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Colored five-vertex models and Lascoux polynomials and atoms

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Abstract. We construct an integrable colored five-vertex model whose partition function is a Lascoux atom based on the five-vertex model of Motegi and Sakai and the colored five-vertex model of Brubaker, the first author, Bump, and Gustafsson. We then modify this model in two different ways to construct a Lascoux polynomial, yielding the first known proven combinatorial interpretation of a Lascoux polynomial and atom. Using this, we prove a conjectured combinatorial interpretation in terms of set-valued tableaux of a Lascoux polynomial and atom due to Pechenik and the second author. We also prove the combinatorial interpretation of the Lascoux atom using set-valued skyline tableaux of Monical.

Keywords: Lascoux polynomial, Lascoux atom, Grothendieck polynomial, colored lattice model, five-vertex model

1 Introduction

Solvable lattice models are often models for simplified physical systems such as water molecules, but are known to have applications to a diverse number of mathematical fields. By tuning the Boltzmann weights, special functions can be expressed as the partition function of the lattice model. Then the Yang–Baxter equation can be used on the model in order to prove functional equations for the partition function, often simplifying intricate combinatorial or algebraic arguments. For example, this approach was applied by Kuperberg in counting the number of alternating sign matrices using a six-vertex model [11]. Similar techniques have also been used to study probabilistic models such as the (totally) asymmetric simple exclusion process, *e.g.*, [21].

We will be focusing on the five-vertex model of Motegi and Sakai [21, 22] (with a gauge transformation on the Boltzmann weights), whose partition function is a *(symmetric β-)Grothendieck polynomial* [6, 14, 15]. This was used to establish a Cauchy identity and skew decomposition for Grothendieck polynomials. Grothendieck polynomials

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arise from the study of the connective K-theory of the *Grassmannian*, the space of k-dimensional subspaces of $\mathbb{C}^n$, where they correspond to the push-forward of the class for any Bott–Samelson resolution of a Schubert variety. These form a basis for the connective K-theory ring of the Grassmannian and are indexed by partitions that fit inside a $k \times (n-k)$ rectangle. Thus, they are the K-theory analog of Schur functions, which are recovered by setting $\beta = 0$. Grothendieck polynomials have been well-studied with a combinatorial interpretation using set-valued tableaux and a Littlewood–Richardson rule [4]. Recently, a crystal structure was applied to set-valued tableaux [19]. The equivariant K-theory of the Grassmannian was studied using integrable systems by Wheeler and Zinn-Justin [27], yielding a construction of double Grothendieck polynomials.

There is a refinement of Schur functions that are known as *key polynomials* given in terms of divided difference operators [13]. Key polynomials are also known as Demazure characters as they can be interpreted as characters of Demazure modules, which also have crystal bases and an explicit combinatorial description [9] and a geometric construction [1, 12]. The K-theory analog of key polynomials are the so-called *Lascoux polynomials* [13], which despite recent attention [10, 18, 19, 20, 23, 24], do not have any known geometric or representation theoretic interpretation and have many conjectural combinatorial interpretations [10, 18, 19, 23, 24], some of which are known to be equivalent [18, 20].

The goal of this paper is to modify the five-vertex model so that the partition function is a Lascoux polynomial. To do this, we need an even smaller piece, the *Lascoux atom* [18], which is essentially the new terms that appear when taking a larger Lascoux polynomial and has a description in terms of divided difference operators. On the solvable lattice model side, we employ the idea of Borodin and Wheeler of using a *colored* lattice model [2], where one can then study the atoms of special functions. Indeed by modifying the colored five-vertex by Brubaker, Bump, the first author, and Gustafsson [3] using the Motegi–Sakai weights, our main result is the construction of an integrable colored five-vertex model whose partition function is a Lascoux atom. Then by a suitable modification of our model, we obtain a Lascoux polynomial. In fact, we provide two such modifications and show they are naturally in bijection.

As an application, we prove [23, Conjecture 6.1], thus establishing the first combinatorial interpretation of Lascoux polynomials and atoms by using a notion of a K-key tableau of a set-valued tableau. We do this by refining the bijection between Gelfand–Tsetlin patterns and states of our five-vertex model to allow markings in certain places, as in [19], in order to obtain a bijection with set-valued tableaux. To make this weight preserving, we need to also twist by the Lusztig involution (an action of the long element of the symmetric group), which utilizes the crystal structure on set-valued tableaux from [19]. Another application is proving the conjectured combinatorial interpretation of [18, Conjecture 5.2]. We do this by (1) showing our model is in bijection with reverse set-valued tableaux; (2) noting the bijection from [18, Theorem 2.4] is governed by the semistandard case of [16] and adding the so-called free entries (which are just markings on certain vertices the state); and (3) using that the semistandard case is known to give Demazure atoms [16, 17].

