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The facial weak order on hyperplane arrangements

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Abstract. We introduce the facial weak order of a real hyperplane arrangement $\mathcal{A}$. It is a partial order on all faces of $\mathcal{A}$ which naturally extends the poset of regions of $\mathcal{A}$. We provide various characterizations of the facial weak order and show that it is a lattice as soon as the poset of regions is a lattice.

Résumé. Nous introduisons l'ordre faible facial d'un arrangement d'hyperplans $\mathcal{A}$. Il s'agit d'un ordre partiel sur toutes les faces de $\mathcal{A}$ qui étend naturellement l'ordre sur les régions de $\mathcal{A}$. Nous présentons plusieurs caractérisations de l'ordre faible facial et montrons que c'est un treillis dès que l'ordre sur les régions est un treillis.

Keywords: Hyperplane arrangements, poset of regions, zonotopes

1 Introduction

A *hyperplane arrangement* is a finite collection $\mathcal{A}$ of linear hyperplanes in $\mathbb{R}^d$. Its *regions* are the closures of the connected components of $\mathbb{R}^d \smallsetminus (\bigcup_{H \in \mathcal{A}} H)$. A region is *simplicial* if the normal vectors to its bounding hyperplanes are linearly independent, and the arrangement is *simplicial* if all its regions are. The *zonotope* of the arrangement $\mathcal{A}$ is a convex polytope dual to the arrangement $\mathcal{A}$, obtained as the Minkowski sum of line segments normal to the hyperplanes of $\mathcal{A}$.

The regions of $\mathcal{A}$ can be ordered as follows. Define the *separation set* $S(R, R')$ between two regions R and R' of $\mathcal{A}$ as the set of hyperplanes of $\mathcal{A}$ separating the two regions R and R'. For a fixed base region B, the *poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ is the set of regions of $\mathcal{A}$ ordered by inclusion of their separation sets $S(B, R)$ with the base region B. A. Björner,

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P. H. Edelman and G. M. Ziegler [5] showed that the poset of regions is a lattice if $\mathcal{A}$ is simplicial, and that the base region B is simplicial if the poset of regions is a lattice. The Hasse diagram of the poset of regions can also be seen as the graph of the zonotope of $\mathcal{A}$, oriented from the base region B to its opposite region $-B$.

A fundamental example is the arrangement containing the reflection hyperplanes of a finite Coxeter group W. The normals to the hyperplanes are the roots of the root system of W, the zonotope is the permutahedron of W (the convex hull of the W-orbit of a well-chosen point), and the poset of regions is isomorphic to the weak order on W.

In this extended abstract, we study the *facial weak order* $\mathrm{FW}(\mathcal{A}, B)$, a poset structure on all faces of the hyperplane arrangement $\mathcal{A}$ or, equivalently, of the zonotope of $\mathcal{A}$. It was first introduced by D. Krob, M. Latapy, J.-C. Novelli, H.-D. Phan, and S. Schwer in [13] for the braid arrangement (or type A Coxeter arrangement) where it was shown to be a lattice. It was then extended to arbitrary Coxeter arrangements by P. Palacios and M. Ronco in [16] and it was shown to be a lattice for arbitrary Coxeter arrangements in [7]. Here, we extend the facial weak order to central hyperplane arrangements.

The first part of this article, contained in Section 3, is dedicated to providing four equivalent definitions for the facial weak order on a given hyperplane arrangement:

- in terms of *separation set* comparisons between the minimal and maximal regions incident to a face (Section 3.1),
- by providing a precise description of its *covering relations* (Section 3.2),
- in terms of *covectors* of the associated oriented matroid (Section 3.3),
- and in terms of *root sets* of the normals to the hyperplanes (Section 3.4), closely related to the geometry of the zonotope (Section 3.5).

In the case of a Coxeter arrangement, this recovers and expands the descriptions in [7].

In Section 4.1, we show that if the poset of regions of a hyperplane arrangement is a lattice, then the facial weak order is a lattice (Theorem 4.1). This is achieved using the BEZ lemma [5, Lemma 2.1] which states that a poset is a lattice as soon as there exists a join $x \vee y$ for every two elements x and y that both cover the same element. This extends the results of [13] for the braid arrangement and of [7] for Coxeter arrangements.

