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Flagged Littlewood–Richardson tableaux and branching rule for orthogonal groups

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Abstract. We give a new combinatorial formula for the branching rule from GL_n to O_n generalizing the Littlewood's restriction formula. The formula is given in terms of Littlewood–Richardson tableaux with certain flag conditions which vanish in a stable range.

Keywords: quantum groups, crystal graphs, classical groups, branching rules

1 Introduction

Let $V^{\lambda}_{\mathrm{GL}_n}$ denote a complex finite-dimensional irreducible representation of the complex general linear group GL_n parametrized by a partition λ of length $\ell(\lambda) \leqslant n$. Suppose that G_n is a closed subgroup Sp_n or O_n, where n is even for $\mathrm{G}_n = \mathrm{Sp}_n$. Let $V^{\mu}_{\mathrm{G_n}}$ be a finite-dimensional irreducible G_n-module parametrized by a partition μ with $\ell(\mu) \leqslant n/2$ for $\mathrm{G}_n = \mathrm{Sp}_n$, and by a partition μ with $\ell(\mu) \leqslant n$ and $\mu'_1 + \mu'_2 \leqslant n$ for $\mathrm{G}_n = \mathrm{O}_n$. Here $\mu' = (\mu'_i)_{i \geqslant 1}$ is the conjugate partition of μ.

Let

$$\left[V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{G_n}}\right] = \dim \mathrm{Hom}_{\mathrm{G}_n}\left(V^{\mu}_{\mathrm{G_n}}, V^{\lambda}_{\mathrm{GL}_n}\right) \tag{1.1}$$

denote the multiplicity of $V^{\mu}_{\mathrm{G_n}}$ in $V^{\lambda}_{\mathrm{GL}_n}$. In [10, 11], Littlewood showed that if $\ell(\lambda) \leqslant n/2$, then

$$\left[V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{Sp}_n}\right] = \sum_{\delta \in \mathscr{P}^{(2)}} c^{\lambda}_{\delta' \mu} \quad , \quad \left[V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{O}_n}\right] = \sum_{\delta \in \mathscr{P}^{(2)}} c^{\lambda}_{\delta \mu} \, , \tag{1.2}$$

where $c^{\alpha}_{\beta\gamma}$ is the Littlewood–Richardson coefficient corresponding to partitions α, β, γ, and $\mathscr{P}^{(2)}$ denotes the set of partition with even parts. There have been numerous works on extending the Littlewood's restriction rules (1.2) for arbitrary λ with $\ell(\lambda) \leqslant n$ (see [1, 3] and also the references therein), but most of which are obtained in an algebraic way and hence given not in a subtraction-free way.

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In [12], Sundaram gave a beautiful combinatorial formula for (1.1) when $\mathrm{G}_n = \mathrm{Sp}_n$, as the sum of the numbers of Littlewood–Richardson (LR) tableaux of shape λ/μ with content δ' satisfying certain constraints on their entries, which vanish in a stable range $\ell(\lambda) \leqslant n/2$. Recently, based on the results in [8, 6], Lecouvey and Lenart obtained another formula for (1.1) when $\mathrm{G}_n = \mathrm{Sp}_n$ in terms of LR tableaux with some flag conditions on their companion tableaux [9]. On the other hand, no orthogonal analogue of these formula has been known so far.

The main result in this abstract is to give a combinatorial formula for (1.1) when $\mathrm{G}_n = \mathrm{O}_n$ for arbitrary λ and μ in terms of LR tableaux with certain flag conditions on their companion tableaux which vanish in a stable range $\ell(\lambda) \leqslant n/2$.

For simplicity, let us state our main result when $n - 2\mu_1' \geqslant 0$. Note that the restriction on $n - 2\mu_1'$ is not significant, since the result for $n - 2\mu_1' < 0$ is almost identical. Let $\mathtt{LR}^{\lambda}_{\delta\mu^\pi}$ be the set of LR tableaux of shape λ/δ with content μ^π, where μ^π is the skew Young diagram obtained by 180°-rotation of μ. Then we have the following (Theorem 4.10).

Theorem 1.1. *For $U \in \mathtt{LR}^{\lambda}_{\delta\mu^\pi}$, let σ_i be the row index of the leftmost $\mu_1' - i + 1$ in U for $1 \leqslant i \leqslant \mu_1'$, and τ_j the row index of the second leftmost $\mu_2' - j + 1$ in U for $1 \leqslant j \leqslant \mu_2'$. Let $m_1 < \cdots < m_{\mu_1'}$ be the sequence given by $m_i = \min\{n - \sigma_i + 1, 2i - 1\}$, and let $n_1 \leqslant \cdots \leqslant n_{\mu_2'}$ be the sequence such that n_j is the j-th smallest number in $\{ j+1, \ldots, n \} \setminus \{ m_{j+1}, \ldots, m_{\mu_1'} \}$. Let $\underline{c}^{\lambda}_{\delta\mu}$ denote the number of $U \in \mathtt{LR}^{\lambda}_{\delta\mu^\pi}$ such that*

$$\tau_j + n_j \leqslant n + 1,$$

for $1 \leqslant j \leqslant \mu_2'$. Then we have

$$\left[V^{\lambda}_{\mathrm{GL}_n} : V^{\mu}_{\mathrm{O}_n} \right] = \sum_{\delta \in \mathscr{P}^{(2)}} \underline{c}^{\lambda}_{\delta\mu} \,.$$

The branching multiplicity (1.1) is equal to the one from D_∞ to $A_{+\infty}$ from a viewpoint of see-saw dual pairs in Howe duality on a Fock space [13]. We use the Kashiwara's crystal base theory of quantum groups and the *spinor model* for crystal graphs of type D_∞ [7] to describe the latter multiplicity. Unlike the case of Sp_n [9], we have to develop in addition a non-trivial combinatorial algorithm on spinor model called *separation* in order to have a description of branching multiplicity in terms of LR tableaux satisfying the condition for $\underline{c}^{\lambda}_{\delta\mu}$. This is a key ingredient in the proof of Theorem 1.1. We can also recover the formula (1.2) in a stable range directly from the above formula. A full version of this paper including detailed proofs has appeared in [5].

