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Beyond Göllnitz' Theorem I: A Bijective Approach

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Abstract. In 2003, Alladi, Andrews and Berkovich proved an identity for partitions where parts occur in eleven colors: four primary colors, six secondary colors, and one quaternary color. Their work answered a longstanding question of how to go beyond a classical theorem of Göllnitz, which uses three primary and three secondary colors. Their main tool was a deep and difficult four parameter q-series identity. In this extended abstract, we take a different approach. Instead of adding an eleventh quaternary color, we introduce forbidden patterns and give a bijective proof of a ten-colored partition identity lying beyond Göllnitz' theorem. Using a second bijection, we show that our identity is equivalent to the identity of Alladi, Andrews, and Berkovich. From a combinatorial viewpoint, the use of forbidden patterns is more natural and leads to a simpler formulation. In fact, in Part II following the full paper, we show how our method can be used to go beyond Göllnitz' theorem to any number of primary colors.

Keywords: Partitions, Rogers–Ramanujan type identities

1 Introduction and Statements of Results

1.1 History

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n. For example, the partitions of 7 are

$$(7),(6,1),(5,2),(5,1,1),(4,3),(4,2,1),(4,1,1,1),(3,3,1),(3,2,2),(3,2,1,1),$$
$$(3,1,1,1,1),(2,2,2,1),(2,2,1,1,1),(2,1,1,1,1,1) \text{ and } (1,1,1,1,1,1,1)\,\cdot$$

The study of partition identities has a long history, dating back to Euler's proof that there are as many partitions of n into distinct parts as partitions of n into odd parts. The corresponding identity is

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}\,, \tag{1.1}$$

where $(x;q)_m = \prod_{k=0}^{m-1}(1-xq^k)$, for any $m \in \mathbb{N}\cup\{\infty\}$ and x,q such that $|q|<1$.

One of the most important identities in the theory of partitions is Schur's theorem [11].

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Theorem 1.1 (Schur 1926)**.** *For any positive integer n, the number of partitions of n into distinct parts congruent to $\pm 1 \mod 3$ is equal to the number of partitions of n where parts differ by at least three and multiples of three differ by at least six.*

There have been a number of proofs of Schur's result over the years, including a q-difference equation proof of Andrews [3] and a simple bijective proof of Bressoud [5].

Another important identity is Göllnitz' theorem [6].

Theorem 1.2 (Göllnitz 1967)**.** *For any positive integer n, the number of partitions of n into distinct parts congruent to $2, 4, 5 \mod 6$ is equal to the number of partitions of n into parts different from 1 and 3, and where parts differ by at least six with equality only if parts are congruent to $2, 4, 5 \mod 6$.*

Like Schur's theorem, Göllnitz's identity can be proved using q-difference equations [4] and an elegant Bressoud-style bijection [10, 12].

Seminal work of Alladi, Andrews, and Gordon in the 90's showed how the theorems of Schur and Göllnitz emerge from more general results on colored partitions [1].

In the case of Schur's theorem, we consider parts in three colors $\{a, b, ab\}$ and the partitions with colored parts different from 1_{ab} and satisfying the minimal difference conditions in the table

$$\begin{array}{|c|cc|c|} \hline \lambda_i \backslash \lambda_{i+1} & a & b & ab \\ \hline a & 1 & 2 & 1 \\ b & 1 & 1 & 1 \\ \hline ab & 2 & 2 & 2 \\ \hline \end{array}\,. \tag{1.2}$$

Here, the part λ_i with color in the row and the part λ_{i+1} with color in the column differ by at least the corresponding entry in the table. An example of such a partition is $(7_{ab}, 5_b, 4_a, 3_{ab}, 1_b)$. The Alladi-Gordon refinement of Schur's partition theorem [2] is stated as follows:

Theorem 1.3. *Let u, v, n be non-negative integers. Denote by $A(u, v, n)$ the number of partitions of n into u distncts parts with color a and v distinct parts with color b, and denote by $B(u, v, n)$ the number of partitions of n satisfying the conditions above, with u parts with color a or ab, and v parts with color b or ab. We then have $A(u, v, n) = B(u, v, n)$ and the identity*

$$\sum_{u,v,n\geq 0} B(u, v, n) a^u b^v q^n = \sum_{u,v,n\geq 0} A(u, v, n) a^u b^v q^n = (-aq; q)_\infty (-bq; q)_\infty \cdot \tag{1.3}$$

Note that a transformation implies Schur's theorem :

$$\begin{cases} \text{dilation :} & q \mapsto q^3 \\ \text{translations :} & a, b \mapsto q^{-2}, q^{-1} \end{cases} . \tag{1.4}$$

In fact, the minimal difference conditions given in (1.2) give after these transformations the minimal differences in Schur's theorem.

In the case of Göllnitz' theorem, we consider parts that occur in six colors $\{a, b, c, ab, ab, bc\}$ and the partitions with colored parts different from $1_{ab}, 1_{ac}, 1_{bc}$ and satisfying the minimal difference conditions in

$$\begin{array}{|c|ccc|ccc|}
\hline
\lambda_i \backslash \lambda_{i+1} & a & b & c & ab & ac & bc \\
\hline
a & 1 & 2 & 2 & 1 & 1 & 2 \\
b & 1 & 1 & 2 & 1 & 1 & 1 \\
c & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
ab & 2 & 2 & 2 & 2 & 2 & 2 \\
ac & 2 & 2 & 2 & 1 & 2 & 2 \\
bc & 1 & 2 & 2 & 1 & 1 & 2 \\
\hline
\end{array} \,. \tag{1.5}$$

The Alladi–Andrews–Gordon refinement of Göllnitz's partition theorem can be stated as follows:

Theorem 1.4. *Let u, v, w, n be non-negative integers. Denote by $A(u, v, w, n)$ the number of partitions of n into u distncts parts with color a, v distinct parts with color b and w distinct parts with color c, and denote by $B(u, v, w, n)$ the number of partitions of n satisfying the conditions above, with u parts with color a, ab or ac, v parts with color b, ab or bc and w parts with color c, ac or bc. We then have $A(u, v, w, n) = B(u, v, w, n)$ and the identity*

$$\sum_{u,v,w,n\geq 0} B(u, v, w, n) a^u b^v c^w q^n = \sum_{u,v,w,n\geq 0} A(u, v, w, n) a^u b^v c^w q^n = (-aq; q)_\infty (-bq; q)_\infty (-cq; q)_\infty \,. \tag{1.6}$$

Note that a transformation implies Göllnitz' theorem :

$$\begin{cases} \text{dilation :} & q \mapsto q^6 \\ \text{translations :} & a, b, c \mapsto q^{-4}, q^{-2}, q^{-1} \end{cases} \,. \tag{1.7}$$

Observe that while Schur's theorem is not a direct corollary of Göllnitz' theorem, Theorem 1.3 *is* implied by Theorem 1.4 by setting $c = 0$. Therefore Göllnitz' theorem may be viewed as a level higher than Schur's theorem, since it requires three primary colors instead of two.

Following the work of Alladi, Andrews, and Gordon, it was an open problem to find a partition identity beyond Göllnitz' theorem, in the sense that it would arise from four primary colors. This was famously solved by Alladi, Andrews, and Berkovich [7]. To describe their result, we consider parts that occur in eleven colors $\{a, b, c, d, ab, ab, ad, bc, bd, cd, abcd\}$. We now take the partitions with the length of the secondary parts greater than one and satisfying the minimal difference conditions in

$\lambda_i \backslash \lambda_{i+1}$	ab	ac	ad	a	bc	bd	b	cd	c	d
ab	2	2	2	2	2	2	2	2	2	2
ac	1	2	2	2	2	2	2	2	2	2
ad	1	1	2	2	2	2	2	2	2	2
a	1	1	1	1	2	2	2	2	2	2
bc	1	1	1	1	2	2	2	2	2	2
bd	1	1	1	1	1	2	2	2	2	2
b	1	1	1	1	1	1	1	2	2	2
cd	1	1	1	1	1	1	1	2	2	2
c	1	1	1	1	1	1	1	1	1	2
d	1	1	1	1	1	1	1	1	1	1

, (1.8)

and such that parts with color $abcd$ differ by at least 4, and the smallest part with color $abcd$ is at least equal to $4 + 2\tau - \chi(1_a \text{ is a part})$, where τ is the number of primary and secondary parts in the partition. The theorem of Alladi, Andrews, and Berkovich is then stated as follows.