This extended abstract is organized as follows. In Section 2, we provide the necessary background on tableaux combinatorics, Grothendieck polynomials, and Lascoux polynomials/atoms. In Section 3 we introduce a new colored lattice model and prove by using a Yang–Baxter equation, that its partition function is equal to a Lascoux atom. In Section 4 we prove [23, Conjecture 6.1] and [18, Conjecture 5.2] by using our main result.

This is an extended abstract of [5], where we refer the reader for additional details.

2 Background

Fix a positive integer n. Let $\mathbf{z} = (z_1, z_2, \dots, z_n)$ be a finite number of indeterminates. Let S_n denote the symmetric group on n elements with simple transpositions $(s_1, \dots, s_{n-1})$. For $w \in S_n$, let $\ell(w)$ denote the *length* of w: the minimal number of simple transpositions whose product equals w. We denote by w_0 the longest element in S_n. Let $\leq$ denote the (strong) Bruhat order on S_n. For more information on the symmetric group, we refer the reader to [25]. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a *partition*, a sequence of weakly decreasing nonnegative integers. Let $\ell(\lambda)$ be the *length*, the number of non-zero entries of λ. We use English convention for our tableaux.

A *(semistandard) set-valued tableau of shape* λ is a filling of the boxes of the Young diagram of λ with finite non-empty sets of positive integers that satisfy

$$\begin{array}{|c|c|}\hline X & Y \\ \hline Z & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{implies } \max X \leq \min Y \text{ and } \max X < \min Z.$$

Let $\mathrm{SVT}^n(\lambda)$ denote the set of all set-valued tableaux of shape λ such that the maximum integer appearing is n. Let $\mathrm{SSYT}^n(\lambda)$ denote the set of *semistandard (Young) tableaux*, where every entry has size 1, of shape λ and maximum entry n. Define the *β-weight* of a set-valued tableau $T \in \mathrm{SVT}^n(\lambda)$ to be

$$\mathrm{wt}_\beta(T) := \beta^{|T|-|\lambda|} z_1^{m_1} \dots z_n^{m_n}$$

where m_i are the number of i's occurring in T and $|T| = m_1 + \cdots + m_n$.

A semistandard Young tableau is called a *key tableau* if the entries of column $i+1$ are a subset of the entries of column i for all $1 \leq i < \lambda_1$. We define a left S_n-action on key tableau K with maximum entry n by applying $w \in S_n$ to each entry of K and sorting columns to be strictly increasing. Let $K_{w\lambda}$ denote the key tableau by applying w to the key tableau of shape λ with every entry of row i filled by i.

Let T be a set-valued tableau. For a semistandard Young tableau S, let $k(S)$ denote the (right) key tableau associated to S (see, *e.g.*, [28, 3] for algorithms to compute this). Let

$\min(T)$ denote the semistandard Young tableau formed by taking the minimum of each entry in T. Let T^* denote the *Lusztig involution* on T using the crystal structure from [19] (see [23, Eq. (6.2)]). We do not require the exact definition of the Lusztig involution, only that $\mathrm{wt}(T^*) = w_0 \,\mathrm{wt}(T)$. From [23, Sec. 6], we define the *(right) K-key tableau* of T to be

$$K(T) := k(\min(T^*)^*).$$

From [19, Sec. 4], a *marked Gelfand–Tsetlin (GT) pattern* is a sequence of partitions $\Lambda = (\lambda^{(j)})_{j=0}^n$, called a *Gelfand–Tsetlin (GT) pattern*, such that $\lambda^{(0)} = \emptyset$ and the skew shape $\lambda^{(j)}/\lambda^{(j-1)}$ does not contain a vertical domino (*i.e.*, is a horizontal strip),[1] with a set M of entries that are "marked," where the entry (i, j), for $2 \leq j \leq n$ and $1 \leq i < \ell(\lambda^{(j)})$, is allowed to be marked if and only if $\lambda^{(j)}_{i+1} < \lambda^{(j-1)}_i$. In particular, an entry (i, j) cannot be marked if the entry to the right equals the entry to the southeast. We depict a marked GT pattern as a triangular array with the top-row corresponding to $\lambda^{(n)}$ and the bottom row $\lambda^{(1)}$ and a marked entry (i, j) as a box around the entry $\lambda^{(j)}_i$.

Next, we recall the bijection ϕ between marked GT patterns and set-valued tableaux, which is defined recursively as follows. Consider a marked GT pattern (Λ, M). Start with $T_0 = \emptyset$. Suppose we are at step j, where the set-valued tableau is T_{j-1} that has entries in $1, \dots, j-1$. For each marked entry (i, j), we add j to the rightmost entry of i-th row of T_{j-1}, and denote this T'_j. Then we consider the horizontal strip $\lambda^{(j)}/\lambda^{(j-1)}$ with all entries being $\{j\}$, which we add to T'_j to obtain a set-valued tableau T_j of shape $\lambda^{(j)}$. We repeat this for every row of Λ and the result is $\phi(\Lambda, M)$. We define the weight of a marked GT pattern $\mathrm{wt}(\Lambda, M) = \mathrm{wt}\big(\phi(\Lambda, M)\big)$.