2 Notations

Let $\mathbb{Z}_+$ denote the set of non-negative integers. Let $\mathscr{P}$ be the set of partitions or Young diagrams. We let $\mathscr{P}_\ell = \{ \lambda \in \mathscr{P} \,|\, \ell(\lambda) \leqslant \ell \}$ for $\ell \geqslant 1$, where $\ell(\lambda)$ is the length of λ, let

$\mathscr{P}^{(2)} = \{\, \lambda \in \mathscr{P} \,|\, \lambda = (\lambda_i)_{i\geqslant 1}, \lambda_i \in 2\mathbb{Z}_+ \ (i \geqslant 1) \,\}$. For a skew Young diagram λ/μ, we define $SST(\lambda/\mu)$ to be the set of semistandard tableaux of shape λ/μ with entries in $\mathbb{N}$. For $T \in SST(\lambda/\mu)$, let $w(T)$ be the word given by reading the entries of T column by column from right to left and from top to bottom in each column, and let $\mathrm{sh}(T)$ denote the shape of T.

Let $\lambda \in \mathscr{P}$ be given. For $T \in SST(\lambda)$ and $a \in \mathbb{N}$, we denote by $a \to T$ the tableau obtained by the column insertion of a into T (cf. [2]). For a word $w = w_1 \ldots w_r$, we define $(w \to T) = (w_r \to (\cdots \to (w_1 \to T)))$. For a semistandard tableau S, we define $(S \to T) = (w(S) \to T)$.

Let λ^π denote the skew Young diagram obtained from λ by 180° rotation. Let H_λ and H_{λ^π} be the tableaux in $SST(\lambda)$ and $SST(\lambda^\pi)$, respectively, where the i-th entry from the top in each column is filled with i for $i \geqslant 1$.

For $\lambda, \mu, \nu \in \mathscr{P}$, let $\mathtt{LR}^{\lambda}_{\mu\nu}$ be the set of Littlewood–Richardson tableaux S of shape λ/μ with content ν. There is a natural bijection from $\mathtt{LR}^{\lambda}_{\mu\nu}$ to the set of $T \in SST(\nu)$ such that $(T \to H_\mu) = H_\lambda$, where each i in the jth row of $S \in \mathtt{LR}^{\lambda}_{\mu\nu}$ corresponds to j in the ith row of T. We call such T a companion tableau of $S \in \mathtt{LR}^{\lambda}_{\mu\nu}$.

We also define $\mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ to be the set of $S \in SST(\lambda/\mu)$ with content ν^π such that $w(T) = w_1 \ldots w_r$ is an anti-lattice word, that is, the number of i in $w_k \ldots w_r$ is greater than or equal to that of $i-1$ for each $k \geqslant 1$ and $1 < i \leqslant \ell(\nu)$. Let us call S a Littlewood–Richardson tableaux of shape λ/μ with content ν^π. As in case of $\mathtt{LR}^{\lambda}_{\mu\nu}$, the map from $S \in \mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ to its companion tableau gives a natural bijection from $\mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ to the set of $T \in SST(\nu^\pi)$ such that $(T \to H_\mu) = H_\lambda$. From now on, all the LR tableaux are assumed to be the corresponding companion tableaux unless otherwise specified.

Let $S \in \mathtt{LR}^{\lambda'}_{\mu'\nu'}$ be given, that is, $(S \to H_{\mu'}) = H_{\lambda'}$. Let $S^1, \ldots, S^p$ denote the columns of S enumerated from the right. For $1 \leqslant i \leqslant p$, let $H^i = (S^i \to H^{i-1})$ with $H^0 = H_{\mu'}$ so that $H^p = H_{\lambda'}$. Define $Q(S \to H_{\mu'}) \in SST(\lambda/\mu)$ to be the tableau such that the horizontal strip $\mathrm{sh}(H^i)'/\mathrm{sh}(H^{i-1})'$ is filled with $1 \leqslant i \leqslant p$. On the other hand, let $U \in \mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ be given, that is, $\mathrm{sh}(U \to H_\mu) = H_\lambda$. Let U_i denote the i-th row of U from the top, and let $H_i = (U_i \to H_{i-1})$ with $H_0 = H_\mu$ for $1 \leqslant i \leqslant p$. Define $Q(U \to H_\mu)$ to be tableau such that the horizontal strip $\mathrm{sh}(H_i)/\mathrm{sh}(H_{i-1})$ is filled with $1 \leqslant i \leqslant p$.

Then we have a bijection

$$\psi : \mathtt{LR}^{\lambda'}_{\mu'\nu'} \longrightarrow \mathtt{LR}^{\lambda}_{\mu\nu^\pi}\,, \tag{2.1}$$

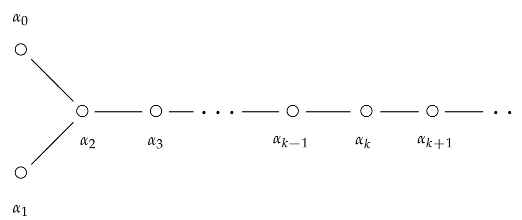
where for $S \in \mathtt{LR}^{\lambda'}_{\mu'\nu'}$, $\psi(S) = U$ is given by a unique $U \in SST(\nu^\pi)$ such that $(U \to H_\mu) = H_\lambda$ and $Q(U \to H_\mu) = Q(S \to H_{\mu'})$.