Theorem 1.5. *Let u, v, w, t, n be non-negative integers. Denote by $A(u, v, w, t, n)$ the number of partitions of n into u distncts parts with color a, v distinct parts with color b, w distinct parts with color c and t distinct parts with color d, and denote by $B(u, v, w, t, n)$ the number of partitions of n satisfying the conditions above, with u parts with color a, ab, ac, ad or $abcd$, v parts with color b, ab, bc, bd or $abcd$, w parts with color c, ac, bc, cd or $abcd$ and t parts with color d, ad, bd, cd or $abcd$. We then have $A(u, v, w, t, n) = B(u, v, w, t, n)$ and the identity*

$$\sum_{u,v,w,t,n\geq 0} B(u, v, w, t, n)a^u b^v c^w d^t q^n = (-aq; q)_\infty(-bq; q)_\infty(-cq; q)_\infty(-dq; q)_\infty \cdot \tag{1.9}$$

Note that the result of Alladi–Andrews–Berkovich uses four primary colors, the full set of secondary colors, along with one quaternary color $abcd$. When $d = 0$, we recover Theorem 1.4. Their main tool was a difficult q-series identity.

In this extended abstract (for the full paper, see [8]), we present a bijective proof of Theorem 1.5. Our proof is divided into two steps. First we prove Theorem 1.6 below, which arises more naturally from our methods than Theorem 1.5. Instead of adding a quaternary color, we lower certain minimum differences and add some forbidden patterns. Then, we show how Theorem 1.6 is equivalent to Theorem 1.5.

The general result beyond Göllnitz's theorem for an arbitrary number of primary colors is given in paper two of this series [9].

1.2 Statement of Results

Suppose that the parts occur in only primary colors a, b, c, d and secondary colors ab, ac, ad, bc, bd, cd. Let us now consider the partitions with the length of the secondary parts greater than one and satisfying the minimal difference conditions in (1.8), where we allow the patterns

$$(k_{cd}, k_{ab}), ((k+1)_{ad}, k_{bc})\,, \tag{1.10}$$

while avoiding the following forbidden patterns for $k \geq 3$:

$$(k_{cd}, k_{ab}, (k-2)_c), (k_{cd}, k_{ab}, (k-2)_d), ((k+1)_{ad}, k_{bc}, (k-1)_a)\cdot \tag{1.11}$$

An example of such a partition is

$$(11_{ad}, 10_{bc}, 8_a, 7_{cd}, 7_{ab}, 4_c, 3_{ad}, 2_{bc}, 1_a)\cdot$$

We can now state the main theorem of this paper.

Theorem 1.6. *Let u, v, w, t, n be non-negative integers. Denote by $A(u, v, w, t, n)$ the number of partitions of n into u distinct parts with color a, v distinct parts with color b, w distinct parts with color c and t distinct parts with color d, and denote by $B(u, v, w, t, n)$ the number of partitions of n satisfying the conditions above, with u parts with color a, ab, ac or ad, v parts with color b, ab, bc or bd, w parts with color c, ac, bc or cd and t parts with color d, ad, bd or cd. We then have $A(u, v, w, t, n) = B(u, v, w, t, n)$, and the corresponding q-series identity is given by*

$$\sum_{u,v,w,t,n\in\mathbb{N}} B(u, v, w, t, n) a^u b^v c^w d^t q^n = (-aq; q)_\infty(-bq; q)_\infty(-cq; q)_\infty(-dq; q)_\infty\cdot \tag{1.12}$$

By specializing the variables in Theorem 1.6, one can deduce many partition identities. For example, by considering the following transformation in (1.12)

$$\begin{cases} \text{dilation :} & q \mapsto q^{12} \\ \text{translations :} & a, b, c, d \mapsto q^{-8}, q^{-4}, q^{-2}, q^{-1} \end{cases}, \tag{1.13}$$

we obtain a corollary of Theorem 1.6.

