are called *Lyndon symmetric functions*. Gessel and Reutenauer [6, Theorem 2.1] showed that L_λ is the quasisymmetric generating function for the set of permutations with cycle type λ.

Corollary 2.2. *The number of permutations π with cycle type λ and descent composition M is equal to $\langle L_\lambda, r_M\rangle$.*

3 Results

3.1 General formula

Given a variable (or integer) $y \in A$ and an integer $k \in \mathbb{Z}$, define the homomorphism $\Theta_{y,k}\colon \Lambda \to \mathbb{Q}[[y]]$ by

$$\Theta_{y,k}(f) := f[k(1-\alpha)]|_{\alpha=-y}.$$

That is, $\Theta_{y,k}$ first sends a symmetric function f to the plethystic substitution $f[k(1-\alpha)]$, where α is a variable, and then evaluates this expression at $\alpha = -y$.

Given a set Π of permutations, define

$$P^{(\mathrm{pk},\mathrm{des})}(\Pi; y, t) := \sum_{\pi\in\Pi} y^{\mathrm{pk}(\pi)+1}t^{\mathrm{des}(\pi)+1};$$

this polynomial encodes the joint distribution of pk and des over Π. If Π has a symmetric quasisymmetric function $Q(\Pi)$, then the following theorem allows us to describe this polynomial in terms of $\Theta_{y,k}(Q(\Pi))$ and in terms of Eulerian polynomials.

Theorem 3.1. *Let $\Pi \subseteq \mathfrak{S}_n$ and suppose that its quasisymmetric generating function $Q(\Pi)$ is a symmetric function with power sum expansion $Q(\Pi) = \sum_{\lambda\vdash n} c_\lambda p_\lambda$. Then*

$$\frac{1}{1+y}\left(\frac{1+yt}{1-t}\right)^{n+1} P^{(\mathrm{pk},\mathrm{des})}\left(\Pi; \frac{(1+y)^2t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right)$$
$$= \sum_{k=0}^{\infty}\Theta_{y,k}(Q(\Pi))t^k = \sum_{\lambda\vdash n} c_\lambda \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}}\prod_{k=1}^{l(\lambda)}(1-(-y)^{\lambda_k}).$$

The full proof of Theorem 3.1 is in [7], but let us sketch the proof here. We begin by stating three preliminary lemmas, one concerning plethysm, one giving the power sum expansion of $(1-tE(y)H)^{-1}$, and one giving the ribbon expansion of $(1-tE(y)H)^{-1}$. The first two lemmas are proved in [7], whereas the third is an immediate consequence of [11, Lemma 4.1].

Lemma 3.2. *Let $f \in A$, let α be a variable, and let $k \in \mathbb{Z}$. Then $f[k(1-\alpha)] = \langle f, H^k E(-\alpha)^k \rangle$.*

Lemma 3.3.

$$\frac{1}{1-tE(y)H} = \sum_{\lambda} \frac{p_\lambda}{z_\lambda} \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}} \prod_{k=1}^{l(\lambda)} (1-(-y)^{\lambda_k}).$$

Lemma 3.4.

$$\frac{1}{1-tE(y)H} = \frac{1}{1-t} + \frac{1}{1+y} \sum_{n=1}^{\infty} \sum_{L \vDash n} \left(\frac{1+yt}{1-t}\right)^{n+1} \left(\frac{(1+y)^2 t}{(y+t)(1+yt)}\right)^{\mathrm{pk}(L)+1} \left(\frac{y+t}{1+yt}\right)^{\mathrm{des}(L)+1} r_L.$$

In Lemma 3.4, $\mathrm{des}(L)$ is defined to be $\mathrm{des}(\pi)$ for any permutation π with descent composition L, and $\mathrm{pk}(L)$ is defined analogously. These are well-defined because the descent number and peak number are descent statistics; they depend only on the descent composition. Lemmas 3.2 to 3.4 allow us to derive three expressions for the scalar product $\langle Q(\Pi), (1-tE(y)H)^{-1} \rangle$:

1. It follows from Lemma 3.2 that

$$\left\langle Q(\Pi), \frac{1}{1-tE(y)H} \right\rangle = \sum_{k=0}^{\infty} \left\langle Q(\Pi), E(y)^k H^k \right\rangle t^k = \sum_{k=0}^{\infty} \Theta_{y,k}(Q(\Pi)) t^k.$$