We also require one additional combinatorial object from [18], where we use the description given in [20]. For a permutation $w \in S_n$, define the *(semistandard) skyline diagram* $w\lambda$ to be the Young diagram of λ but the rows permuted by w. In particular, we have $w\lambda = (\lambda_{w(1)}, \dots, \lambda_{w(n)})$. A *set-valued skyline tableau of shape* $w\lambda$ is a filling of a skyline diagram $w\lambda$ with finite nonempty sets of positive integers such that entries do not repeat in a column; rows weakly decrease in the set-valued sense; *anchors*, the largest entry in a box, satisfy the triple conditions of [16, Sec. 2.1]; the *free* entries, the non-anchor entries in a box, are in the top-most row possible; and anchors in the first column equal their row index. Let $\mathrm{SSLT}(w\lambda)$ denote the set of set-valued skyline tableaux of shape $w\lambda$. Define the β-weight for a set-valued skyline tableau the same as for a set-valued tableau.

The combinatorial definition of a *Grothendieck polynomial* is due to Buch [4, Theorem 3.1] and is generating function of semistandard set-valued tableaux:

$$\mathfrak{G}_\lambda(\mathbf{z}; \beta) = \sum_{T \in \mathrm{SVT}^n(\lambda)} \mathrm{wt}_\beta(T).$$

[1] This is equivalent to the usual interlacing condition on GT patterns.

Using ϕ, a Grothendieck polynomial is also equal to the generating function over marked GT patterns [19, Prop. 4.5]. A definition as a ratio of determinants was given by Ikeda and Naruse [7]. We will consider another algebraic definition of the Grothendieck polynomials using the *Demazure–Lascoux operator* ϖ_i, which defines an action of the 0-Hecke algebra on $\mathbb{Z}[\beta][\mathbf{z}]$ by

$$\varpi_i f(\mathbf{z};\beta) = \frac{(z_i + \beta z_i z_{i+1}) f(\mathbf{z};\beta) - (z_{i+1} + \beta z_i z_{i+1}) f(s_i \mathbf{z};\beta)}{z_i - z_{i+1}}.$$

In particular, the Demazure–Lascoux operators satisfy the braid relations and $\varpi_i^2 = \varpi_i$. Hence, for any permutation $w \in S_n$, one may define $\varpi_w := \varpi_{i_1}\varpi_{i_2}\cdots\varpi_{i_\ell}$ for any choice of reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$. Thus, we can write $\mathfrak{G}_\lambda(\mathbf{z};\beta) = \varpi_{w_0} z_1^{\lambda_1}\cdots z_n^{\lambda_n}$ [7, 8, 14] When $\beta = 0$, we obtain the *Schur function* $s_\lambda(\mathbf{z})$.

Next, following [18], we define the *Demazure–Lascoux atom operator* $\overline{\varpi}_i := \varpi_i - 1$, which satisfies the braid relations and $\overline{\varpi}_i^2 = -\overline{\varpi}_i$. We define the *Lascoux polynomials* [13] and *Lascoux atoms* [18] by

$$L_{w\lambda}(\mathbf{z};\beta) := \varpi_w z_1^{\lambda_1}\cdots z_n^{\lambda_n}, \qquad \overline{L}_{w\lambda}(\mathbf{z};\beta) := \overline{\varpi}_w z_1^{\lambda_1}\cdots z_n^{\lambda_n},$$

which satisfies $\mathfrak{G}_\lambda(\mathbf{z};\beta) = L_{w_0\lambda}(\mathbf{z};\beta)$ and [18, Theorem 5.1]:

$$L_{w\lambda}(\mathbf{z};\beta) = \sum_{u\le w} \overline{L}_{u\lambda}(\mathbf{z};\beta). \tag{2.1}$$

It is an open problem to find a geometric or representation-theoretic interpretation for general Lascoux polynomials. We prove the following conjectured combinatorial interpretations of Lascoux polynomials and atoms in this extended abstract

Conjecture 2.1 ([23, Conjecture 6.1] and [18, Conjecture 5.2])**.** *We have*

$$\overline{L}_{w\lambda}(\mathbf{z};\beta) = \sum_{\substack{T\in \mathrm{SVT}^n(\lambda)\\ K(T)=K_{w\lambda}}} \mathrm{wt}_\beta(T) = \sum_{S\in \mathrm{SSLT}(w\lambda)} \mathrm{wt}_\beta(S), \qquad L_{w\lambda}(\mathbf{z};\beta) = \sum_{\substack{T\in \mathrm{SVT}^n(\lambda)\\ K(T)\le K_{w\lambda}}} \mathrm{wt}_\beta(T).$$

Note that the second Lascoux atom formula is also the K-theoretic analog of Demazure characters being described by skyline tableaux [16, 17].