Corollary 1.7. *For any positive integer n, the number of partitions of n into distinct parts congruent to $-2^3, -2^2, -2^1, -2^0 \mod 12$ is equal to the number of partitions of n into parts not congruent to $1, 5 \mod 12$ and different from $2, 3, 6, 7, 9$, such that the difference between two consecutive parts is greater than 12 up to the following exceptions:*

- $\lambda_i - \lambda_{i+1} = 9 \Longrightarrow \lambda_i \equiv \pm 3 \mod 12$ *and* $\lambda_i - \lambda_{i+2} \geq 24$,
- $\lambda_i - \lambda_{i+1} = 12 \Longrightarrow \lambda_i \equiv -2^3, -2^2, -2^1, -2^0 \mod 12$,

except that the pattern $(27, 18, 4)$ is allowed.

For example, with $n = 49$, the partitions of the first kind are

$$(35, 10, 4), (34, 11, 4), (28, 11, 10), (23, 22, 4),$$

$$(23, 16, 10), (22, 16, 11) \text{ and } (16, 11, 10, 8, 4)$$

and the partitions of the second kind are

$$(35, 14), (34, 15), (33, 16), (45, 4), (39, 10), (38, 11) \text{ and } (27, 18, 4) \cdot$$

Corollary 1.1 may be compared with **Theorem 3** of [7], which is Theorem 1.5 transformed by (1.13) but with the dilation $q \mapsto q^{15}$ instead of $q \mapsto q^{12}$.

The extended abstract is organized as follows. In Section 2, we will present some tools that will be useful for the bijections. After that, in Section 3, we will give the bijection for Theorem 1.6. Finally, in Section 4, we will present the bijection between the partitions with forbidden patterns considered in Theorem 1.6 and the partitions with quaternary parts given in Theorem 1.5.

2 The setup

Denote by $\mathcal{C} = \{a, b, c, d\}$ the set of primary colors and $\mathcal{C}_{\rtimes} = \{ad, ab, ac, bc, bd, cd\}$ the set of secondary colors, and recall the order on $\mathcal{C} \sqcup \mathcal{C}_{\rtimes}$:

$$ab < ac < ad < a < bc < bd < b < cd < c < d \cdot \tag{2.1}$$

We can then define the strict lexicographic order $\succ$ on colored parts by

$$k_p \succ l_q \Longleftrightarrow k - l \geq \chi(p \leq q) \cdot \tag{2.2}$$

Explicitly, this gives the order

$$1_{ab} \prec 1_{ac} \prec 1_{ad} \prec 1_a \prec 1_{bc} \prec 1_{bd} \prec 1_b \prec 1_{cd} \prec 1_c \prec 1_d \prec 2_{ab} \prec \cdots \cdot \tag{2.3}$$

We denote by $\mathcal{P}$ the set of positive integers with primary color. We can easily see that for any $pq \in \mathcal{C}_{\rtimes}$, *with* $p < q$, and any $k \geq 1$, we have that

$$(2k)_{pq} = k_q + k_p \tag{2.4}$$
$$(2k+1)_{pq} = (k+1)_p + k_q \cdot \tag{2.5}$$

Here, the sum of two parts with primary colors consists of a part whose size and color are respectively the sum of the sizes and the commutative product of colors of the primary colored parts. One can check that any part greater than 1 with a secondary color pq can be uniquely written as the sum of two consecutive parts in $\mathcal{P}$ with colors p and q. We then denote by $\mathcal{S}$ the set of secondary parts greater than 1, and define the functions α and β on $\mathcal{S}$ by

$$\alpha : \begin{cases} 2k_{pq} & \mapsto \quad k_q \\ (2k+1)_{pq} & \mapsto \quad (k+1)_p \end{cases} \qquad \text{and} \qquad \beta : \begin{cases} 2k_{pq} & \mapsto \quad k_p \\ (2k+1)_{pq} & \mapsto \quad k_q \end{cases}, \tag{2.6}$$

respectively named upper and lower halves.