2. It follows from Lemma 3.3 that

$$\left\langle Q(\Pi), \frac{1}{1-tE(y)H} \right\rangle = \sum_{\lambda \vdash n} c_\lambda \frac{A_{l(\lambda)}(t)}{(1-t)^{l(\lambda)+1}} \prod_{k=1}^{l(\lambda)} (1-(-y)^{\lambda_k}).$$

3. It follows from Lemma 3.4 and Theorem 2.1 that

$$\left\langle Q(\Pi), \frac{1}{1-tE(y)H} \right\rangle = \frac{1}{1+y} \left(\frac{1+yt}{1-t}\right)^{n+1} P^{(\mathrm{pk},\mathrm{des})}\left(\Pi; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right).$$

Equating these three expressions completes the proof of Theorem 3.1.

3.2 Cyclic permutations

We say that a permutation of length n is a *cyclic permutation* (or a *cycle*) if it has cycle type (n). We denote the set of cyclic permutations of length n by $\mathfrak{C}_n$. The formula below follows immediately from Theorem 3.1 and (2.1), which is the quasisymmetric generating function for $\mathfrak{C}_n$.

Theorem 3.5. *Let $n \geq 1$. Then*

$$P^{(\mathrm{pk,des})}\left(\mathfrak{C}_n; \frac{(1+y)^2t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right) = \frac{(1+y)}{n(1+yt)^{n+1}} \sum_{d|n} \mu(d)(1-(-y)^d)^{n/d}(1-t)^{n-n/d}A_{n/d}(t).$$

Although this formula may seem complicated, it allows for easy computation of the polynomials $P^{(\mathrm{pk,des})}(\mathfrak{C}_n; y, t)$ in the following way. First, we replace y with u and t with v in the formula. Then set $y = \frac{(1+u)^2v}{(u+v)(1+uv)}$ and $t = \frac{u+v}{1+uv}$; solving these two equations yield $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$. Thus, we have

$$P^{(\mathrm{pk,des})}(\mathfrak{C}_n; y, t) = \frac{(1+u)}{n(1+uv)^{n+1}} \sum_{d|n} \mu(d)(1-(-u)^d)^{n/d}(1-v)^{n-n/d}A_{n/d}(v).$$

(All of the formulas given later in this paper can be inverted in a similar manner, but we will omit the details.) For example, for $n = 5$, we have

$$\begin{aligned}
P^{(\mathrm{pk,des})}(\mathfrak{C}_5; y, t) &= \frac{(1+u)}{5(1+uv)^6}((1+u)^5A_5(v) - (1+u^5)(1-v)^4A_1(v)) \\
&= \frac{1+u}{5(1+uv)^6}((1+u)^5(v+26v^2+66v^3+26v^4+v^5) - (1+u^5)(1-v)^4v)) \\
&= (y+5y^2)t^2 + (y+8y^2+3y^3)t^3 + (y+5y^2)t^4.
\end{aligned}$$

where the last equality was obtained by substituting in $u = \frac{1+t^2-2yt-(1-t)\sqrt{(1+t)^2-4yt}}{2(1-y)t}$ and $v = \frac{(1+t)^2-2yt-(1+t)\sqrt{(1+t)^2-4yt}}{2yt}$ and then simplifying using Maple.

Observe that in $P^{(\mathrm{pk,des})}(\mathfrak{C}_5; y, t)$, the coefficient of t^2 is equal to the coefficient of t^4. In fact, for any $n \in \mathbb{P}$ not congruent to 2 modulo 4, the number of cyclic permutations of length n with j peaks and k descents is equal to the number of cyclic permutations of length n with j peaks and $n-1-k$ descents. This is not easily apparent from the formula obtained in Theorem 3.5, but is a special case of the following result, which we prove in [7].

Theorem 3.6. *Suppose that λ is a partition of n with no parts congruent to 2 modulo 4 and that every odd part of λ occurs only once. Then the number of permutations of cycle type λ with j peaks and k descents is equal to the number of permutations of cycle type λ with j peaks and $n-1-k$ descents.*

Given $\Pi \subseteq \mathfrak{S}_n$, let us define

$$A(\Pi;t) := \sum_{\pi\in\Pi} t^{\mathrm{des}(\pi)+1} \quad \text{and} \quad P^{\mathrm{pk}}(\Pi;t) := \sum_{\pi\in\Pi} t^{\mathrm{pk}(\pi)+1}.$$