For a general arrangement $\mathcal{A}$, the facial weak order may not be a lattice, but its topology still admits a nice description that we study in Section 4.5. There is a wide variety of simplicial complexes associated to a hyperplane arrangement. Typically, complexes that depend on the matroid structure of $\mathcal{A}$ are homotopy equivalent to a wedge of (several) spheres, *e.g.* the independence complex, the reduced broken circuit complex, or the lattice of flats [2]. On the other hand, complexes that depend on the *oriented* matroid structure of $\mathcal{A}$ tend to be homotopy equivalent to a single sphere or are contractible, *e.g.* the complexes of acyclic, convex, or free sets [9], the poset of regions [8], or the poset of cellular strings [1]. We compute the homotopy types of intervals of the facial weak order (Theorem 4.12). Keeping with the aforementioned trends, we prove that every interval of the facial weak order is either contractible or homotopy equivalent to a sphere.

2 The poset of regions of a hyperplane arrangement

2.1 Hyperplane arrangements

We consider an *arrangement* $\mathcal{A}$ of linear hyperplanes in $\mathbb{R}^d$. We assume that $\mathcal{A}$ is *essential*, *i.e.* that the intersections of all its hyperplanes is the origin. We denote by $\mathscr{R}_\mathcal{A}$ the set of *regions* of $\mathcal{A}$, *i.e.* closures of connected components of $\mathbb{R}^d \smallsetminus \bigcup_{H \in \mathcal{A}} H$, and by $\mathscr{F}_\mathcal{A}$ the set of *faces* of $\mathcal{A}$, *i.e.* any intersection of regions. A region is *simplicial* if it is generated by d rays, and the arrangement $\mathcal{A}$ is *simplicial* if all its regions are.

Example 2.1. Well-known examples of simplicial arrangements are *Coxeter arrangements*. These are the arrangements formed by the reflection hyperplanes of a Coxeter group W. We refer the reader to the books [12, 3] for comprehensive surveys on Coxeter groups. Figure 1 (left) gives an example in type A_2 together with its faces. It has six dimension 2 regions denoted by R_i (in blue), six dimension 1 rays denoted by F_i (in red), and one dimension 0 face $\{0\}$ at the center (in green). The other arrangements of Figure 1 are the Coxeter arrangements of types A_3, B_3 and H_3 in $\mathbb{R}^3$.

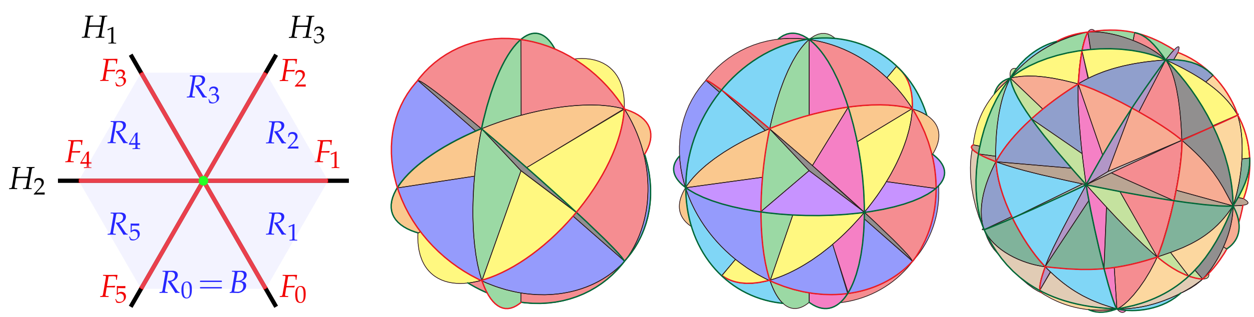


Figure 1: The type A_2, A_3, B_3 and H_3 Coxeter arrangements.

2.2 Poset of regions

For two regions $R, R' \in \mathscr{R}_\mathcal{A}$, we denote by $S(R, R')$ the set of hyperplanes of $\mathcal{A}$ that separate R and R'. We fix a base region $B \in \mathscr{R}_\mathcal{A}$ and let $S(R) := S(B, R)$.

Definition 2.2. The *poset of regions* $\mathrm{PR}(\mathcal{A}, B) := (\mathscr{R}_\mathcal{A}, \leq_{\mathrm{PR}})$ is defined for two regions $R, R' \in \mathscr{R}_\mathcal{A}$ by $R \leq_{\mathrm{PR}} R' \iff S(R) \subseteq S(R')$.

The poset of regions is graded by the cardinality of the separation set $|S(R)|$. The base region B is its minimum element and has rank $|S(B)| = |\varnothing| = 0$, and its opposite region $-B$ is its maximum element and has rank $|S(-B)| = |\mathcal{A}|$.