By considering the lexicographic order $\succ$, the minimal differences described in the table (1.8) can be viewed as an order $\triangleright$ on $\mathcal{P} \sqcup \mathcal{S}$ defined by the relation

$$k_p \triangleright l_q \Longleftrightarrow \begin{cases} k_p \succeq (l+1)_q & \text{if} \quad p \text{ or } q \in \mathcal{C} \\ k_p \succ (l+1)_q & \text{if} \quad p \text{ and } q \in \mathcal{C}_{\rtimes} \end{cases}. \tag{2.7}$$

We recall that the table (1.8) and the minimal differences for Theorem 1.6 differ only when we have a pair (p,q) of secondary colors such that $(p,q) \in \{(cd,ab),(ad,bc)\}$. In these cases, the difference for Theorem 1.6 is one less. We then define in the same way a relation $\gg$ on $\mathcal{P} \sqcup \mathcal{S}$, for the minimal differences of Theorem 1.6, and obtain by (2.7)

$$k_p \gg l_q \Longleftrightarrow \begin{cases} k_p \succeq (l+1)_q & \text{if} \quad p \text{ or } q \in \mathcal{C} \\ k_p \succ (l+1)_q & \text{if} \quad p \text{ and } q \in \mathcal{C}_{\rtimes} \quad \text{and} \quad (p,q) \notin \{(cd,ab),(ad,bc)\} \\ k_p \succ l_q & \text{if} \quad (p,q) \in \{(cd,ab),(ad,bc)\} \end{cases}. \tag{2.8}$$

We denote by $\mathcal{O}$ the set of partitions with parts in $\mathcal{P}$ and well-ordered by $\succ$. We then have that $\lambda \in \mathcal{O}$ if and only if there exist $\lambda_1 \succ \cdots \succ \lambda_t \in \mathcal{P}$ such that $\lambda = (\lambda_1,\ldots,\lambda_t)$. We set $c(\lambda_i)$ to be the color of λ_i in $\mathcal{C}$, and $C(\lambda) = c(\lambda_1)\cdots c(\lambda_t)$ as a commutative product of colors in $< \mathcal{C} >$. In the same way, we denote by $\mathcal{E}$ the set of partitions with parts in $\mathcal{P} \sqcup \mathcal{S}$ and well-ordered by $\gg$ and set colors $c(\nu_i) \in \mathcal{C} \sqcup \mathcal{C}_{\rtimes}$ depending on whether ν_i is in $\mathcal{P}$ or $\mathcal{S}$, and we also define $C(\nu) = c(\nu_1)\cdots c(\nu_t)$ *seen* as a commutative product of colors in $\mathcal{C}$. In fact, a secondary color is just a product of two primary colors. For both kinds of partitions, their size is the sum of their part sizes. We also denote by $\mathcal{E}_1$ the subset of partitions of $\mathcal{E}$ with the forbidden patterns,

$$((k+2)_{cd},(k+2)_{ab},k_c),((k+2)_{cd},(k+2)_{ab},k_d),((k+2)_{ad},(k+1)_{bc},k_a), \tag{2.9}$$

except the pattern $(3_{ad},2_{bc},1_a)$ which is allowed. We finally define $\mathcal{E}_2$ as the subset of partitions of $\mathcal{E}$ with parts well-ordered by $\triangleright$ in (2.7), and we observe that $\mathcal{E}_2$ is indeed a subset of $\mathcal{E}_1$.

Finally, in what follows, adding an integer to a colored part only changes its size and does not affect its color.

3 Bressoud's algorithm

Here we adapt the algorithm given by Bressoud in his bijective proof of Schur's partition theorem [5]. The bijection is easy to describe and execute, but its justification is more subtle and is given in Section 4 of the full paper [8].

