Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #19, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

K-theoretic crystals for set-valued tableaux of rectangular shapes

Oliver Pechenik*[1] and Travis Scrimshaw†[2]

[1] *Department of Mathematics, University of Michigan, Ann Arbor, MI, USA*
[2] *School of Mathematics and Physics, The University of Queensland, St. Lucia, Australia*

Abstract. With C. Monical (2018), we introduced a notion of K-crystals and conjectured they exist for all rectangular shapes λ. Here, we establish this conjecture, yielding the first combinatorial formula (as the sum over flagged set-valued tableaux) for the Lascoux polynomials $L_{w\lambda}$. We then prove corresponding cases of conjectures of Ross–Yong (2015) and Monical (2016).

Keywords: Lascoux polynomial, crystal, Kohnert diagram, skyline tableau

1 Introduction

In classical Schubert calculus, we can study the cohomology ring of the Grassmannian $\mathrm{Gr}(k, n)$, the set of k-dimensional subspaces of $\mathbb{C}^n$, using the basis of Poincaré duals of the Schubert varieties X_λ that decompose $\mathrm{Gr}(k, n)$. The cohomology classes $[X_\lambda]$ can be represented by Schur polynomials s_λ, where the partition λ sits inside a $k \times (n-k)$ rectangle. A more modern approach is to use connective K-theory, where the Schubert class $[X_\lambda]$ is given as the push-forward of the class for any Bott–Samelson resolution of X_λ. Here, representatives are symmetric (or stable) β-Grothendieck polynomials [3].

We can describe s_λ combinatorially as a generating function for semistandard (Young) tableaux of shape λ and representation-theoretically as the character of the highest weight representation $V(\lambda)$ of the Lie algebra $\mathfrak{sl}_n$ of traceless $n \times n$ matrices. We can also compute s_λ by applying a product of Demazure operators π_{w_0} for the reverse permutation w_0 to the monomial $\mathbf{x}^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. We can refine s_λ to the key polynomials $\kappa_{w\lambda} := \pi_w \mathbf{x}^\lambda$ for any permutation w, which are characters of Demazure modules $V_w(\lambda)$.

Combinatorially, A. Buch [1] showed the symmetric Grothendieck polynomial $\mathfrak{G}_\lambda$ is the generating function for semistandard set-valued tableaux of shape λ. A. Lascoux [7] deformed the Demazure operators to Demazure–Lascoux operators ϖ_w, so that $\mathfrak{G}_\lambda = \varpi_{w_0} \mathbf{x}^\lambda$. The analogous deformation of key polynomials, the so-called Lascoux

*pechenik@umich.edu. Partially supported by the NSF Mathematical Sciences Postdoctoral Research Fellowship #1703696.

†tcscrims@gmail.com. Partially supported by the Australian Research Council DP170102648.

polynomials $L_{w\lambda} = \varpi_w \mathbf{x}^\lambda$, remain mysterious as currently there is no known geometric, representation-theoretic, or combinatorial interpretation, despite recent work [17, 5, 13, 15]. Yet, combinatorial formulas have been conjectured [13, 17, 5].

One way to connect the combinatorics and representation theory associated to key polynomials is using M. Kashiwara's crystal bases (see, *e.g.*, [2, 4]). Indeed, Kashiwara showed that the Demazure module $V_w(\lambda)$ has a crystal basis and can be described as a subcrystal $B_w(\lambda)$(called a Demazure crystal) of the highest weight crystal $B(\lambda)$ [4]. For the quantum group $U_q(\mathfrak{sl}_n)$, the crystal $B(\lambda)$ may be realized as the set of semistandard tableaux of shape λ and the subcrystal $B_w(\lambda)$ is characterized by key tableaux [10].

In our previous paper with C. Monical [14], we initiated an analogous approach to Demazure crystals for Lascoux polynomials. We first gave a $U_q(\mathfrak{sl}_n)$-crystal structure to semistandard set-valued tableaux. Then we proposed an enriched crystal structure with the property that the Lascoux polynomials appear as the characters of our K-theoretic analogs of Demazure subcrystals. We coined this enriched structure a K-crystal. We established the existence of K-crystals for single rows and columns, but we discovered that no such structure exists for general shapes. Nonetheless, we conjectured [14, Conjecture 7.12] that K-crystals exist for all rectangular shapes. Our first main result is a proof of this conjecture. Our proof gives rise to a combinatorial formula for the class of Lascoux polynomials indexed by a weight in the Weyl group orbit of a multiple of a fundamental weight (*i.e.*, a rectangular shape partition). We then use this formula to establish the corresponding cases of Ross–Yong–Kirillov and Monical conjectures.