3 Spinor model

3.1 Definitions

Let us recall the spinor model of type D_∞, which is a combinatorial model for the crystal of an integrable irreducible highest weight module over the quantum group of type D_∞ (see [7] and [5, Section 2.1] for more details).

Let $\mathfrak{g}$ be the Kac-Moody Lie algebra of type D_∞. We assume that the index set for simple roots is $I = \mathbb{Z}_+$, and the weight lattice is $P = \mathbb{Z}\Lambda_0 \oplus \left(\bigoplus_{i\geqslant 1}\mathbb{Z}\epsilon_i\right)$. The associated Dynkin diagram, set of simple roots $\Pi = \{\,\alpha_i \,|\, i \geqslant 0\,\}$, and fundamental weight Λ_i ($i \geqslant 0$) are given by



$$\Pi = \{\, \alpha_0 = -\epsilon_1 - \epsilon_2,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i \geqslant 1)\,\}, \quad \Lambda_i = \begin{cases} \Lambda_0 + \epsilon_1, & \text{if } i = 1, \\ 2\Lambda_0 + \epsilon_1 + \cdots + \epsilon_i, & \text{if } i > 1. \end{cases}$$

Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{g}$ associated to $\Pi\backslash\{\alpha_0\}$, which is of type $A_{+\infty}$.

For $n \geqslant 1$, let

$$\mathcal{P}(\mathrm{O}_n) = \{\, \mu = (\mu_1, \cdots, \mu_n) \,|\, \mu_i \in \mathbb{Z}_+,\ \mu_1 \geqslant \ldots \geqslant \mu_n,\ \mu'_1 + \mu'_2 \leqslant n \,\}.$$

For $\mu \in \mathcal{P}(\mathrm{O}_n)$, put $\Lambda(\mu) = n\Lambda_0 + \mu'_1\epsilon_1 + \mu'_2\epsilon_2 + \cdots$. Then we have $P_+ = \{\, \Lambda(\mu) \,|\, \mu \in \bigsqcup_n \mathcal{P}(\mathrm{O}_n)\,\}$ the set of dominant integral weights for $\mathfrak{g}$. We denote by $\mathbf{B}(\Lambda)$ the crystal of an integrable irreducible highest weight module over the quantum group $U_q(\mathfrak{g})$ with highest weight $\Lambda \in P_+$.

Let T be a tableau of two-column skew shape $(2^{b+c}, 1^a)/(1^b)$ for $a, b, c \in \mathbb{Z}_+$. We denote the left and right columns of T by $T^{\mathtt{L}}$ and $T^{\mathtt{R}}$ respectively. Suppose that T is semistandard and we can slide down $T^{\mathtt{R}}$ by k positions to have a semistandard tableau T' of shape $(2^{b+c}, 1^{a-k})/(1^{b-k})$. We define $\mathfrak{r}_T$ to to be the maximal such k.

Let

$$\begin{aligned} \mathbf{T}(a) &= \left\{\, T \,|\, T \in SST\big((2^{b+c}, 1^a)/(1^b)\big),\ b, c \in 2\mathbb{Z}_+,\ \mathfrak{r}_T \leqslant 1 \,\right\} \quad (a \in \mathbb{Z}_+), \\ \overline{\mathbf{T}}(0) &= \bigsqcup_{b,c\in 2\mathbb{Z}_+} SST\big((2^{b+c+1})/(1^b)\big), \qquad \mathbf{T}^{\mathrm{sp}} = \bigsqcup_{a\in\mathbb{Z}_+} SST((1^a)), \\ \mathbf{T}^{\mathrm{sp}+} &= \{\, T \,|\, T \in \mathbf{T}^{\mathrm{sp}},\ \mathfrak{r}_T = 0\,\}, \quad \mathbf{T}^{\mathrm{sp}-} = \{\, T \,|\, T \in \mathbf{T}^{\mathrm{sp}},\ \mathfrak{r}_T = 1\,\}, \end{aligned}$$

where the integer $\mathfrak{r}_T$ of $T \in \mathbf{T}^{\mathrm{sp}}$ is defined by the residue of $\mathrm{ht}(T)$ modulo 2. It is shown that $\mathbf{T}(a)$, $\overline{\mathbf{T}}(0)$ and $\mathbf{T}^{\mathrm{sp}}$ have $\mathfrak{g}$-crystal structure [7, Proposition 4.1] (cf. [5, Section 2.3]) such that

$$\mathbf{T}(a) \cong \mathbf{B}(\Lambda_a) \ \ (a \geqslant 2), \quad \mathbf{T}(0) \cong \mathbf{B}(2\Lambda_0), \quad \overline{\mathbf{T}}(0) \cong \mathbf{B}(2\Lambda_1), \quad \mathbf{T}(1) \cong \mathbf{B}(\Lambda_0 + \Lambda_1),$$
$$\mathbf{T}^{\mathrm{sp}+} \cong \mathbf{B}(\Lambda_0)\,, \quad \mathbf{T}^{\mathrm{sp}-} \cong \mathbf{B}(\Lambda_1).$$