Theorem 2.3 ([5, Theorems 3.1 and 3.4])**.** *If $\mathcal{A}$ is a simplicial arrangement then the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice for any base region B. Moreover, if the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice then the base region B is a simplicial region.*

3 The facial weak order of a hyperplane arrangement

We now provide various equivalent definitions of the facial weak order, a natural extension of the poset of regions to all faces of the arrangement. These definitions are illustrated in Figure 2 for the type A_2 Coxeter arrangement presented in Figure 1 (left).

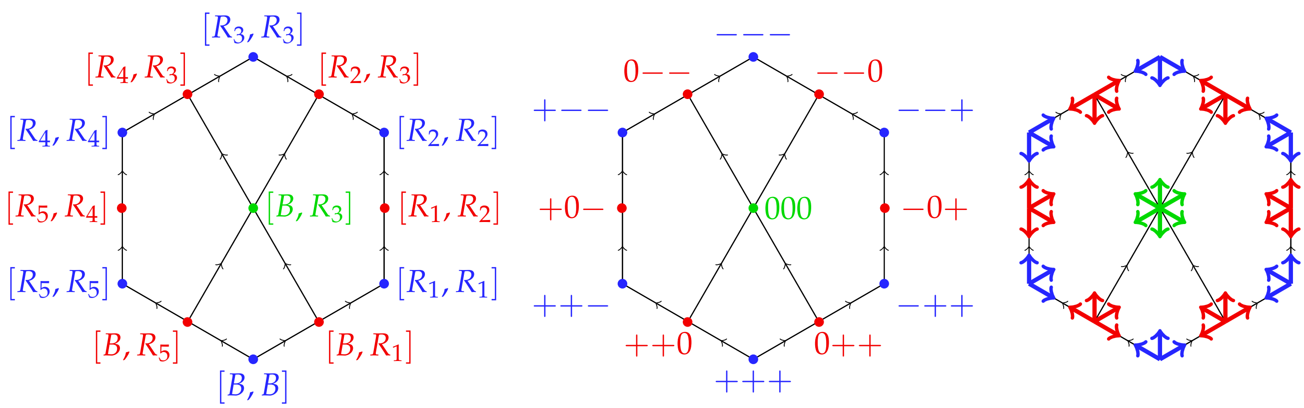


Figure 2: The type A_2 facial weak order seen with facial intervals (left), sign vectors (middle), and root sets (right).

3.1 Facial intervals

One of the interesting facts about the poset of regions is that it allows each face in $\mathscr{F}_\mathcal{A}$ to be described by a unique interval in $\mathrm{PR}(\mathcal{A}, B)$.

Proposition 3.1. *For any face $F \in \mathscr{F}_\mathcal{A}$, the set $\{R \in \mathscr{R}_\mathcal{A} \mid F \subseteq R\}$ is an interval of the poset of regions $\mathrm{PR}(\mathcal{A}, B)$. We denote it by $[m_F, M_F]$ and call it the* facial interval *of F.*

Definition 3.2. The *facial weak order* $\mathrm{FW}(\mathcal{A}, B) := (\mathscr{F}_\mathcal{A}, \leq_{\mathrm{FW}})$ is defined for two faces $F, G \in \mathscr{F}_\mathcal{A}$ with facial intervals $[m_F, M_F]$ and $[m_G, M_G]$ by $F \leq_{\mathrm{FW}} G \iff m_F \leq_{\mathrm{PR}} m_G$ and $M_F \leq_{\mathrm{PR}} M_G$.

Remark 3.3. In other words, the facial weak order is the subposet induced by facial intervals in the poset of all intervals of the poset of regions $\mathrm{PR}(\mathcal{A}, B)$, where the order is defined componentwise: $[x, X] \leq [y, Y]$ if and only if $x \leq_{\mathrm{PR}} y$ and $X \leq_{\mathrm{PR}} Y$.

3.2 Cover relations

We can alternatively describe the facial weak order by its cover relations.

Proposition 3.4. *The cover relations of the facial weak order $\leq_{\mathrm{FW}}$ are precisely the pairs of faces $F, G \in \mathscr{F}_\mathcal{A}$ such that F is a facet of G and $m_F = m_G$, or G is a facet of F and $M_F = M_G$.*

3.3 Covectors

We now use sign vectors to provide yet another description of the facial weak order. The underlying context is the more general theory of oriented matroids [4], to which the facial weak order extends. We fix a vector $\boldsymbol{e}_H$ normal to each hyperplane $H \in \mathcal{A}$, so that $H = \{v \in V \mid \langle \boldsymbol{e}_H \mid v \rangle = 0\}$ and $\langle \boldsymbol{e}_H \mid v \rangle > 0$ for any v in the interior $\mathring{B}$ of the base region B.