3.1 From $\mathcal{O}$ to $\mathcal{E}_1$

Let us consider the following machine Φ:

Step 1: For a sequence $\lambda = \lambda_1, \ldots, \lambda_t$, take the smallest $i < t$ such that $\lambda_i, \lambda_{i+1} \in \mathcal{P}$ and $\lambda_i \succ \lambda_{i+1}$ but $\lambda_i \not\ggcurly \lambda_{i+1}$, if it exists, and replace

$$\begin{array}{lcll} \lambda_i & \leftarrowtail & \lambda_i + \lambda_{i+1} & \text{as a part in } \mathcal{S} \\ \lambda_j & \leftarrow & \lambda_{j+1} & \text{for all} \quad i < j < t \end{array} \tag{3.1}$$

and move to **Step 2**. We call such a pair of parts a *troublesome* pair. We observe that λ loses two parts in $\mathcal{P}$ and gains one part in $\mathcal{S}$. The new sequence is $\lambda = \lambda_1, \ldots, \lambda_{t-1}$. Otherwise, exit from the machine.

Step 2: For $\lambda = \lambda_1, \ldots, \lambda_t$, take the smallest $i < t$ such that $(\lambda_i, \lambda_{i+1}) \in \mathcal{P} \times \mathcal{S}$ and $\lambda_i \not\ggcurly \lambda_{i+1}$ if it exists, and replace

$$(\lambda_i, \lambda_{i+1}) \looparrowright (\lambda_{i+1} + 1, \lambda_i - 1) \in \mathcal{S} \times \mathcal{P} \tag{3.2}$$

and redo **Step 2**. We say that the parts λ_i, λ_{i+1} are *crossed*. Otherwise, move to **Step 1**.

Let $\Phi(\lambda)$ be the resulting sequence after putting any $\lambda = (\lambda_1, \ldots, \lambda_t) \in \mathcal{O}$ in Φ. This transformation preserves the size and the commutative product of primary colors of partitions. Below, we apply this machine on the partition

$$(11_c, 8_d, 6_a, 4_d, 4_c, 4_b, 3_a, 2_b, 2_a, 1_d, 1_c, 1_b, 1_a).$$

$$\begin{array}{c} 11_c \\ 8_d \\ 6_a \\ \underline{4_d} \\ \underline{4_c} \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightarrowtail \begin{array}{c} 11_c \\ 8_d \\ \mathbf{6_a} \\ \mathbf{8_{cd}} \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \looparrowright \begin{array}{c} 11_c \\ \mathbf{8_d} \\ \mathbf{9_{cd}} \\ 5_a \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \looparrowright \begin{array}{c} 11_c \\ 10_{cd} \\ 7_d \\ \underline{5_a} \\ \underline{4_b} \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightarrowtail \begin{array}{c} 11_c \\ 10_{cd} \\ \mathbf{7_d} \\ \mathbf{9_{ab}} \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \looparrowright \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ \underline{3_a} \\ \underline{2_b} \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightarrowtail \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ \underline{2_a} \\ \underline{1_d} \\ 1_c \\ 1_b \\ 1_a \end{array} \rightarrowtail \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ \underline{1_c} \\ \underline{1_b} \\ 1_a \end{array} \rightarrowtail \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} .$$

3.2 From $\mathcal{E}_1$ to $\mathcal{O}$

Let us consider the following machine Ψ:

Step 1: For a sequence $\nu = \nu_1, \dots, \nu_t$, take the greastest $i \leq t$ such that $\nu_i \in \mathcal{S}$ if it exists. If $\nu_{i+1} \in \mathcal{P}$ and $\beta(\nu_i) \not\succ \nu_{i+1}$, then replace

$$(\nu_i, \nu_{i+1}) \looparrowright (\nu_{i+1} + 1, \nu_i - 1) \in \mathcal{P} \times \mathcal{S} \tag{3.3}$$

and redo **Step 1**. We say that the parts ν_i, ν_{i+1} are *crossed*. Otherwise, move to **Step 2**. If there are no more parts in $\mathcal{S}$, exit from the machine.