Let us remark on why our proposed K-crystal structure exists only for rectangular shapes. With C. Monical [14], we proposed a slightly weaker structure for general λ that depends on a choice of a reduced expression for w_0. The key distinction appears to be that in the rectangular case the minimal-length coset representatives that index Lascoux polynomials are all fully-commutative (*i.e.*, all reduced words differ only by commutations). However, for more general shapes, such as $\lambda = (2, 1)$ in [14, Figure 6, 7], one needs to apply the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ to get all possible reduced expressions. Subsequently, we believe that, in general, K-crystal structures depend on choosing a commutation class of the reduced words for the appropriate parabolic w_0 (see also [14, §7.3]). This fact seems related to a similar dependence for Schubert classes in cohomology theories more general than connective K-theory (see, *e.g.*, [11]). In the rectangular case, we have a flagging condition to characterize the tableaux in the K-Demazure crystal, and we expect a key tableau condition to work for general shapes.

This extended abstract of [16] (where we refer the reader to for more details) is organized as follows. In Section 2, we recall the necessary background. In Section 3, we construct a K-crystal structure on set-valued tableaux of rectangular shapes. In Section 4 (resp. Section 5), we prove the conjectural combinatorial interpretation of Lascoux polynomials for rectangular shapes due to Ross–Yong–Kirillov (resp. Monical). In Section 6, we describe our conjecture for key tableaux of set-valued tableaux and their relationship with Lascoux polynomials.

2 Background

Let S_n be the symmetric group with simple transpositions $\{s_i \mid 1 \le i < n\}$ and longest element $w_0 = [n, \ldots, 2, 1]$. Let $v \le w$ be the (strong) Bruhat order, which means there is a reduced word for v that is a subword of a reduced word for w. Let $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ be a countable vector of indeterminants. For a tuple $\alpha = (\alpha_1, \alpha_2, \ldots)$, define $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

We use the English convention for both partitions and tableaux. Consider a partition λ as a word of length n by appending 0's as necessary, whence it carries a natural S_n-action. Let $\operatorname{Stab}_n(\lambda) = \{w \in S_n \mid w\lambda = \lambda\}$ denote the stabilizer of λ. Let S_n^λ denote the set of minimal length coset representatives $\{\lfloor w \rfloor \mid w \in S_n\}$ of $S_n / \operatorname{Stab}_n(\lambda)$.

A *(semistandard) set-valued tableau of shape* λ is a filling T of the boxes of λ by finite nonempty sets of positive integers so that for every set A to the left of a set B in the same row, we have $\max A \le \min B$, and for C below A in the same column, we have $\max A < \min C$. We say an integer $a \in T$ if there exists a box of T containing a set A with $a \in A$. A set-valued tableau is a *semistandard Young tableau* if all sets have size 1. Let $\mathrm{SV}^n(\lambda)$ denote the set of all set-valued tableaux of shape λ with entries at most n.

We recall the crystal structure on $\mathrm{SV}^n(\lambda)$ from [14]. We refer to [2] for more details on crystals. First, we recall the *crystal operators* $e_i, f_i \colon \mathrm{SV}^n(\lambda) \to \mathrm{SV}^n(\lambda) \sqcup \{0\}$, where $i \in I := \{1, \ldots, n-1\}$. Begin by constructing a sequence by writing $+$ (resp. $-$) above each column of T containing i but not $i+1$ (resp. $i+1$ but not i) and canceling ordered pairs $-+$. If every $+$ (resp. $-$) thereby cancels, then $f_i T = 0$ (resp. $e_i T = 0$). Otherwise,

- if there exists a box b' immediately to the right (resp. left) of b that contains an i (resp. $i+1$), then remove the i (resp. $i+1$) from b' and add an $i+1$ (resp. i) to b;
- otherwise replace the i in b with an $i+1$ (resp. $i+1$ in b with an i);

where b is the box of the rightmost uncanceled $+$ (resp. leftmost uncanceled $-$), and the result is $f_i T$ (resp. $e_i T$). See Figure 1 for an example.