For any face $F \in \mathscr{F}_{\mathcal{A}}$, we let $F(H) \in \{-, 0, +\}$ be such that $\operatorname{sign}\langle \boldsymbol{e}_H \mid v \rangle = F(H)$ for any v in the interior $\mathring{F}$ of F. The *sign vector* of F is $\sigma(F) := (F(H))_{H \in \mathcal{A}}$. The facial weak order is then obtained by the componentwise order on the sign vectors.

Proposition 3.5. *For any $F, G \in \mathscr{F}_{\mathcal{A}}$, we have $F \le_{\mathrm{FW}} G \iff F(H) \ge G(H)$ for all $H \in \mathcal{A}$.*

3.4 Root sets

We now interpret the facial weak order on certain subsets of normal vectors of the hyperplanes. By analogy with the Coxeter setting, we call these normal vectors *roots*, and we denote $\Phi^+_{\mathcal{A}} := \{\boldsymbol{e}_H \mid H \in \mathcal{A}\}$, $\Phi^-_{\mathcal{A}} := \{-\boldsymbol{e}_H \mid H \in \mathcal{A}\}$ and $\Phi_{\mathcal{A}} := \Phi^+_{\mathcal{A}} \cup \Phi^-_{\mathcal{A}}$. For $X \subseteq \Phi_{\mathcal{A}}$, we still denote by $X^+ := X \cap \Phi^+_{\mathcal{A}}$ the positive part and $X^- := X \cap \Phi^-_{\mathcal{A}}$ the negative part.

Definition 3.6. The *root set* of a face $F \in \mathscr{F}_{\mathcal{A}}$ is

$$\boldsymbol{R}(F) := \big\{ \boldsymbol{e} \in \Phi_{\mathcal{A}} \mid \langle \boldsymbol{e} \mid x \rangle \le 0, \text{ for some } x \in \mathring{F} \big\}.$$

Proposition 3.7. *For $F, G \in \mathscr{F}_{\mathcal{A}}$, we have*

$$F \le_{\mathrm{FW}} G \iff \boldsymbol{R}(F)^+ \subseteq \boldsymbol{R}(G)^+ \quad \textit{and} \quad \boldsymbol{R}(F)^- \supseteq \boldsymbol{R}(G)^-.$$

3.5 Zonotope

We finally provide an alternative geometric interpretation of the root sets in terms of the geometry of zonotopes associated to hyperplane arrangements.

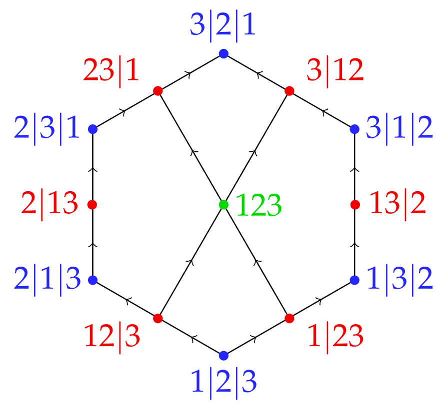
Definition 3.8. The *zonotope* $\mathbf{Z}_{\mathcal{A}}$ is the Minkowski sum of segments normal to the hyperplanes of $\mathcal{A}$, *i.e.* $\mathbf{Z}_{\mathcal{A}} := \sum_{H \in \mathcal{A}} [-\boldsymbol{e}_H, \boldsymbol{e}_H] = \{\sum_{H \in \mathcal{A}} \lambda_H \boldsymbol{e}_H \mid -1 \le \lambda_H \le 1 \text{ for all } H \in \mathcal{A}\}$.

This zonotope depends upon the choice of the normal vectors $\boldsymbol{e}_H$ of the hyperplanes $H \in \mathcal{A}$, but its combinatorics does not. Namely, P. H. Edelman gives in [8, Lemma 3.1] a bijection between the non-empty faces of the zonotope $\mathbf{Z}_{\mathcal{A}}$ and the the faces $\mathscr{F}_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ using the map in the following lemma.