Let $\mu \in \mathcal{P}(\mathrm{O}_n)$ be given. Let $q_\pm$ and $r_\pm$ be non-negative integers such that

$$\begin{cases} n - 2\mu_1' = 2q_+ + r_+, & \text{if } n - 2\mu_1' \geqslant 0, \\ 2\mu_1' - n = 2q_- + r_-, & \text{if } n - 2\mu_1' < 0, \end{cases}$$

where $r_\pm = 0, 1$. Let $\overline{\mu} = (\overline{\mu}_i) \in \mathscr{P}$ be such that $\overline{\mu}_1' = n - \mu_1'$ and $\overline{\mu}_i' = \mu_i'$ for $i \geqslant 2$ and let $M_+ = \mu_1'$ and $M_- = \overline{\mu}_1'$. Put

$$\widehat{\mathbf{T}}(\mu, n) = \begin{cases} \mathbf{T}(\mu_1) \times \cdots \times \mathbf{T}(\mu_{M_+}) \times \mathbf{T}(0)^{\times q_+} \times (\mathbf{T}^{\mathrm{sp}+})^{\times r_+}, & \text{if } n - 2\mu_1' \geqslant 0, \\ \mathbf{T}(\overline{\mu}_1) \times \cdots \times \mathbf{T}(\overline{\mu}_{M_-}) \times \overline{\mathbf{T}}(0)^{\times q_-} \times (\mathbf{T}^{\mathrm{sp}-})^{\times r_-}, & \text{if } n - 2\mu_1' < 0. \end{cases}$$

We give the $\mathfrak{g}$-crystal structure on $\widehat{\mathbf{T}}(\mu, n)$ by the tensor product rule of crystals by identifying $(\ldots, T_2, T_1) \in \widehat{\mathbf{T}}(\mu, n)$ with $T_1 \otimes T_2 \otimes \ldots$.

Let

$$\mathbf{T}(\mu, n) = \{\, \mathbf{T} = (\ldots, T_2, T_1) \in \widehat{\mathbf{T}}(\mu, n) \mid T_{i+1} \prec T_i \ (i \geqslant 1) \,\},$$

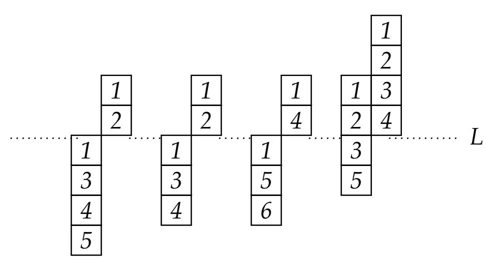
where $T_{i+1} \prec T_i$ means that the pair (T_{i+1}, T_i) satisfies the *admissible conditions* given in [7, Definition 3.4]. It is shown in [7] that $\mathbf{T}(\mu, n)$ is a connected component in $\widehat{\mathbf{T}}(\mu, n)$ including the unique highest weight element of weight $\Lambda(\mu)$. Hence we have the following.

Theorem 3.1. *[7, Theorem 4.3–4.4] For $\mu \in \mathcal{P}(\mathrm{O}_n)$, we have*

$$\mathbf{T}(\mu, n) \cong \mathbf{B}(\Lambda(\mu)).$$

We call $\mathbf{T}(\mu, n)$ the *spinor model for* $\mathbf{B}(\Lambda(\mu))$.

Example 3.2. *Let $n = 8$ and $\mu = (4, 3, 3, 2) \in \mathcal{P}(\mathrm{O}_8)$. Then $\Lambda(\mu) = \Lambda_4 + 2\Lambda_3 + \Lambda_2$. Let $\mathbf{T} = (T_4, T_3, T_2, T_1)$ given by*



$T_4 \qquad T_3 \qquad T_2 \qquad T_1$

where the dotted line denotes the common horizontal line L. In this case, $T_4 \prec T_3 \prec T_2 \prec T_1$ (cf. [7, Definition 3.4]) and thus $\mathbf{T} \in \mathbf{T}(\mu, 8)$.

3.2 Separation lemma

For simplicity, we assume that n is even and $\mu \in \mathcal{P}(\mathrm{O}_n)$ satisfies $n - 2\mu_1' \geqslant 0$. The same result (Lemma 3.4) also holds for the other cases (see [5, Section 3.3–3.4] for more details).

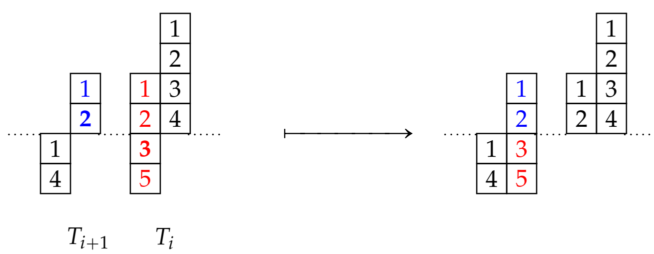
Definition 3.3. *Let*

$$\mathbf{H}(\mu, n) = \{\, \mathbf{T} \mid \mathbf{T} \in \mathbf{T}(\mu, n),\ \widetilde{e}_i \mathbf{T} = \mathbf{0}\ (i \neq 0)\,\},$$

and call $\mathbf{T} \in \mathbf{H}(\mu, n)$ *an* $\mathfrak{l}$-highest weight element *in* $\mathbf{T}(\mu, n)$. *In other words, we have* $\mathbf{T} \in \mathbf{H}(\mu, n)$ *if and only if* $\mathbf{T} \equiv_{\mathfrak{l}} H_\lambda$ *for some* $\lambda \in \mathscr{P}$. *Here* $\equiv_{\mathfrak{l}}$ *means the* $\mathfrak{l}$*-crystal equivalence or Knuth equivalence.*