Step 2: For $\nu = \nu_1, \dots, \nu_t$, take the the greatest $i \leq t$ such that $\nu_i \in \mathcal{S}$. By **Step 1**, it satisfies $\beta(\nu_i) \succ \nu_{i+1}$. Then replace

$$\begin{array}{lcll} \nu_{j+1} & \leftarrow & \nu_j & \text{for all} \quad t \geq j > i \\ (\nu_i) & \rightrightarrows & (\alpha(\nu_i), \beta(\nu_i)) & \text{as a pair of parts in } \mathcal{P}\,, \end{array} \tag{3.4}$$

and move to **Step 1**. We say that the part ν_i *splits*. We observe that ν gains two parts in $\mathcal{P}$ and loses one part in $\mathcal{S}$. The new sequence is $\nu = \nu_1, \dots, \nu_{t+1}$.

Let $\Psi(\nu)$ be the resulting sequence after putting any $\nu = (\nu_1, \dots, \nu_t) \in \mathcal{E}_1$ in Ψ. This transformation preserves the size and the product of primary colors of partitions. For example, applying this to $(11_c, 10_{cd}, 10_{ab}, 6_d, 5_{ab}, 3_{ad}, 2_{bc}, 1_a)$ gives

$$\begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 1_c + 1_b \\ 1_a \end{array} \rightrightarrows \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 2_a + 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightrightarrows \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 3_a + 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightrightarrows \begin{array}{c} 11_c \\ 10_{cd} \\ 5_b + 5_a \\ 6_d \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \looparrowright \begin{array}{c} 11_c \\ 10_{cd} \\ 7_d \\ 5_a + 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightrightarrows \begin{array}{c} 11_c \\ 5_d + 5_c \\ 7_d \\ 5_a \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \looparrowright \begin{array}{c} 11_c \\ 8_d \\ 5_c + 4_d \\ 5_a \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \looparrowright \begin{array}{c} 11_c \\ 8_d \\ 6_a \\ 4_d + 4_c \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} \rightrightarrows \begin{array}{c} 11_c \\ 8_d \\ 6_a \\ 4_d \\ 4_c \\ 4_b \\ 3_a \\ 2_b \\ 2_a \\ 1_d \\ 1_c \\ 1_b \\ 1_a \end{array} .$$

4 Bijective proof of Theorem 1.5

In this section, we will describe a bijection for Theorem 1.5. For brevity, we refer to the partitions in Theorem 1.5 as quaternary partitions.

4.1 From $\mathcal{E}_1$ to quaternary partitions

We consider the patterns $((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})$ and we sum them as follows :

$$\begin{aligned} (k+1)_{ad} + k_{bc} &= (2k+1)_{abcd} \\ k_{cd} + k_{ab} &= 2k_{abcd} \,\cdot \end{aligned} \tag{4.1}$$

Let us now take a partition ν in $\mathcal{E}_1$. We then identify all the patterns $(M^i, m^i) \in \{((k+1)_{ad}, k_{bc}), (k_{cd}, k_{ab})\}$ and suppose that

$$\nu = \nu_1, \dots, \nu_x, M^1, m^1, \nu_{x+1}, \dots, \nu_y, M^2, m^2, \nu_{y+1}, \dots, M^t, m^t, \dots, \nu_s \cdot$$

As long as we have a pattern ν_j, M^i, m^i, we cross the parts by replacing them using

$$\nu_j, M^i, m^i \longmapsto M^i + 1, m^i + 1, \nu_j - 2 \cdot \tag{4.2}$$

At the end of the process, we obtain a final sequence

$$N^1, n^1, N^2, n^2, \dots, N^t, n^t, \nu'_1, \dots, \nu'_s \cdot$$

Finally, the associated pair of partitions is set to be $(K^1, \dots, K^t), \nu' = (\nu'_1, \dots, \nu'_t)$, where $K^i = N^i + n^i$ according to (4.1). To sum up the previous transformation, we only remark that, for each quaternary part K^i obtained by summing of the original pattern M^i, m^i, we add twice the number of the remaining primary and secondary parts in ν to the left of the pattern that gave K^i, while we subtract from these parts two times the number of quaternary parts obtained by patterns that occur to their right.