Lemma 3.9 ([15])**.** *The map $\tau : F \mapsto \{\sum_{F \not\subseteq H} F(H)\boldsymbol{e}_H + \sum_{F \subseteq H} \lambda_H \boldsymbol{e}_H \mid -1 \le \lambda_H \le 1\}$ is a bijection from the faces $\mathscr{F}_{\mathcal{A}}$ to the non-empty faces of the zonotope $\mathbf{Z}_{\mathcal{A}}$. Moreover, F is the outer normal cone of $\tau(F)$, so that the fan $\mathbf{F}_{\mathcal{A}}$ of the arrangement $\mathcal{A}$ is the normal fan of $\mathbf{Z}_{\mathcal{A}}$.*

Proposition 3.10. *For any face F of $\mathscr{F}_{\mathcal{A}}$, the cone $\mathbb{R}_{\ge 0}\boldsymbol{R}(F)$ generated by the root set $\boldsymbol{R}(F)$ is the inner primal cone of the face $\tau(F)$ in the zonotope $\mathbf{Z}_{\mathcal{A}}$.*

3.6 Examples on Coxeter arrangements



In the type A Coxeter group, the facial weak order is the *pseudo-permutahedron* of [13]. While a region of the braid arrangement corresponds to a permutation of $[n]$, a face of the braid arrangement corresponds to an *ordered partition* of $[n]$. The pseudo-permutahedron is defined equivalently as:

- the transitive closure of its cover relations given for an ordered partition $\lambda = \lambda_1 | \dots | \lambda_k$ of $[n]$ by:
$$\lambda_1|\cdots|\lambda_i|\lambda_{i+1}|\cdots|\lambda_k \lessdot_{\mathrm{FW}} \lambda_1|\cdots|\lambda_i\lambda_{i+1}|\cdots|\lambda_k \qquad \text{if } \lambda_i \ll \lambda_{i+1},$$
$$\lambda_1|\cdots|\lambda_i\lambda_{i+1}|\cdots|\lambda_k \lessdot_{\mathrm{FW}} \lambda_1|\cdots|\lambda_i|\lambda_{i+1}|\cdots|\lambda_k \qquad \text{if } \lambda_{i+1} \ll \lambda_i,$$
where $X \ll Y$ means $\max(X) < \min(Y)$ or equivalently $x < y$ for all $x \in X, y \in Y$. This is the type A version of Proposition 3.4.
- the componentwise comparison $\lambda \le_{\mathrm{FW}} \lambda'$ if $\mathrm{inv}(\lambda, i, j) \le \mathrm{inv}(\lambda', i, j)$ for all $i < j$ on *inversion maps* of ordered partitions defined by $\mathrm{inv}(\lambda, i, j) := \mathrm{sign}\left(\lambda^{-1}(i) - \lambda^{-1}(j)\right)$. This is a type A combinatorial version of the more geometric Proposition 3.7.

Note that the pseudo-permutahedron has particularly relevant connections to F. Chapoton's Hopf algebra on all faces of the permutahedra [6], generalizing C. Malvenuto and C. Reutenauer's classical Hopf algebra on permutations [14].

For finite Coxeter arrangements, the facial weak order was defined in [16] in terms of its cover relations of Proposition 3.4 and studied in details in [7], where the characterizations of Definition 3.2 and Propositions 3.7 and 3.10 were presented. Note that in this situation, the faces of the Coxeter arrangement correspond to the cosets of all parabolic subgroups of W. To a coset xW_I of a parabolic subgroups W_I corresponds the facial interval $[x, xw_{\circ,I}]$ of the weak order on W, the subset $x(\Phi_I \cup \Phi^+)$ of the root system of W and the normal cone of the face $\mathrm{conv}\{xw(p) \mid w \in W_I\}$ of the W-permutahedron $\mathrm{conv}\{w(p) \mid w \in W\}$ (where p is inside the fundamental chamber of W).

4 Properties of the facial weak order

4.1 Lattice

The main result of this paper is the following statement. It extends the result of [7] for Coxeter arrangements. We also conjecture that the converse holds, see Conjecture 4.6.

Theorem 4.1. *If* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice, then* $\mathrm{FW}(\mathcal{A}, B)$ *is a lattice.*

Remark 4.2. Following Remark 3.3, note that the poset of intervals of a lattice is also a lattice with meet $[x, X] \wedge [y, Y] = [x \wedge y, X \wedge Y]$ and join $[x, X] \vee [y, Y] = [x \vee y, X \vee Y]$. However, when the poset of regions is a lattice, the facial weak order is a subposet but not a sublattice of the lattice of intervals of the poset of regions.