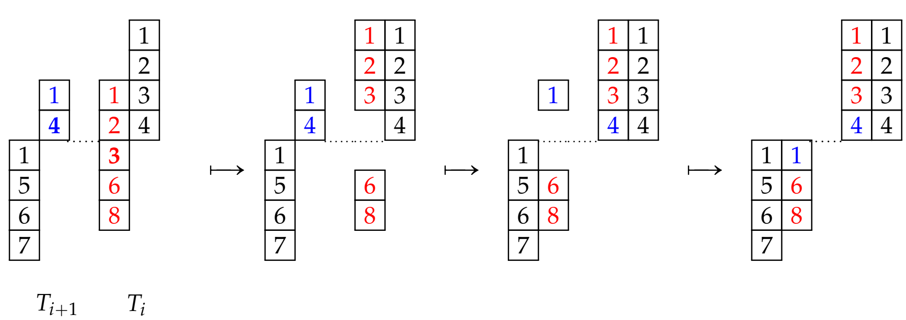
Let $\mathbf{T} = (T_l, \ldots T_1) \in \mathbf{H}(\mu, n)$ with $\mathrm{sh}(T_i) = (2^{b_i + c_i}, 1^{a_i})/(1^{b_i})$ for $1 \leqslant i \leqslant l$. We denote by $T_i^{\mathtt{R}}(k)$ (resp. $T_i^{\mathtt{L}}(k)$) the k-th entry of $T_i^{\mathtt{R}}$ (resp. $T_i^{\mathtt{L}}$) from the bottom. Let us introduce an algorithm on (T_{i+1}, T_i), which is roughly speaking sliding the *tail* of T_i to the left by one position.

(S1) If $T_{i+1}^{\mathtt{R}}(1) < T_i^{\mathtt{L}}(a_i)$, then we move the subtableau $\{T_i^{\mathtt{L}}(k) : 1 \leqslant k \leqslant a_i\}$ of $T_i^{\mathtt{L}}$ to be located below $T_{i+1}^{\mathtt{R}}$. For example,



Here $T_{i+1}^{\mathtt{R}}(1) = 2 < T_i^{\mathtt{L}}(a_i) = 3$ with $a_i = 2$.

(S2) If $T_{i+1}^{\mathtt{R}}(1) > T_i^{\mathtt{L}}(a_i)$, then we slide up the subtableau $\{T_i^{\mathtt{L}}(k) : k \geqslant a_i\}$ of $T_i^{\mathtt{L}}$ by two positions and put $T_{i+1}^{\mathtt{R}}(1)$ below it. Also we slide down the subtableau $T_{i+1}^{\mathtt{R}} \setminus \{T_{i+1}^{\mathtt{R}}(1)\}$ of $T_{i+1}^{\mathtt{R}}$ by two positions and put the subtableau $\{T_i^{\mathtt{L}}(k) : 1 \leqslant k \leqslant a_i - 1\}$ below it. For example,



Here $T_{i+1}^{\mathtt{R}}(1) = 4 > T_i^{\mathtt{L}}(a_i) = 3$ with $a_i = 3$.

Note that the single-column tableaux $T^{\mathtt{L}}_{i+1}$ and $T^{\mathtt{R}}_{i}$ are invariant under the above algorithm.

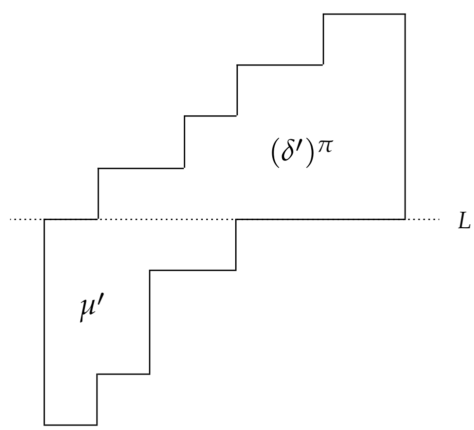
We identify $\mathbf{T}$ with $(T^{\mathtt{L}}_{l}, T^{\mathtt{R}}_{l}, \dots T^{\mathtt{L}}_{1}, T^{\mathtt{R}}_{1})$. Let $\widetilde{\mathbf{T}}$ be the sequence of single-column tableaux obtained from $\mathbf{T}$ by applying the above algorithm to each pair (T_{i+1}, T_i) from $l-1$ to 1, and then removing $T^{\mathtt{L}}_{l}$. By [5, Lemma 3.10, Corollary 3.11], we have $\widetilde{\mathbf{T}} \in \mathbf{H}(\widetilde{\mu}, n-1)$, where $\widetilde{\mu} = (\mu_2, \mu_3, \dots)$. Hence we can apply the above algorithm to $\widetilde{\mathbf{T}}$ again, and repeat this process as far as possible to get a tableau $\overline{\mathbf{T}}$.