Specializing Theorem 3.5 at $y=0$ yields

$$A(\mathfrak{C}_n;t) = \frac{1}{n}\sum_{d|n}\mu(d)(1-t)^{n-n/d}A_{n/d}(t),$$

which is the $q=1$ evaluation of a formula by Gessel and Reutenauer [6, Corollary 6.2]. Specializing at $y=1$, on the other hand, yields the formula

$$P^{\mathrm{pk}}\left(\mathfrak{C}_n;\frac{4t}{(1+t)^2}\right) = \frac{1}{n(1+t)^{n+1}}\sum_{\substack{d|n\\ d\text{ odd}}}\mu(d)2^{n/d+1}(1-t)^{n-n/d}A_{n/d}(t).$$

3.3 Involutions

A permutation π is called an *involution* if π^2 is the identity permutation, or equivalently, if π has no cycles of size larger than 2. We denote the set of involutions of length n by $\mathfrak{I}_n$. The quasisymmetric generating function for all involutions weighted by length and number of fixed points is known to be

$$\sum_{n=0}^{\infty}\sum_{\pi\in\mathfrak{I}_n} F_{\mathrm{Comp}(\pi)}z^{\mathrm{fix}(\pi)}x^n = \prod_i \frac{1}{1-zxx_i}\prod_{i<j}\frac{1}{1-x^2x_ix_j}; \tag{3.1}$$

see [6, Equation (7.1)].

For $\Pi \subseteq \mathfrak{S}_n$, let

$$P^{(\mathrm{pk},\mathrm{des},\mathrm{fix})}(\Pi;y,t,z) := \sum_{\pi\in\Pi} y^{\mathrm{pk}(\pi)+1}t^{\mathrm{des}(\pi)+1}z^{\mathrm{fix}(\pi)}.$$

In [7], we use Theorem 3.1 and (3.1) to derive the following generating function formula for the polynomials $P^{(\mathrm{pk},\mathrm{des},\mathrm{fix})}(\mathfrak{I}_n;y,t,z)$.

Theorem 3.7.

$$\begin{aligned}\frac{1}{1-t}+\frac{1}{1+y}\sum_{n=1}^{\infty}\left(\frac{1+yt}{1-t}\right)^{n+1}P^{(\mathrm{pk},\mathrm{des},\mathrm{fix})}\left(\mathfrak{I}_n;\frac{(1+y)^2t}{(y+t)(1+yt)},\frac{y+t}{1+yt},z\right)x^n\\ =\sum_{k=0}^{\infty}\frac{(1+zxy)^k(1+x^2y)^{k^2}t^k}{(1-zx)^k(1-x^2)^{\binom{k}{2}}(1-x^2y^2)^{\binom{k+1}{2}}}\end{aligned}$$

Given $\Pi \subseteq \mathfrak{S}_n$, let

$$P^{(\mathrm{des,fix})}(\Pi;t,z) := \sum_{\pi\in\Pi} t^{\mathrm{des}(\pi)+1} z^{\mathrm{fix}(\pi)} \quad \text{and} \quad P^{(\mathrm{pk,fix})}(\Pi;t,z) := \sum_{\pi\in\Pi} t^{\mathrm{pk}(\pi)+1} z^{\mathrm{fix}(\pi)}.$$

Specializing Theorem 3.7 at $y=0$ yields

$$\frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{P^{(\mathrm{des,fix})}(\mathfrak{I}_n;t,z)}{(1-t)^{n+1}} x^n = \sum_{k=0}^{\infty} \frac{t^k}{(1-zx)^k (1-x^2)^{\binom{k}{2}}},$$

which is equivalent to Equation (5.5) of Désarménien and Foata [1] and Equation (7.3) of Gessel and Reutenauer [6]. Specializing at $y=1$ yields the formula

$$\frac{1}{1-t} + \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{1+t}{1-t}\right)^{n+1} P^{(\mathrm{pk,fix})}\left(\mathfrak{I}_n; \frac{4t}{(1+t)^2}, z\right) x^n$$
$$= \sum_{k=0}^{\infty} \frac{(1+zx)^k (1+x^2)^{k^2} t^k}{(1-zx)^k (1-x^2)^{\binom{k}{2}} (1-x^2)^{\binom{k+1}{2}}}.$$

3.4 Derangements

Let $\mathfrak{D}_n$ denote the set of *derangements*—permutations with no fixed points—of length n. The quasisymmetric generating function for all permutations weighted by length and number of fixed points is known to be

$$\sum_{n=0}^{\infty} \sum_{\pi\in\mathfrak{S}_n} F_{\mathrm{Comp}(\pi)} z^{\mathrm{fix}(\pi)} x^n = \frac{H(zx)}{H(x)(1-p_1 x)}; \tag{3.2}$$

see the proof of [6, Theorem 8.4]. Setting $z=0$ in (3.2) specializes to derangements.