3 Colored lattice models and Lascoux atoms

We will build colored models that represent Lascoux atoms, generalizing the work in [3], whose partition function was a Demazure atom. The model we consider is a colored version of the lattice model of Motegi and Sakai [21], whose partition function is $\mathfrak{G}_\lambda(\mathbf{z};\beta)$.

$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_1^\dagger$	$\mathtt{b}_1^\circ$	$\mathtt{b}_2$	$\mathtt{c}_1$
1	$1+\beta z$	1	1	1	z	1

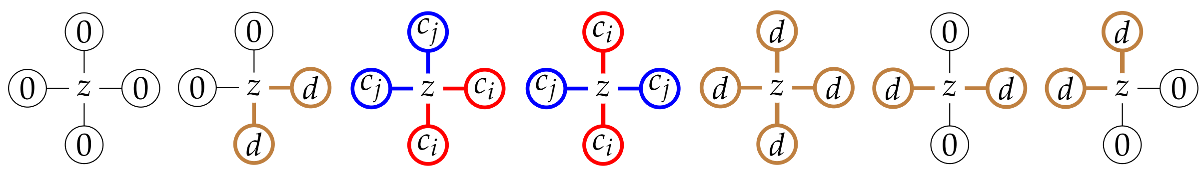


Figure 1: The colored Boltzmann weights with $c_i > c_j$ and d being any color.

Consider a rectangular grid with n horizontal lines and m vertical lines (we consider a crossing of the lines to be vertices). We also fix an n-tuple of colors $\mathbf{c} = (c_1 > c_2 > \cdots > c_n > 0)$. Let $w \in S_n$, and let $w\mathbf{c} = (c_{w(1)}, c_{w(2)}, \dots, c_{w(n)})$ be the colors permuted by w. We label the bottom and right (half) edges by 0, the left (half) edges by $ww_0\mathbf{c}$ from top to bottom, and the top edges by λ: the i-th 1 in the $\{0,1\}$-sequence of λ, counted from the left, is labeled by c_i. An *admissible state* is an assignment of labels on the interior edges of the grid such that all vertices are of the form given in Figure 1, which also specifies their *Boltzmann weights*. Let $\mathfrak{S}_\lambda$ denote the set of all possible admissible states of the model. The *(Boltzmann) weight* $\mathrm{wt}(S)$ of an admissible state $S \in \mathfrak{S}_\lambda$ is the product of all of the Boltzmann weights of all vertices with $z = z_i$ in the i-th row numbered starting from top. Let $\overline{\mathfrak{S}}_{\lambda,w}$ denote the set of all possible admissible states for this model. The *partition function* of a model $\mathcal{M}$

$$Z(\mathcal{M}; \mathbf{z}; \beta) := \sum_{S \in \mathcal{M}} \mathrm{wt}(S)$$

is the sum of the Boltzmann weights of all possible admissible states of $\mathcal{M}$.

Using the colors and the admissible configurations, we can think of an admissible state in $\mathfrak{S}_{\lambda,w}$ as corresponding to a wiring diagram of w, where the different strands are represented by different colors. Indeed, we can think of $\mathtt{a}_2$, $\mathtt{b}_2$, and $\mathtt{c}_1$ as a single strand passing through the vertex (possibly turning), $\mathtt{b}_1^\dagger$ as two strands crossing at the vertex (thus corresponding to a simple transposition), and $\mathtt{b}_1$ as two strands both passing near the vertex but not crossing.

Our model is amenable to study via the Yang–Baxter equation. We introduce the R-matrix for this model, which we call the *colored R-matrix*, and the admissible configurations with their Boltzmann weights are given in Figure 2. Furthermore, we can see that the R-matrix generally corresponds to the vertices of the L-matrix rotated by 45° clockwise and the weights of the L-matrix take $z = \frac{z_j - z_i}{1+\beta z_i}$ and are scaled by $(1+\beta z_i)z_j$. To distinguish them from the usual vertices given by the L matrix, we draw them tilted on their side. Together with the previously introduced vertices, they satisfy the RLL version