Lemma 3.4 (Separation lemma)**.** *Under the above hypothesis, $\overline{\mathbf{T}}$ satisfies the following conditions:*

(1) $\overline{\mathbf{T}} \in SST(\eta)$, *where η is the skew Young diagram given in* (3.1),

(2) $\overline{\mathbf{T}}$ *is Knuth equivalent to* $\mathbf{T}$, *that is,* $\overline{\mathbf{T}} \equiv_{\mathfrak{K}} \mathbf{T}$,

(3) *Let* $\overline{\mathbf{T}}^{\mathtt{body}}$ *and* $\overline{\mathbf{T}}^{\mathtt{tail}}$ *be the subtableaux of* $\overline{\mathbf{T}}$ *located above and below the horizontal line L, respectively. Then* $\overline{\mathbf{T}}^{\mathtt{body}} = H_{(\delta')^{\pi}}$ *for some* $\delta \in \mathscr{P}^{(2)}$, *and* $\overline{\mathbf{T}}^{\mathtt{tail}} \in \mathrm{LR}^{\lambda'}_{\delta'\mu'}$ *if* $\mathbf{T} \equiv_{\mathfrak{K}} H_{\lambda'}$ *for some* $\lambda \in \mathscr{P}$.

$\eta =$  (3.1)

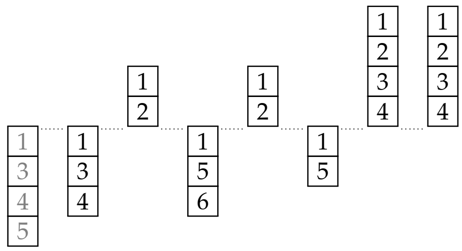
Note that $\mathbf{T} \equiv_{\mathfrak{K}} \overline{\mathbf{T}} \equiv_{\mathfrak{K}} \overline{\mathbf{T}}^{\mathtt{body}} \otimes \overline{\mathbf{T}}^{\mathtt{tail}}$ by (2), and it is not difficult to check that (3) implies that the map

$$\mathbf{T} \longmapsto \overline{\mathbf{T}}^{\mathtt{tail}} \tag{3.2}$$

is injective (see [5, Lemma 6.5]). We will describe the image of the injection (3.2) in Section 4.

Example 3.5. Let $n = 8$ and $\mu = (4,3,3,2) \in \mathcal{P}(\mathrm{O}_8)$. Let $\mathbf{T} = (T_4, T_3, T_2, T_1) \in \mathbf{H}(\mu, 8)$ given as in Example 3.2. Then since $T^{\mathtt{R}}_4(1) = 2 > 1 = T^{\mathtt{L}}_3(3)$, $T^{\mathtt{R}}_3(1) = 2 > 1 = T^{\mathtt{L}}_2(3)$ and $T^{\mathtt{R}}_2(1) = 4 > 3 = T^{\mathtt{L}}_1(2)$, we apply the algorithm (S2) to each pair (T_{i+1}, T_i) for $i = 1,2,3$.

Consequently we have $\widetilde{\mathbf{T}}$ given by



with the left-most column (in gray) removed. By [5, Lemma 3.10, Corollary 3.11], we have $\widetilde{\mathbf{T}} \in \mathbf{H}(\widetilde{\mu}, 7)$ with $\widetilde{\mu} = (3, 3, 2)$.

Repeating this process, we have

$$\overline{\mathbf{T}} = \begin{array}{cccccccc} & & & & & & 1 & 1 \\ & & & & & & 2 & 2 \\ & & & & 1 & 1 & 3 & 3 \\ & & & & 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 & & & & \\ 3 & 3 & 5 & 5 & & & & \\ 4 & 4 & 6 & & & & & \\ 5 & & & & & & & \end{array},$$

where

$$\overline{\mathbf{T}}^{\texttt{body}} = \begin{array}{cccc} & & 1 & 1 \\ & & 2 & 2 \\ 1 & 1 & 3 & 3 \\ 2 & 2 & 4 & 4 \end{array}, \qquad \overline{\mathbf{T}}^{\texttt{tail}} = \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 3 & 3 & 5 & 5 \\ 4 & 4 & 6 & \\ 5 & & & \end{array}$$

4 Combinatorial formula of branching multiplicities

4.1 Branching from D_∞ to $A_{+\infty}$

In this section, we assume $n \in \mathbb{Z}_+$. Let $\mu \in \mathcal{P}(\mathrm{O}_n)$, $\lambda \in \mathscr{P}_n$ and $\delta \in \mathscr{P}_n^{(2)}$ be given. We denote by $\delta^{\texttt{rev}} = (\delta_1^{\texttt{rev}}, \ldots, \delta_n^{\texttt{rev}})$ the reverse sequence of $\delta = (\delta_1, \ldots, \delta_n)$, that is, $\delta_i^{\texttt{rev}} = \delta_{n-i+1}$, for $1 \leqslant i \leqslant n$. We put $p = \mu_1'$, $q = \mu_2'$, and $r = (\overline{\mu})_1'$ if $n - 2\mu_1' < 0$.

Let

$$\texttt{LR}_\lambda^\mu(\mathfrak{d}) = \{\, \mathbf{T} \mid \mathbf{T} \in \mathbf{H}(\mu, n),\ \mathbf{T} \equiv_{\mathfrak{l}} H_{\lambda'} \,\}\,, \quad c_\lambda^\mu(\mathfrak{d}) = |\texttt{LR}_\lambda^\mu(\mathfrak{d})|\,.$$

Note that $c_\lambda^\mu(\mathfrak{d})$ is equal to the multiplicity of irreducible highest weight $\mathfrak{l}$-module with highest weight $\sum_{i \geqslant 1} \lambda_i' \epsilon_i$ in the irreducible highest weight $\mathfrak{g}$-module with highest weight $\Lambda(\mu)$.