Identifying $\mathbb{Z}^n$ with the multiplicative group generated by $(x_1, \ldots, x_n)$, we define the weight function $\mathrm{wt} \colon \mathrm{SV}^n(\lambda) \to \mathbb{Z}^n$ by $\mathrm{wt}(T) = \prod_{i=1}^n x_i^{c_i}$, where c_i is the number of $A \in T$ such that $i \in A$. Define $|\mathrm{wt}(T)| = \sum_{i=1}^n c_i$.

Theorem 2.1 ([14, Theorem 3.9]). *$\mathrm{SV}^n(\lambda)$ is isomorphic to a direct sum of highest weight crystals.*

For $1 \le i < n$, the *Demazure–Lascoux operator* ϖ_i acts on $\mathbb{Z}[\beta][x_1, \ldots, x_n]$ by

$$\varpi_i f = \pi_i\big((1 + \beta x_{i+1}) \cdot f\big) = \pi_i f + \beta \cdot \pi_i(x_{i+1} \cdot f), \text{ where } \pi_i f = \frac{x_i \cdot f - x_{i+1} \cdot s_i f}{x_i - x_{i+1}},$$

is the *Demazure operator*. The Demazure–Lascoux operators (and Demazure operators) satisfy the braid relations. Thus for any $w \in S_n$, one may unambiguously define $\varpi_w := \varpi_{i_1} \cdots \varpi_{i_\ell}$, where $s_{i_1} \cdots s_{i_\ell}$ is some reduced expression for w (and similarly for π_w).

Since ϖ_w does not depend on the choice of reduced expression, we can define the *Lascoux polynomials* [7] for any $a \in \mathbb{Z}^n_{\geq 0}$ as

$$L_a(\mathbf{x}; \beta) := \varpi_w \mathbf{x}^\lambda,$$

where λ is the sorting of a to a partition and $w \in S_n^\lambda$ is the unique element such that $a = w\lambda$. The *symmetric Grothendieck polynomial* can be defined as the n variable truncation of $L_{w_0\lambda}(\mathbf{x}; \beta)$ and is known [1, Theorem 3.1] to be given combinatorially by

$$L_{w_0\lambda}(\mathbf{x}; \beta) = \sum_{T \in \mathsf{SV}^n(\lambda)} \operatorname{wt}_\beta(T), \quad \text{where } \operatorname{wt}_\beta(T) := \beta^{|\operatorname{wt}(T)| - |\lambda|} \operatorname{wt}(T) \text{ is the } \beta\text{-weight}. \tag{2.1}$$

We now recall two conjectural combinatorial descriptions of Lascoux polynomials.

The first conjectural combinatorial rule was introduced in [17]. To state it, we begin by recalling the notion of a *K-Kohnert diagram* to be a subset D of $\mathbb{Z}^n_{>0}$, which we realize as boxes, and a subset $M \subseteq D$ of boxes that are marked. Now start with some $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ and draw the initial K-Kohnert diagram as a *skyline diagram* by putting a box at each position $\{(i, y) \mid i \in [n], 1 \leq y \leq a_i\}$ (in Cartesian coordinates), marking no boxes. Then we successively apply any sequence of the following operations.

***Kohnert move*:** Move any unmarked box at the top of a column into the rightmost open position to its left and in the same row so that it does not pass over a marked box.[1]

***K-Kohnert move*:** Perform a Kohnert move but leave a marked box behind.

Let $\mathcal{D}_a$ denote the resulting set of K-Kohnert diagrams obtainable from the original skyline diagram for a. Define the β-weight of $D \in \mathcal{D}_a$ by $\operatorname{wt}_\beta(D) := \beta^e \prod_{i=1}^n x_i^{c_i}$, where e (resp. c_i) is the number of marked boxes (resp. boxes in column i) in D.

Conjecture 2.2 ([17, Conjecture 1.4],[5, Fn. 14])**.** *We have* $L_a(\mathbf{x}; \beta) = \sum_{D \in \mathcal{D}_a} \operatorname{wt}_\beta(D)$.