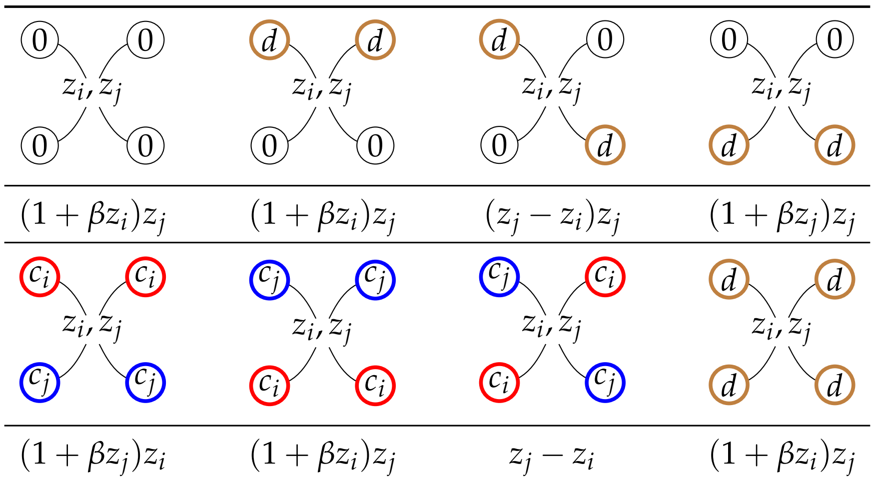
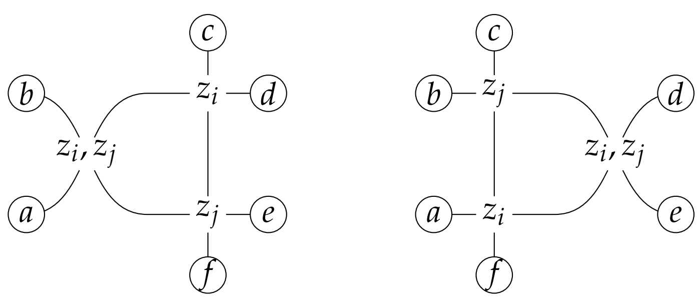


Figure 2: The colored R-matrix with $c_i > c_j$ and d being any color. Note that the weights are not symmetric with respect to color.

of the Yang–Baxter equation (hence, this model is integrable). It is a finite computation to verify this since it requires at most 3 colors, which can easily be done by computer.

Proposition 3.1. *Consider the L-matrix given in Figure 1 and R-matrix given in Figure 2. The partition function of the following two models*



(3.1)

are equal for any boundary conditions $a, b, c, d, e, f \in \{0, c_1, \ldots, c_n\}$.

By using the Yang–Baxter equation and the train argument (see [5] for more information on the train argument), we can derive the following equation for the partition functions of our lattice model. That is to say, we add an R-matrix to one row, pass it through to the other side using the Yang–Baxter equation (see Figure 3), and then obtain a functional equation that corresponds to acting by a Demazure–Lascoux operator.

Lemma 3.2. *Let* $w \in S_n$*, and consider* s_i *be such that* $s_i w > w$*. Then we have*

$$Z(\overline{\mathfrak{S}}_{\lambda,s_i w}; \mathbf{z}; \beta) = \frac{(1 + \beta z_i) z_{i+1} \big(Z(\overline{\mathfrak{S}}_{\lambda,w}; \mathbf{z}; \beta) - Z(\overline{\mathfrak{S}}_{\lambda,w}; s_i \mathbf{z}; \beta) \big)}{z_i - z_{i+1}}.$$

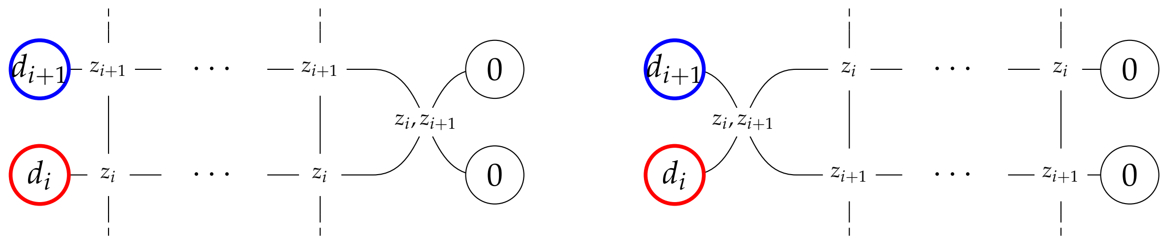
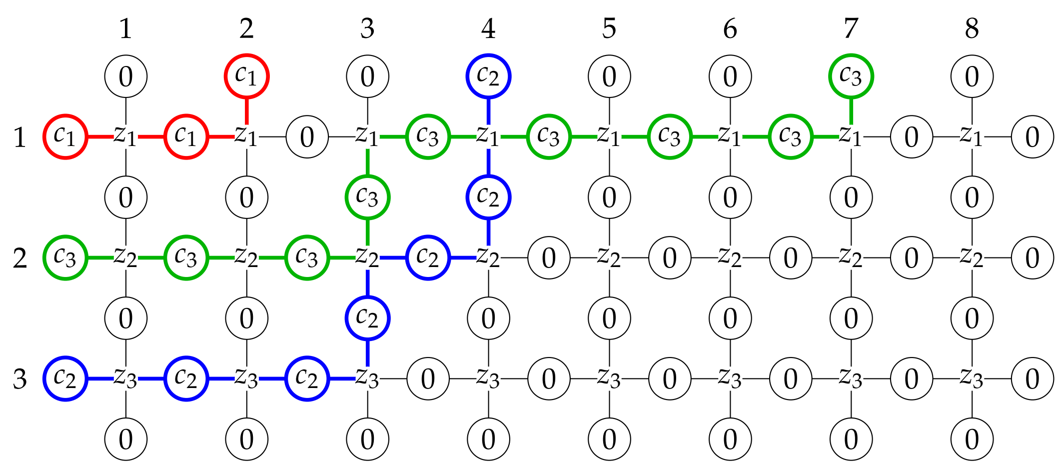


Figure 3: Left: The model $\overline{\mathfrak{S}}_{\lambda,w}$ with an R-matrix attached on the right. Right: The model after using the Yang–Baxter equation in the same model.