Definition 4.1. For $S \in \texttt{LR}_{\delta'\mu'}^{\lambda'}$, let $s_1 \leqslant \cdots \leqslant s_p$ denote the entries in the first row, and $t_1 \leqslant \cdots \leqslant t_q$ the entries in the second row of S. Let $1 \leqslant m_1 < \cdots < m_p < n$ be the sequence defined inductively from p to 1 as follows:

$$m_i = \max\{\, k \mid \delta_k^{\texttt{rev}} \in X_i,\ \delta_k^{\texttt{rev}} < s_i \,\},$$

where

$$X_i = \begin{cases} \{\, \delta_i^{\mathtt{rev}}, \dots, \delta_{2i-1}^{\mathtt{rev}} \,\} \backslash \{\delta_{m_{i+1}}^{\mathtt{rev}}, \dots, \delta_{m_p}^{\mathtt{rev}}\}, & \text{if } 1 \leqslant i \leqslant r, \\ \{\, \delta_i^{\mathtt{rev}}, \dots, \delta_{n-p+i}^{\mathtt{rev}} \,\} \backslash \{\delta_{m_{i+1}}^{\mathtt{rev}}, \dots, \delta_{m_p}^{\mathtt{rev}}\}, & \text{if } r < i \leqslant p, \end{cases}$$

Here we assume that $r = p$ when $n - 2\mu_1' \geqslant 0$.

Let $n_1 < \cdots < n_q$ be the sequence such that n_j is the j-th smallest integer in $\{j+1, \cdots, n\} \backslash \{m_{j+1}, \cdots, m_p\}$ for $1 \leqslant j \leqslant q$.

Then we define $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ to be a subset of $\mathtt{LR}^{\lambda'}_{\delta'\mu'}$ consisting of S satisfying

$$t_j > \delta_{n_j}^{\mathtt{rev}},$$

for $1 \leqslant j \leqslant q$. We put $\overline{c}^{\lambda}_{\delta\mu} = |\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}|$.

The following is the main result in this abstract, which characterizes the image of injection (3.2).

Theorem 4.2. *For $\mu \in \mathcal{P}(\mathrm{O}_n)$ and $\lambda \in \mathscr{P}_n$, we have a bijection*

$$\begin{array}{ccc} \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d}) & \longrightarrow & \displaystyle\bigsqcup_{\delta \in \mathscr{P}_n^{(2)}} \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'} \,. \\ \mathbf{T} & \longmapsto & \overline{\mathbf{T}}^{\mathtt{tail}} \end{array}$$

Corollary 4.3. *Under the above hypothesis, we have*

$$c^{\mu}_{\lambda}(\mathfrak{d}) = \sum_{\delta \in \mathscr{P}_n^{(2)}} \overline{c}^{\lambda}_{\delta\mu} \,. \tag{4.1}$$

Let us give the alternative description of $c^{\mu}_{\lambda}(\mathfrak{d})$ which is simpler than $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$.

Definition 4.4. For $U \in \mathtt{LR}^{\lambda}_{\delta\mu^{\pi}}$, let $\sigma_1 > \cdots > \sigma_p$ denote the entries in the rightmost column and $\tau_1 > \cdots > \tau_q$ the second rightmost column of U, respectively. Let $m_1 < \cdots < m_p$ be the sequence defined by

$$m_i = \begin{cases} \min\{n - \sigma_i + 1, 2i - 1\}, & \text{if } 1 \leqslant i \leqslant r, \\ \min\{n - \sigma_i + 1, n - p + i\}, & \text{if } r < i \leqslant p. \end{cases}$$

and let $n_1 < \cdots < n_q$ be the sequence such that n_j is the j-th smallest number in $\{\, j+1, \dots, n \,\} \backslash \{\, m_{j+1}, \dots, m_p \,\}$. Then we define $\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$ to be the subset of $\mathtt{LR}^{\lambda}_{\delta\mu^{\pi}}$ consisting of U such that

$$\tau_j + n_j \leqslant n + 1, \tag{4.2}$$

for $1 \leqslant j \leqslant q$. We put $\underline{c}^{\lambda}_{\delta\mu} = |\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}|$.

Remark 4.5. Recall that $\mathtt{LR}^{\lambda}_{\delta\mu^\pi}$ is the set of $S \in SST(\lambda/\delta)$ with content μ^π such that $w(T) = w_1 \dots w_r$ is an anti-lattice word, while in Definition 4.4, $\mathtt{LR}^{\lambda}_{\delta\mu^\pi}$ is given by the set of the companion tableaux U of S.

Theorem 4.6. *For $\mu \in \mathcal{P}(\mathrm{O}_n)$, $\lambda \in \mathscr{P}_n$ and $\delta \in \mathscr{P}^{(2)}_n$, the bijection $\psi : \mathtt{LR}^{\lambda'}_{\mu'\nu'} \longrightarrow \mathtt{LR}^{\lambda}_{\mu\nu^\pi}$ in (2.1) induces a bijection from $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ to $\underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$.*

Corollary 4.7. *Under the above hypothesis, we have*

$$c^{\mu}_{\lambda}(\mathfrak{d}) = \sum_{\delta \in \mathscr{P}^{(2)}_n} \underline{c}^{\lambda}_{\delta\mu} \,. \tag{4.3}$$

In particular, if $\ell(\lambda) \leqslant \frac{n}{2}$, then we have the Littlewood's restriction formula (1.2) for $\mathrm{G_n} = \mathrm{O}_n$ from (4.3).

Example 4.8. Let $n = 8$, $\mu = (2,2,2,1,1) \in \mathcal{P}(\mathrm{O}_8)$, $\lambda = (5,4,4,3,2,2) \in \mathscr{P}_8$, and $\delta = (4,2,2,2,2) \in \mathscr{P}^{(2)}_8$.