We prove the following in [7] using Theorem 3.1 and (3.2).

Theorem 3.8. *We have*

$$\frac{1}{1-t} + \frac{1}{1+y}\sum_{n=1}^{\infty} \left(\frac{1+yt}{1-t}\right)^{n+1} P^{(\mathrm{pk,des,fix})}\left(\mathfrak{S}_n; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}, z\right) x^n$$
$$= \sum_{k=0}^{\infty} \frac{(1+zxy)^k}{(1-zx)^k} \frac{(1-x)^k}{(1+xy)^k} \frac{t^k}{1-k(1+y)x}$$

and

$$\frac{1}{1-t} + \frac{1}{1+y}\sum_{n=1}^{\infty} \left(\frac{1+yt}{1-t}\right)^{n+1} P^{(\mathrm{pk,des})}\left(\mathfrak{D}_n; \frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}\right) x^n$$
$$= \sum_{k=0}^{\infty} \frac{(1-x)^k}{(1+xy)^k} \frac{t^k}{1-k(1+y)x}.$$

We can also specialize the formulas in Theorem 3.8 at $y = 0$ and $y = 1$ to obtain generating function formulas for the polynomials $P^{(\mathrm{des},\mathrm{fix})}(\mathfrak{S}_n; t, z)$, $P^{(\mathrm{pk},\mathrm{fix})}(\mathfrak{S}_n; t, z)$, $A(\mathfrak{D}_n; t)$, and $P^{\mathrm{pk}}(\mathfrak{D}_n; t)$; we omit these formulas here. In particular, the formula obtained for $P^{(\mathrm{des},\mathrm{fix})}(\mathfrak{S}_n; t, z)$ is the $q = 1$ evaluation of a formula by Gessel and Reutenauer [6, Equation (8.3)].

3.5 Cycle type

Theorem 3.8 gave a formula for counting permutations by the number of fixed points jointly with the peak number and descent number. We now refine this result by giving a formula for counting permutations by peaks, descents, and cycle type.

For a permutation π, let $N_i(\pi)$ denote the number of i-cycles in π. Similarly, given a partition λ, let $N_i(\lambda)$ denote the number of parts of size i in π. Recall that the Lyndon symmetric function

$$L_\lambda = h_{m_1}[L_1]h_{m_2}[L_2]\cdots$$

is the quasisymmetric generating function for the set of permutations with cycle type $\lambda = (1^{m_1}2^{m_2}\cdots)$. Then

$$1 + \sum_{n=1}^{\infty}\sum_{\lambda\vdash n} L_\lambda x^n \prod_{i=1}^{\infty} z_i^{N_i(\lambda)} = \sum_{m_1,m_2,\ldots}\prod_{i=1}^{\infty} h_{m_i}[L_i](z_i x^i)^{m_i} = \prod_{i=1}^{\infty}\sum_{m_i=0}^{\infty} h_{m_i}[L_i](z_i x^i)^{m_i} \tag{3.3}$$

is the quasisymmetric generating function for all permutations refined by cycle type and length.

Define

$$F_n^{(\mathrm{pk},\mathrm{des})}(y, t, z_1, z_2, \ldots) := \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{pk}(\pi)+1} t^{\mathrm{des}(\pi)+1} \prod_{i=1}^{\infty} z_i^{N_i(\pi)},$$

$$F_n^{\mathrm{pk}}(t, z_1, z_2, \ldots) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{pk}(\pi)+1} \prod_{i=1}^{\infty} z_i^{N_i(\pi)},$$

and

$$F_n^{\mathrm{des}}(t, z_1, z_2, \ldots) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)+1} \prod_{i=1}^{\infty} z_i^{N_i(\pi)}.$$

We prove the next theorem in [7] using Theorem 3.1 and (3.3).