Theorem 3.3. *We have* $\overline{L}_{w\lambda}(\mathbf{z};\beta) = Z(\overline{\mathfrak{S}}_{\lambda,w};\mathbf{z};\beta)$.

Example 3.4. A state for the colored system $\overline{\mathfrak{S}}_{\lambda,s_2s_1}$, with $m = 8$, $n = 3$, and $\lambda = (4,2,1)$:



We use colors $c_1 > c_2 > c_3$. The Boltzmann weight of this state is $(1+\beta z_1)z_1^3 z_2^2 z_3^2$.

Let $\mathfrak{S}_{\lambda,w}$ denote the same model as $\overline{\mathfrak{S}}_{\lambda,w}$ with additional colored configurations $\mathtt{b}_1'$, defined as $\mathtt{b}_1$ except for $c_j < c_i$, whenever the colors $c_j < c_i$ do not cross in $\overline{\mathfrak{S}}_{\lambda,w}$; *i.e.*, whenever (i,j) is not an inversion of ww_0. In this case, we can remove configurations $\mathtt{b}^\dagger$ and $\mathtt{b}_1$ for colors $c_i > c_j$ from the model without changing the possible states. We extend the definition of the above R-matrix, except we require that the bottom left two configurations only appear when colors c_i and c_j cross, and we add the following additional two configurations when they do not cross:

$(1+\beta z_i)z_j$ $\qquad$ $(1+\beta z_j)z_i$ $\qquad$ (3.2)

(We have simply interchanged the bottom left two Boltzmann weights from the R-matrix in Figure 2.) Indeed, this allows us to show the model is integrable.

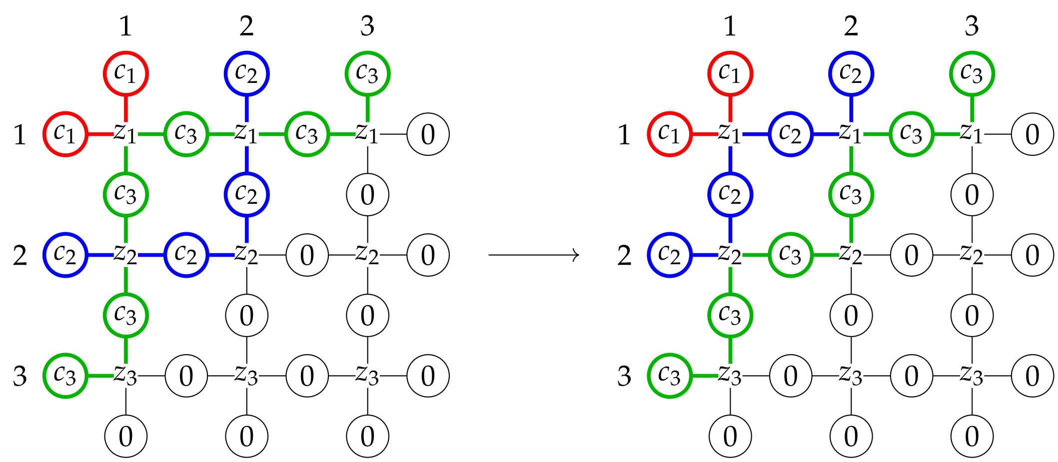
Proposition 3.5. *Consider the modified L-matrix and R-matrix given above for $\mathfrak{S}_{\lambda,w}$. Then Proposition 3.1 holds for this (colored) model $\mathfrak{S}_{\lambda,w}$.*

Theorem 3.6. *We have $L_{w\lambda}(\mathbf{z};\beta) = Z(\mathfrak{S}_{\lambda,w};\mathbf{z};\beta)$.*

This can be proven using the train argument or a direct combinatorial argument by replacing first touching corners with crossings and Theorem 3.3.

Note that we could use the train proof and then use the combinatorial proof to show Equation (2.1) as a consequence (thus yielding an alternative proof of [18, Theorem 5.1]).