The second conjectural combinatorial rule is from [13]. We fill a skyline diagram with finite nonempty sets of positive integers that satisfy the following conditions. Call the largest entry in a box the *anchor* and the other entries *free*. (S.1): Entries do not repeat in a row. (S.2): If B is below A, then $\min B \geq \max A$ (*i.e.*, the columns are weakly increasing top-to-bottom in the set-valued sense). (S.3): For every triple of boxes of the form

$$\begin{array}{|c|}\hline C \\ \hline \end{array} \cdots \begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array} \cdots \begin{array}{|c|}\hline C \\ \hline \end{array}$$

right column weakly taller left column strictly taller

[1] In [17], it is misstated that a Kohnert move could move the unmarked box over a marked box.

the anchors a, b, c of A, B, C, respectively, must satisfy either $c < a$ or $b < c$. (S.4): Every free entry is in the leftmost cell of its row such that (S.2) is not violated. (S.5): Anchors in the bottom row equal their column index. We call such a tableau a *(semistandard) set-valued skyline tableau*, and let SLT_a denote those of shape a. We define the weight, excess, and β-weight for a set-valued skyline tableau the same way as for a set-valued tableau.

Let $\overline{\varpi}_i = \varpi_i - 1$. Define the *Lascoux atom* to be $\overline{L}_{w\lambda}(\mathbf{x}; \beta) := \overline{\varpi}_w x^\lambda$.

Conjecture 2.3 ([13, Conjecture 5.2])**.** *We have* $\overline{L}_{w\lambda} = \sum_{S \in \mathrm{SLT}_{w\lambda}} \mathrm{wt}_\beta(S)$.

Note Conjeture 2.3 is equivalent to [13, Conjecture 5.3] by [15]. Also from [13, Theorem 5.1],

$$L_{w\lambda}(\mathbf{x}; \beta) = \sum_{v \leq w} \overline{L}_{v\lambda}(\mathbf{x}; \beta), \tag{2.2}$$

where the inequality is (strong) Bruhat order on permutations.

3 K-crystals for rectangular shapes

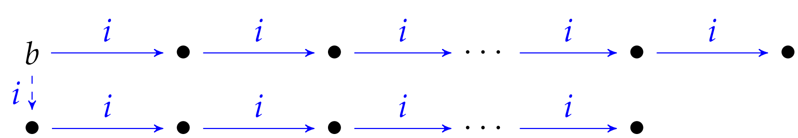
We aim to prove the proposed K-theory analog of crystals from [14] exists on $\mathrm{SV}^n(\lambda)$ when λ is a rectangle. Recall that a $U_q(\mathfrak{sl}_n)$-crystal B is called a *K-crystal* if it is enhanced with *K-crystal operators*, $e_i^K, f_i^K \colon B \to B \sqcup \{0\}$ that satisfy the following properties:

(K.1) The set B is generated by a unique element $u \in B$ that satisfies $e_i u = 0$ and $e_i^K u = 0$ for all $i \in I$. The element u is called the *minimal highest weight element*.

(K.2) The *K-Demazure crystal* $B_w := \{b \in B \mid (e_{i_\ell}^K)^{\max} e_{i_\ell}^{\max} \cdots (e_{i_1}^K)^{\max} e_{i_1}^{\max} b = u\}$ does not depend on the choice of reduced expression $w = s_{i_1} \cdots s_{i_\ell}$. Moreover, $B_{w_0} = B$.

(K.3) Let $\lambda = \mathrm{wt}(u)$. The *β-character* $\mathrm{ch}_\beta(B_w) := \sum_{b \in B_w} \beta^{|\mathrm{wt}(b)| - |\lambda|} \mathbf{x}^{\mathrm{wt}(b)} = L_{w\lambda}(\mathbf{x}; \beta)$.

Our construction of the K-crystal operators are based off the heuristics given in [14], which come from the following K-theory analog of the decomposition of a crystal into i-strings (*i.e.*, restricting to the action of e_i and f_i for a fixed $i \in I$) based on the definition of the Demazure–Lusztig operators. Indeed, by considering only the action of a fixed $i \in I$, the K-crystal is expected to decompose into (maximal) subcrystals of the form



where the solid (resp. dashed) arrow represents the f_i (resp. f_i^K) action. Such a subcrystal was coined an *i-K-string* in [14]. We say an i-K-string has *length* $\ell := \max\{k \mid f_i^k b \neq 0\}$.