Although all details are out of reach in this extended abstract for space reasons, we want to give the main ideas of the proof of Theorem 4.1. Let us underline in particular that this proof is completely different from that of [7] in the special situation of Coxeter arrangements, as we have to get rid of all the Coxeter technology. It is based on two main ingredients:

(i) first, the BEZ lemma [5, Lemma 2.1] which states that a poset is a lattice as soon as there exists a join $x \vee y$ for every two elements x, y that both cover the same element.

(ii) second, Proposition 4.3 below that enables to lift the join in the facial weak order of a certain subarrangement of $\mathcal{A}$ into the join of the facial weak order of $\mathcal{A}$.

To properly state the latter property, recall that a *subarrangement* of an arrangement $\mathcal{A}$ is a subset $\mathcal{A}'$ of $\mathcal{A}$. There is a natural map $\mathscr{F}_{\mathcal{A}} \to \mathscr{F}_{\mathcal{A}'}$ that projects each face G in $\mathscr{F}_{\mathcal{A}}$ to the smallest face $G_{\mathcal{A}'}$ in $\mathscr{F}_{\mathcal{A}'}$ such that the relative interior of G is contained in the relative interior of $G_{\mathcal{A}'}$. Note that this map is surjective and preserves the facial weak order: if $F \le_{\mathrm{FW}} G$ in $\mathcal{A}$, then $F_{\mathcal{A}'} \le_{\mathrm{FW}} G_{\mathcal{A}'}$ in $\mathcal{A}'$. Here, we particularly focus on the subarrangement $\mathcal{A}_F := \{H \in \mathcal{A} \mid F \subseteq H\}$ defined by all hyperplanes which contain a certain face $F \in \mathscr{F}_{\mathcal{A}}$. This subarrangement $\mathcal{A}_F$ is known as the *support* of F or the *localization* of $\mathcal{A}$ to F. We use the shorthand G_F for $G_{\mathcal{A}_F}$ in this situation. Note that the surjection $G \to G_F$ restricts to a bijection between $\{G \in \mathscr{F}_{\mathcal{A}} \mid F \subseteq G\}$ and $\mathscr{F}_{\mathcal{A}_F}$. We take advantage of subarrangements through the following instrumental statement.

Proposition 4.3. *For any three faces $X, Y, Z \in \mathscr{F}_{\mathcal{A}}$ such that $Z \subseteq X \cap Y$, if there exists a face W containing Z such that $W_Z = X_Z \vee_{\mathrm{FW}} Y_Z$ in $\mathrm{FW}(\mathcal{A}_Z, B_Z)$ then $W = X \vee_{\mathrm{FW}} Y$ in $\mathrm{FW}(\mathcal{A}, B)$.*

We can now sketch the proof of Theorem 4.1 using these two ingredients. According to (i), we consider two cover relations $Z \lessdot_{\mathrm{FW}} X$ and $Z \lessdot_{\mathrm{FW}} Y$ of the same element. We know from Proposition 3.4 that this is equivalent to $|\dim Z - \dim X| = 1$, $Z \le_{\mathrm{FW}} X$, and either $Z \subseteq X$ or $X \subseteq Z$ and similarly for Y. By symmetry of X and Y, we thus obtain the following three cases:

(1) $X \cup Y \subseteq Z$ and $\dim X = \dim Y = \dim Z - 1$,

(2) $Z \subseteq X \cap Y$ and $\dim X = \dim Y = \dim Z + 1$, and

(3) $X \subseteq Z \subseteq Y$ and $\dim X + 1 = \dim Y - 1 = \dim Z$.

In each case we consider the subarrangement associated to the largest face contained in all three faces. Namely, the subarrangement $\mathcal{A}_{X \cap Y} = \{H \in \mathcal{A} \mid X \cap Y \subseteq H\}$ for case (1), the subarrangement $\mathcal{A}_Z = \{H \in \mathcal{A} \mid Z \subseteq H\}$ for case (2), and finally the subarrangement $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ for case (3) . We then prove in each of the three cases that the faces corresponding to X and Y in these subarrangements admit a join, and we lift it to the join of X and Y using Proposition 4.3.

Note that, in contrast to the special case of Coxeter arrangements treated in [7], the present approach to Theorem 4.1 does not provide explicit formulas for meets and joins in the facial weak order of a hyperplane arrangements. The computation relies on identifying the right subarrangement and computing meets and joins in subarrangements.