Example 3.7. We consider replacing the $\mathtt{b}^{\dagger}$ corresponding to c_1 and c_3 in $\mathfrak{S}_{\emptyset,1}$ for $m = n = 3$ with colors $c_1 > c_2 > c_3$. This introduces a double crossing of the colors c_2 and c_3. We can then resolve this to a valid state in $\mathfrak{S}_{\emptyset,w_0}$ by



Finally, we construct another variation on the model $\overline{\mathfrak{S}}_{\lambda,w}$ where instead of adding $\mathtt{b}'_1$ for certain colors (and removing the corresponding $\mathtt{b}_1$ and $\mathtt{b}^{\dagger}$), we replace $\mathtt{b}_1$ with $\mathtt{b}'_1$ for all colors. Let $\mathfrak{S}'_{\lambda,w}$ denote this modified model. We also use the following R-matrix given by Figure 2 except we *replace* the two bottom left configurations by Equation (3.2). This satisfies the Yang–Baxter equation (the proof is the same as Proposition 3.1):

Proposition 3.8. *Consider the modified L-matrix and R-matrix given above for $\mathfrak{S}'_{\lambda,w}$. Then Proposition 3.1 holds for this model $\mathfrak{S}'_{\lambda,w}$.*

Theorem 3.9. *We have $L_{w\lambda}(\mathbf{z};\beta) = Z(\mathfrak{S}'_{\lambda,w};\mathbf{z};\beta)$.*

Proposition 3.10. *There exists a weight-preserving bijection $\xi \colon \mathfrak{S}'_{\lambda,w} \to \mathfrak{S}_{\lambda,w}$.*

Proof sketch. Construct a bijection: For each pair of crossing colors, replace the northeast $\mathtt{b}'_1$ with $\mathtt{b}^{\dagger}$ and the other $\mathtt{b}'_1$ and $\mathtt{b}^{\dagger}$ with $\mathtt{b}_1$. □

4 Lascoux atoms to K-key and set-valued skyline tableaux

In this section, we sketch the proof of Conjecture 2.1.

First, recall that we can equate λ with a $\{0,1\}$-sequence of length m by considering the Young diagram inside of an $n \times (m-n)$ rectangle and starting at the bottom left, each up step we write a 1 and each right step we write a 0. For example, with $\lambda = 522100$ (so $n = 6$) with $m = 14$, the corresponding $\{0,1\}$-sequence is 11010110001000. Next, by forgetting about the color (*i.e.*, replacing every color with a 1), we obtain the model of Motegi and Sakai [21, 22]. Furthermore, every admissible state corresponds to a GT pattern $(\lambda^{(i)})_{i=0}^{n}$ by letting $\lambda^{(i)}$ be the $(n-i)$-th row of vertical edges (with the top (half) edges being the 0-th row of the model) to be the $\{0,1\}$-sequence of a partition [22, Sec. 3]. We denote this bijection $\mathfrak{P}$ from $\mathfrak{S}_\lambda$ to GT patterns with top row λ. However, the bijection $\mathfrak{P}$ is not weight preserving, but instead we also have to twist by w_0, *i.e.*, map $z_i \mapsto z_{n+1-i}$.

We refine the states of $\overline{\mathfrak{S}}_{\lambda,w}$ to allow markings at the configurations $\mathtt{a}_2$. More precisely, a *marked state* is a pair (S, M) with $S \in \overline{\mathfrak{S}}_{\lambda,w}$ such that M is some subset of all configurations of $\mathtt{a}_2$. We note that $\mathfrak{P}$ naturally extends to a bijection between marked states $\bigsqcup_{w \in S_n} \overline{\mathfrak{S}}_{\lambda,w}$ and marked GT patterns with top row λ as each configuration $\mathtt{a}_2$ in a state S corresponds to a position where a marking is possible in $\mathfrak{P}(S)$. As before, the weight gets twisted by w_0. Thus, for any state $S \in \overline{\mathfrak{S}}_{\lambda,w}$, we have

$$\mathrm{wt}(S) = \sum_{(S,M)} \beta^{|M|}\, \mathrm{wt}(S, M) = \sum_{(\mathfrak{P}(S),M)} \beta^{|M|} w_0\, \mathrm{wt}(\mathfrak{P}(S), M), \tag{4.1}$$

where the first sum is over all possible markings of S and the second sum is over all possible markings of $\mathfrak{P}(S)$. We also can do the same refinement for $\mathfrak{S}_{\lambda,w}$. The following two theorems are proved by noting that markings correspond to adding larger entries in a marked GT pattern.

Theorem 4.1. *Conjecture 2.1 is true, that is to say*

$$\overline{L}_{w\lambda}(\mathbf{z};\beta) = \sum_{\substack{T \in \mathrm{SVT}^n(\lambda) \\ K(T) = K_{w\lambda}}} \mathrm{wt}_\beta(T) = \sum_{S \in \mathrm{SSLT}(w\lambda)} \mathrm{wt}_\beta(S), \qquad L_{w\lambda}(\mathbf{z};\beta) = \sum_{\substack{T \in \mathrm{SVT}^n(\lambda) \\ K(T) \leq K_{w\lambda}}} \mathrm{wt}_\beta(T).$$

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References

[1] H. Andersen. "Schubert varieties and Demazure's character formula". *Invent. Math.* **79**.3 (1985), pp. 611–618. Link.