Let us consider the Littlewood–Richardson tableau $U \in \mathtt{LR}^{\lambda}_{\delta\mu^\pi}$ given by

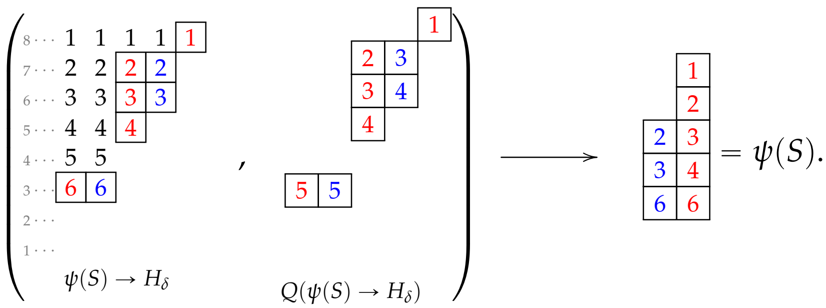
$$U = \begin{array}{|c|c|} \hline \multicolumn{1}{c|}{} & 1 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 6 & 6 \\ \hline \end{array}\,, \tag{4.4}$$

where $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (6,4,3,2,1)$ and $(\tau_1, \tau_2, \tau_3) = (6,3,2)$. Note that the shape of U is $\mu^\pi = (1,1,2,2,2)$ and the content is $\lambda/\delta = (1,2,2,1,0,2)$. Then the sequences $(m_i)_{1 \leqslant i \leqslant 5}$ and $(n_j)_{1 \leqslant j \leqslant 3}$ (Definition 4.4) are given by $(1,3,5,7,8)$ and $(2,4,6)$ respectively. It is easy to check that U satisfies the condition (4.2). Hence $U \in \underline{\mathtt{LR}}^{\lambda}_{\mu\delta}$.

On the other hand, let S be the Littlewood–Richardson tableau in $\overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ (recall Definition 4.1) with the enumeration of the columns as follows:

$$S = \begin{array}{ccccc} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} & \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} & \begin{array}{|c|} \hline 3 \\ \hline \end{array} & \begin{array}{|c|} \hline 5 \\ \hline \end{array} \\ S^5 & S^4 & S^3 & S^2 & S^1 \end{array}\,,$$

where $(s_1, s_2, s_3, s_4, s_5) = (1,3,3,3,5)$ and $(t_1, t_2, t_3) = (2,4,4)$. Then $\psi(S)$ (2.1) is obtained by



Note that $\psi(S)$ is same with U (4.4). Under the above correspondence, we observe that σ_i $(1 \leqslant i \leqslant 5)$ and τ_j $(1 \leqslant j \leqslant 3)$ record the positions of s_i and t_j in δ', respectively, and vice versa. This implies that $S \in \overline{\mathtt{LR}}^{\lambda'}_{\delta'\mu'}$ if and only if $\psi(S) \in \underline{\mathtt{LR}}^{\lambda}_{\delta\mu}$.

Remark 4.9. (1) We may have an analogue of Theorem 4.2 for type B and C, that is, a multiplicity formula with respect to the branching from B_∞ and C_∞ to $A_{+\infty}$, respectively (see Remark 4.14 in [5] for more details).

(2) When n is odd, there is a bijection between $\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ and a set of LR tableaux with certain conditions, where λ appears as an inner shape of LR tableaux [4]. This alternative description of $\mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ is used to construct a bijection between the set of pairs of standard tableau of shape λ and $\mathbf{T} \in \mathtt{LR}^{\mu}_{\lambda}(\mathfrak{d})$ and the set of vacillating tableaux of shape μ.

4.2 Branching from GL_n to O_n

We assume that the base field is $\mathbb{C}$. Let $V^{\lambda}_{\mathrm{GL}_n}$ denote the finite-dimensional irreducible GL_n-module corresponding to $\lambda \in \mathscr{P}_n$, and $V^{\mu}_{\mathrm{O}_n}$ the finite-dimensional irreducible module O_n-module corresponding to $\mu \in \mathcal{P}(\mathrm{O}_n)$.

Then we have the following new combinatorial description of $\left[V^{\lambda}_{\mathrm{GL_n}} : V^{\mu}_{\mathrm{O}_n}\right]$.

Theorem 4.10. *For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathcal{P}(\mathrm{O}_n)$, we have*

$$\left[V^{\lambda}_{\mathrm{GL_n}} : V^{\mu}_{\mathrm{O_n}}\right] = \sum_{\delta \in \mathscr{P}^{(2)}_n} \overline{c}^{\lambda}_{\delta\mu} = \sum_{\delta \in \mathscr{P}^{(2)}_n} \underline{c}^{\lambda}_{\delta\mu}.$$

Proof. It follows from the branching rule of see-saw pairs $(D_\infty, A_{+\infty})$ and $(\mathrm{GL}_n, \mathrm{O}_n)$ [8, Theorem 5.3]

$$\left[V^{\lambda}_{\mathrm{GL_n}} : V^{\mu}_{\mathrm{O_n}}\right] = c^{\mu}_{\lambda}(\mathfrak{d}),$$

and Corollaries 4.3 and 4.7. □

Remark 4.11. As an application of the branching multiplicity, we obtain a new combinatorial realization for the Lusztig t-weight multiplicity $K_{\mu 0}(t)$ of type B_n and D_n with highest weight μ and weight 0 or generalized exponents (see [5, Section 5]). This gives an orthogonal analogue of the result for type C_n in [9].

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