4.2 The poset of regions inside the facial weak order

We now want to compare the poset of regions with the facial weak order. Note first that it is immediate from Definition 3.2 that the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is the subposet of the facial weak order $\mathrm{FW}(\mathcal{A}, B)$ induced by the regions of $\mathscr{R}_{\mathcal{A}}$. Recall now that a *sublattice* L' of a lattice L is an induced subposet such that $u \vee v \in L'$ and $u \wedge v \in L'$ for any $u, v \in L'$. Based on a BEZ-like characterization of sublattices [18, Lemma 9-2.11], we obtain the following statement.

Proposition 4.4. *For a simplicial arrangement $\mathcal{A}$, the lattice of regions is a sublattice of the facial weak order* $\mathrm{FW}(\mathcal{A}, B)$.

Remark 4.5. Following Remark 3.3, observe that a lattice is always the sublattice of its lattice of intervals induced by singletons. The difficulty here is that the facial weak order is not itself a sublattice of the lattice of intervals of the poset of regions, as observed in Remark 4.2.

Note that we conjecture that Proposition 4.4 holds for any arrangement, not only simplicial ones. This would imply in particular the following conjecture.

Conjecture 4.6. *For any hyperplane arrangement $\mathcal{A}$ and any base region B of $\mathcal{A}$, the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *is a lattice if and only if the facial weak order* $\mathrm{FW}(\mathcal{A}, B)$ *is a lattice.*

4.3 Further lattice properties of the facial weak order

We now explore some additional lattice properties of the facial weak order. We start with a convenient self-duality, inherited from that of the poset of regions.

Proposition 4.7. *The facial weak order* $\mathrm{FW}(\mathcal{A}, B)$ *is self-dual under* $F \mapsto -F := \{-v \mid v \in F\}$.

We next aim to find all the join-irreducible elements of the facial weak order. An element x of a finite lattice L is *join-irreducible* if $x \neq \bigvee L'$ for all $L' \subseteq L \smallsetminus \{x\}$. Equivalently, x is join-irreducible if and only if it covers exactly one element $x_\star$ of L. We denote by $\mathrm{JIrr}(\mathrm{FW})$ and $\mathrm{JIrr}(\mathrm{PR})$ the sets of join-irreducible elements in the facial weak order and in the poset of regions. The following statement describes the join-irreducibles of the facial weak order in terms of that of the poset of regions. A similar statement holds for meet-irreducible elements by Proposition 4.7.

Proposition 4.8. *Suppose $\mathcal{A}$ is a simplicial hyperplane arrangement and let F be a face with facial interval $[m_F, M_F]$. Then $F \in \mathrm{JIrr}(\mathrm{FW})$ if and only if $M_F \in \mathrm{JIrr}(\mathrm{PR})$ and* $\mathrm{codim}(F) \in \{0, 1\}$.

A lattice is *semidistributive* if $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$, and $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$. The poset of regions of a simplicial hyperplane arrangement is known to be semidistributive [20, Theorem 3]. This property extends to the facial weak order.

Theorem 4.9. *For a simplicial arrangement $\mathcal{A}$, its facial weak order is a semidistributive lattice.*

4.4 Lattice congruences

Recall that a *lattice congruence* of a lattice $(L, \leq, \wedge, \vee)$ is an equivalence relation on L that respects the meet and the join, *i.e.* such that $x \equiv x'$ and $y \equiv y'$ implies $x \wedge y \;\equiv\; x' \wedge y'$ and $x \vee y \;\equiv\; x' \vee y'$. It turns out that lattice congruences of the poset of regions can be transported to lattice congruences of the facial weak order as follows.

Theorem 4.10. *Any lattice congruence* $\equiv_{\mathrm{PR}}$ *of the poset of regions* $\mathrm{PR}(\mathcal{A}, B)$ *defines a lattice congruence* $\equiv_{\mathrm{FW}}$ *of the facial weak order by* $F \equiv_{\mathrm{FW}} G \iff m_F \equiv_{\mathrm{PR}} m_G$ *and* $M_F \equiv_{\mathrm{PR}} M_G$.

Remark 4.11. Following Remark 3.3, note that a congruence $\equiv$ of a lattice always defines a congruence $\equiv$ of its lattice of intervals given by $[x, X] \equiv [y, Y] \iff x \equiv y$ and $X \equiv Y$. It is surprising that the same recipe works although the facial weak order is not a sublattice of the lattice of intervals of the poset of regions, as observed in Remark 4.2.