[2] A. Borodin and M. Wheeler. "Coloured stochastic vertex models and their spectral theory". 2018. arXiv:1808.01866.

[3] B. Brubaker, V. Buciumas, D. Bump, and H. Gustafsson. "Coloured five-vertex models and Demazure atoms". 2019. arXiv:1902.01795.

[4] A. Buch. "A Littlewood-Richardson rule for the K-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78. Link.

[5] V. Buciumas, T. Scrimshaw, and K. Weber. "Colored five-vertex models and Lascoux polynomials and atoms". *Journal of the London Mathematical Society* (2020). Link.

[6] S. Fomin and A. Kirillov. *Grothendieck polynomials and the Yang-Baxter equation*. DIMACS, Piscataway, NJ, 1994, pp. 183–189.

[7] T. Ikeda and H. Naruse. "K-theoretic analogues of factorial Schur P- and Q-functions". *Adv. Math.* **243** (2013), pp. 22–66. Link.

[8] T. Ikeda and T. Shimazaki. "A proof of K-theoretic Littlewood-Richardson rules by Bender-Knuth-type involutions". *Math. Res. Lett.* **21**.2 (2014), pp. 333–339. Link.

[9] M. Kashiwara. "The crystal base and Littelmann's refined Demazure character formula". *Duke Math. J.* **71**.3 (1993), pp. 839–858. Link.

[10] A. Kirillov. "Notes on Schubert, Grothendieck and key polynomials". *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016), Paper No. 034, 1–56.

[11] G. Kuperberg. "Another proof of the alternating-sign matrix conjecture". *Internat. Math. Res. Notices* 3 (1996), pp. 139–150. Link.

[12] V. Lakshmibai, C. Musili, and C. S. Seshadri. "Geometry of G/P. IV. Standard monomial theory for classical types". *Proc. Indian Acad. Sci. Sect. A Math. Sci.* **88**.4 (1979), pp. 279–362.

[13] A. Lascoux. *Transition on Grothendieck polynomials*. World Sci. Publ., River Edge, NJ, 2001, pp. 164–179. Link.

[14] A. Lascoux and M.-P. Schützenberger. "Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux". *C. R. Acad. Sci. Paris Sér. I Math.* **295**.11 (1982), pp. 629–633.

[15] A. Lascoux and M.-P. Schützenberger. *Symmetry and flag manifolds*. Vol. 996. Lecture Notes in Math. Springer, Berlin, 1983, pp. 118–144. Link.

[16] S. Mason. "A decomposition of Schur functions and an analogue of the Robinson-Schensted-Knuth algorithm". *Sém. Lothar. Combin.* **57** (2006/08), Art. B57e, 24.

[17] S. Mason. "An explicit construction of type A Demazure atoms". *J. Algebraic Combin.* **29**.3 (2009), pp. 295–313. Link.

[18] C. Monical. "Set-valued skyline fillings". 2016. arXiv:1611.08777.

[19] C. Monical, O. Pechenik, and T. Scrimshaw. "Crystal structures for symmetric Grothendieck polynomials". 2018. arXiv:1807.03294.

[20] C. Monical, O. Pechenik, and D. Searles. "Polynomials from combinatorial K-theory". *Canad. J. Math.* (2019). To appear. Link.

[21] K. Motegi and K. Sakai. "Vertex models, TASEP and Grothendieck polynomials". *J. Phys. A* **46**.35 (2013), pp. 355201, 26. Link.

[22] K. Motegi and K. Sakai. "K-theoretic boson-fermion correspondence and melting crystals". *J. Phys. A* **47**.44 (2014), p. 445202. Link.

[23] O. Pechenik and T. Scrimshaw. "K-theoretic crystals for set-valued tableaux of rectangular shapes". 2019. arXiv:1904.09674.

[24] C. Ross and A. Yong. "Combinatorial rules for three bases of polynomials". *Sém. Lothar. Combin.* **74** (2015), Art. B74a, 11.

[25] B. E. Sagan. *The symmetric group*. Second. Vol. 203. Graduate Texts in Mathematics. Representations, combinatorial algorithms, and symmetric functions. Springer-Verlag, New York, 2001, pp. xvi+238. Link.

[26] *Sage Mathematics Software (Version 8.7)*. `http://www.sagemath.org`. The Sage Developers. 2019.

[27] M. Wheeler and P. Zinn-Justin. "Littlewood-Richardson coefficients for Grothendieck polynomials from integrability". *J. Reine Angew. Math.* **2019**.757 (2017), pp. 159 –195. Link.

[28] M. J. Willis. "A direct way to find the right key of a semistandard Young tableau". *Ann. Comb.* **17**.2 (2013), pp. 393–400. Link.