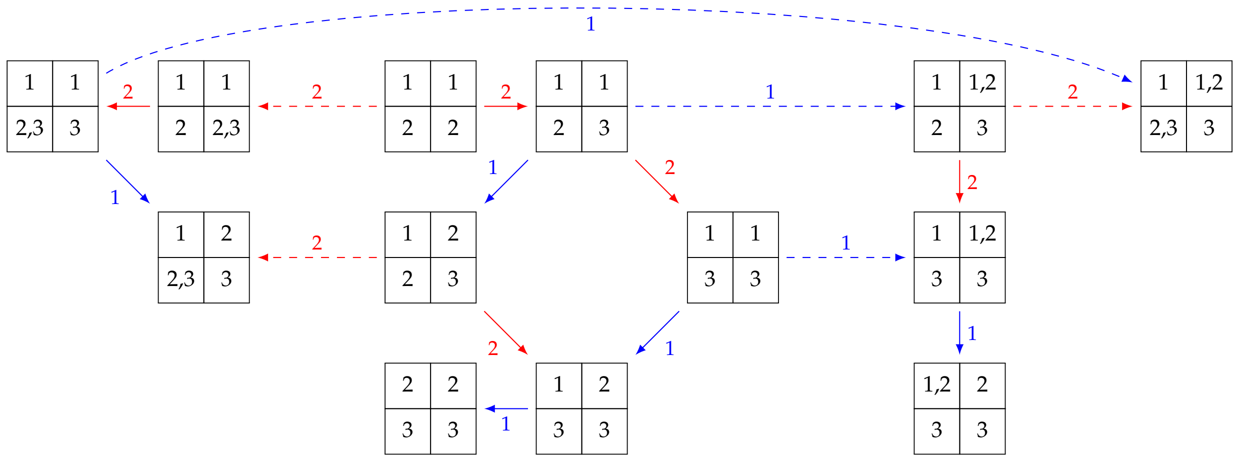


Figure 1: The K-crystal for $\mathrm{SV}^3\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right)$ with the K-crystal operators as dashed lines.

Note that $f_i^{\ell-1} f_i^K b \neq 0$ and $f_i^{\ell} f_i^K b = 0$. It is easy to see that for b such that $e_i b = 0$, the β-character of the i-K-string starting at b equals $\varpi_i \operatorname{wt}(b)$.

Lemma 3.1. *Let λ be an $r \times s$ rectangle. For any $w \in S_n^{\lambda}$, there exists a reduced word of w equal to $(s_{i_k} \cdots s_{r-k+1} s_{r-k}) \cdots (s_{i_1} \cdots s_r s_{r-1})(s_{i_0} \cdots s_{r+1} s_r)$ for some $k < r$ and $i_k < \cdots < i_0 \leq n$.*

Definition 3.2. Let $T \in \mathrm{SV}^n(\lambda)$ and fix some $i \in I$.

f_i^K**:** If $i \notin T$ or $e_i T \neq 0$, then $f_i^K T = 0$. Otherwise, let b be the rightmost box that contains an i corresponding to an unpaired $+$. If i and $i+1$ are both in a box weakly to the right of b, then $f_i^K T = 0$. Otherwise, define $f_i^K T$ by adding an $i+1$ to b.

e_i^K**:** If there does not exist a box with both an i and $i+1$ or $e_i T \neq 0$, then $e_i^K T = 0$. Otherwise, let b be the rightmost box that has both an i and $i+1$. If there exists an i corresponding to an unpaired $+$ strictly to the right of b, then $e_i^K T = 0$. Otherwise, define $e_i^K T$ by removing the $i+1$.

For an example, see Figure 1; additional examples may be found in [14]. Note that it is clear that if $f_i^K T \neq 0$ (resp. $e_i^K T \neq 0$), then $f_i^K T \in \mathrm{SV}^n(\lambda)$ (resp. $e_i^K T \in \mathrm{SV}^n(\lambda)$).