With the example $11_c, \underline{10_{cd}, 10_{ab}}, 6_d, 5_{ab}, \underline{3_{ad}, 2_{bc}}, 1_a$,

$$\begin{matrix} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{matrix} \mapsto \begin{matrix} 11_c \\ 10_{cd}, 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad}, 2_{bc} \\ 1_a \end{matrix} \mapsto \begin{matrix} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 5_{ab} \\ 3_{ad}, 2_{bc} \\ 1_a \end{matrix} \mapsto \begin{matrix} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 4_{ad}, 3_{bc} \\ 3_{ab} \\ 1_a \end{matrix} \mapsto \begin{matrix} 11_{cd}, 11_{ab} \\ 9_c \\ 5_{ad}, 4_{bc} \\ 4_d \\ 3_{ab} \\ 1_a \end{matrix} \mapsto \begin{matrix} 11_{cd}, 11_{ab} \\ 6_{ad}, 5_{bc} \\ 7_c \\ 4_d \\ 3_{ab} \\ 1_a \end{matrix} \cdot$$

we obtain $[(22_{abcd}, 11_{abcd}), (7_c, 4_d, 3_{ab}, 1_a)]$.

4.2 From quaternary partitions to $\mathcal{E}_1$

Recall by (4.1) that K_{abcd} splits as follows :

$$\begin{aligned} (k+1)_{ad} + k_{bc} &= (2k+1)_{abcd} \\ k_{cd} + k_{ab} &= 2k_{abcd} \end{aligned}$$

Let us then consider partitions $(K^1, \dots, K^t)$ and $\nu = (\nu_1, \dots, \nu_s) \in \mathcal{E}_2$, with quaternary part K^u such that $K^t \geq 4 + 2s - \chi(1_a \in \nu)$ and $K^u - K^{u+1} \geq 4$. We also set $K^u = (k^u, l^u)$ the decomposotion according to (4.1). We then proceed as follows by beginning with K^t and ν_1,

Step 1: If we do not encounter $K^{u+1} = (k^{u+1}, l^{u+1})$ and $\nu_i \neq 1_a$ and $\nu_i + 2 \triangleright k^u - 1$, then replace

$$\begin{aligned} \nu_i &\longmapsto \nu_i + 2 \\ (k^u, l^u) &\longmapsto (k^u - 1, l^u - 1) \end{aligned}$$

and move to $i+1$ and redo **Step 1**. Otherwise, move to **Step 2**.

Step 2 If we encounter $K^{u+1} = k^{u+1} \gg l^{u+1}$, then split (k^u, l^u) into $k^u \gg l^u$. If not, it means that we have met ν_i such that $\nu_i + 2 \not\triangleright k^u - 1$. Then we split $k^u \gg l^u$. We can now move to **Step 1** with $u-1$ and $i = 1$.

With the example $[(22_{abcd}, 11_{abcd}), (7_c, 4_d, 3_{ab}, 1_a)]$, we obtain

$$\begin{array}{c} 11_{cd}, 11_{ab} \\ 6_{ad}, 5_{bc} \\ 7_c \\ 4_d \\ 3_{ab} \\ 1_a \end{array} \mapsto \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 5_{ad}, 4_{bc} \\ 4_d \\ 3_{ab} \\ 1_a \end{array} \mapsto \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 4_{ad}, 3_{bc} \\ 3_{ab} \\ 1_a \end{array} \mapsto \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 5_{ab} \\ 3_{ad}, 2_{bc} \\ 1_a \end{array} \mapsto \begin{array}{c} 11_{cd}, 11_{ab} \\ 9_c \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} \mapsto \begin{array}{c} 11_c \\ 10_{cd}, 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array} \mapsto \begin{array}{c} 11_c \\ 10_{cd} \\ 10_{ab} \\ 6_d \\ 5_{ab} \\ 3_{ad} \\ 2_{bc} \\ 1_a \end{array}.$$

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