Our interest in Theorem 4.10 is that it provides a lattice structure on the fan associated to the lattice congruence $\equiv_{\mathrm{PR}}$. Recall that, when the poset of regions $\mathrm{PR}(\mathcal{A}, B)$ is a lattice, each lattice congruence $\equiv_{\mathrm{PR}}$ of $\mathrm{PR}(\mathcal{A}, B)$ defines a fan $\mathbf{F}_{\equiv_{\mathrm{PR}}}$ whose maximal cones are obtained by glueing together the regions of the arrangement $\mathcal{A}$ that belong to the same congruence class of $\equiv_{\mathrm{PR}}$. While the quotient $\mathrm{PR}(\mathcal{A}, B)/{\equiv_{\mathrm{PR}}}$ defines a lattice structure on the maximal cones of $\mathbf{F}_{\equiv_{\mathrm{PR}}}$, the quotient $\mathrm{FW}(\mathcal{A}, B)/{\equiv_{\mathrm{FW}}}$ defines a lattice structure on all cones of $\mathbf{F}_{\equiv_{\mathrm{PR}}}$. In particular, if the fan $\mathbf{F}_{\equiv_{\mathrm{PR}}}$ is polytopal (this remains an open question in general, see [17]), then $\mathrm{FW}(\mathcal{A}, B)/{\equiv_{\mathrm{FW}}}$ is an order on all faces of the corresponding polytope. For instance, quotients of the facial weak order for finite Coxeter arrangements provide lattice structures on all faces of the generalized associahedra of [11] which are polytopal realizations of the Cambrian fans [19].

4.5 Poset topology

We finally determine the homotopy type of intervals of the facial weak order. Given elements x, y of a poset P, let (x, y) (resp. $[x, y]$) denote the open interval (resp. closed interval) of x and y and let $P_{<x}$ (resp. $P_{>x}$) denote the set of elements $z \in P$ such that $z < x$ (resp. $z > x$). Recall that the *order complex* $\Delta(P)$ of a poset P is the simplicial complex of chains $x_0 < \cdots < x_d$ of elements of P. The link of a face F of a simplicial complex Δ is the subcomplex of faces G for which $F \cap G = \varnothing$ and $F \cup G$ is a face of Δ. The join $\Delta \star \Delta'$ of two complexes with disjoint ground sets is the simplicial complex with faces $F \sqcup F'$ where $F \in \Delta$ and $F' \in \Delta'$. The link of a face $x_0 < \ldots < x_d$ in an order complex $\Delta(P)$ is isomorphic to the join of the order complexes of $P_{<x_0}$, (x_0, x_1), $\cdots$, (x_{d-1}, x_d), $P_{>x_d}$. Hence, the local topology of $\Delta(P)$ is completely determined by the topology of open intervals and principal order ideals and filters of P.

P. H. Edelman and J. W. Walker determined the local topology of the poset of regions [10] showing that the facial interval of a face X is homotopy equivalent to a sphere of dimension $\operatorname{codim}(X) - 2$ and every other interval is contractible. We now determine the homotopy type of intervals of the facial weak order.

Theorem 4.12. *Let $\mathcal{A}$ be an arrangement with base region B. Let X, Y be covectors such that $X \leq_{\mathrm{FW}} Y$ and set $Z = X \cap Y$. If $X \leq_{\mathrm{FW}} Z \leq_{\mathrm{FW}} Y$ and $Z = X_{-Z} \cap Y$, then the order complex of the open interval (X, Y) in $\mathrm{FW}(\mathcal{A}, B)$ is homotopy equivalent to a sphere of dimension $\dim(X) + \dim(Y) - 2\dim(Z) - 2$. Every other interval is contractible.*

Knowing the homotopy type of intervals allows us to calculate the Möbius function for the facial weak order. Recall that the *Möbius function* of a poset P is the function $\mu : P \times P \to \mathbb{Z}$ defined inductively by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius function can be restated using its homotopy type. A contractible interval (x, y) is homotopy equivalent to a point and therefore $\mu(x, y) = 0$. An interval (x, y) homotopy equivalent to the sphere $\mathbb{S}^n$ gives $\mu(x, y) = n$. As a consequence of Theorem 4.12 we have the following corollary.

Corollary 4.13. *Let $\mathcal{A}$ be an arrangement with base region B. Let X, Y be covectors such that $X \leq_{\mathrm{FW}} Y$ and set $Z = X \cap Y$. Then*

$$\mu(X,Y) = \begin{cases} (-1)^{\dim(X)+\dim(Y)} & X \leq_{\mathrm{FW}} Z \leq_{\mathrm{FW}} Y \text{ and } Z = X_{-Z} \cap Y \\ 0 & \text{otherwise.} \end{cases}$$

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