Theorem 3.9.

$$\frac{1}{1-t} + \frac{1}{1+y}\sum_{n=1}^{\infty}\left(\frac{1+yt}{1-t}\right)^{n+1} F_n^{(\mathrm{pk},\mathrm{des})}\left(\frac{(1+y)^2 t}{(y+t)(1+yt)}, \frac{y+t}{1+yt}, z_1, z_2, \ldots\right) x^n$$
$$= \sum_{k=0}^{\infty} t^k \prod_{i=1}^{\infty} \exp\left(\sum_{m_i=1}^{\infty} \frac{(z_i x^i)^{m_i}}{i m_i} \sum_{d\mid i} \mu(d)\big(k(1-(-y)^{d m_i})\big)^{i/d}\right).$$

Now, define the numbers

$$f_{i,k} := \frac{1}{i}\sum_{d|i}\mu(d)k^{i/d} \quad \text{and} \quad g_{i,k} := \frac{1}{2i}\sum_{\substack{d|i \\ i \text{ odd}}}\mu(d)(2k)^{i/d}.$$

We can specialize Theorem 3.9 at $y=0$ to recover the formula

$$\frac{1}{1-t}+\sum_{n=1}^{\infty}\frac{F_n^{\mathrm{des}}(t,z_1,z_2,\dots)}{(1-t)^{n+1}}x^n=\sum_{k=0}^{\infty}t^k\prod_{i=1}^{\infty}\left(\frac{1}{1-z_ix^i}\right)^{f_{i,k}},$$

which was originally proved by Fulman [3]. Specializing at $y=1$ recovers the formula

$$\frac{1}{1-t}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1+t}{1-t}\right)^{n+1}F_n^{\mathrm{pk}}\left(\frac{4t}{(1+t)^2},z_1,z_2,\dots\right)x^n=\sum_{k=0}^{\infty}t^k\prod_{i=1}^{\infty}\left(\frac{1+z_ix^i}{1-z_ix^i}\right)^{g_{i,k}},$$

originally due to Diaconis, Fulman, and Holmes [2]; see also [4].

4 Other descent statistics

Let us call $i\in[n-1]$ a *left peak* of π if i is a peak or if $i=1$ and $\pi(1)>\pi(2)$. A *birun* of π is a maximal monotone consecutive subsequence, and an *up-down run* of π is a birun of π or the letter $\pi(1)$ if $\pi(1)>\pi(2)$. Let $\mathrm{lpk}(\pi)$ the number of left peaks of π, and $\mathrm{udr}(\pi)$ the number of up-down runs of π.

Define

$$P^{\mathrm{lpk}}(\Pi;t) := \sum_{\pi\in\Pi}t^{\mathrm{lpk}(\pi)} \quad \text{and} \quad P^{\mathrm{udr}}(\Pi;t) := \sum_{\pi\in\Pi}t^{\mathrm{udr}(\pi)}.$$

The theorem below is an analogue of Theorem 3.1 for the lpk and udr statistics; we omit the proof.

Theorem 4.1. *Let $\Pi\subseteq\mathfrak{S}_n$ and suppose that its quasisymmetric generating function $Q(\Pi)$ is a symmetric function with power sum expansion $Q(\Pi)=\sum_{\lambda\vdash n}c_\lambda p_\lambda$. Then*

$$\frac{(1+t)^n}{(1-t)^{n+1}}P^{\mathrm{lpk}}\left(\Pi;\frac{4t}{(1+t)^2}\right)=\sum_{k=0}^{\infty}\Theta_{1,k}(Q(\Pi)[X+1])t^k=\sum_{\lambda\vdash n}c_\lambda\frac{B_{o(\lambda)}(t)}{(1-t)^{o(\lambda)+1}}$$

and

$$\begin{aligned}\frac{(1+t^2)^n}{2(1-t)^2(1-t^2)^{n-1}}&P^{\mathrm{udr}}\left(\Pi;\frac{2t}{1+t^2}\right)\\&=\sum_{k=0}^{\infty}\Theta_{1,k}(Q(\Pi))t^{2k}+\sum_{k=0}^{\infty}\Theta_{1,k}(Q(\Pi)[X+1])t^{2k+1}\\&=\sum_{\substack{\lambda\vdash n\\ \text{all parts odd}}}c_\lambda 2^{l(\lambda)}\frac{A_{l(\lambda)}(t^2)}{(1-t^2)^{l(\lambda)+1}}+t\sum_{\lambda\vdash n}c_\lambda\frac{B_{o(\lambda)}(t^2)}{(1-t^2)^{o(\lambda)+1}},\end{aligned}$$

where $o(\lambda)$ is the number of odd parts of λ and $B_n(t)$ is the nth type B Eulerian polynomial (defined, e.g., in [11, Section 2.3]).

In [7], we use Theorem 4.1 to obtain formulas for the distributions of lpk and udr over cyclic permutations, derangements, and involutions, and for counting permutations by these statistics jointly with cycle type.

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