Lemma 3.3. *Let λ be an $r \times s$ rectangle. Let $T, T' \in \mathrm{SV}^n(\lambda)$. We have $e_i^K T' = T$ if and only if $T' = f_i^K T$. Moreover, for any $i \in I$, we have that $\mathrm{SV}^n(\lambda)$ is a union of i-K-strings.*

Consider some $w \in S_n^{\lambda}$, and let $i_k < \cdots < i_0$ be from the reduced expression of w given by Lemma 3.1. For all $k < j < r$, define $i_j = r - j$. Define $F(\lambda; w)$ to be the subset of $\mathrm{SV}^n(\lambda)$ such that row $r - j$ has all entries at most $i_j + 1$. We call such a set-valued tableau a *flagged set-valued tableau*. This flagging characterizes the K-Demazure crystal.

Lemma 3.4. *Let λ be an $r \times s$ rectangle. For $w \in S_n$, then $\mathrm{SV}^n_w(\lambda) = \mathrm{SV}^n_{\lfloor w \rfloor}(\lambda) = F(\lambda; \lfloor w \rfloor)$.*

Theorem 3.5. *Let λ be an $r \times s$ rectangle. Then $\mathrm{SV}^n(\lambda)$ is a K-crystal.*

We also have the following K-theoretic analog of [4, Proposition 3.3.4].

Corollary 3.6. *Let λ be an $r \times s$ rectangle. Consider an i-K-string S of $\mathrm{SV}^n(\lambda)$, and let b be the highest weight element of S. Then, the set $\mathrm{SV}^n_w(\lambda) \cap S$ is either empty, S, or $\{b\}$.*

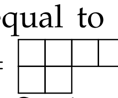
We also have the following interpretation of certain Lascoux polynomials as instances of *(β-)Grothendieck polynomials*, which recall from [6, 8, 9, 3] are defined by

$$\mathfrak{G}_{w_0 s_{i_1} \cdots s_{i_\ell}} := \partial^\beta_{i_1} \cdots \partial^\beta_{i_\ell} x_1^{n-1} \cdots x_{n-1}^1 x_n^0, \quad \text{where } \partial^\beta_i f = \frac{(1+\beta x_i) \cdot f - (1+\beta x_{i+1}) \cdot s_i f}{x_i - x_{i+1}}.$$

Corollary 3.7. *Let λ be an $r \times s$ rectangle. Let $w = (s_k \cdots s_2 s_1) \cdots (s_{k+r-1} \cdots s_{r+1} s_r)$ for some $k \geq 1$, and let $\widetilde{w} = s_{m-1}(s_{m-2}s_{m-1}) \cdots (s_{r+1} \cdots s_{m-1})(s_r \cdots s_{k-1}) \cdots (s_1 \cdots s_{k-1}) \in S_m$ where $m = s + k + 1$. Then, we have $L_{w\lambda}(\mathbf{x}; \beta) = \mathfrak{G}_{w_0 \widetilde{w}^{-1}}(\mathbf{x}; \beta)$.*

The permutations $w_0 \widetilde{w}^{-1}$ appearing in Corollary 3.7 are vexillary (*i.e.*, 2143-avoiding). Since the greatest term of $L_{w\lambda}(\mathbf{x}; 0)$ in reverse lexicographic order is $\mathbf{x}^{w\lambda}$ and the greatest term of $\mathfrak{G}_{w_0 \widetilde{w}^{-1}}(\mathbf{x}; 0)$ in the same order is the Lehmer code of $w_0 \widetilde{w}^{-1}$, we see $w\lambda$ is the Lehmer code of $w_0 \widetilde{w}^{-1}$. Hence, $w_0 \widetilde{w}^{-1}$ are Grassmannian, and so the Grothendieck polynomials from Corollary 3.7 are actually *symmetric* Grothendieck polynomials, but symmetric only in some initial segment of the variables $\mathbf{x}$.

Example 3.8. For λ be a 2×2 rectangle, $L_{s_1 s_2 \lambda}(\mathbf{x}; \beta) = \mathfrak{G}_{w_0 (s_2 s_1) s_2 (s_4 s_3) s_4}(x_1, \ldots, x_5; \beta)$, and $L_{s_2 s_1 s_3 s_2 \lambda}(\mathbf{x}; \beta) = \mathfrak{G}_{w_0 (s_3 s_2 s_1)(s_3 s_2)(s_5 s_4 s_3)(s_5 s_4) s_5}(x_1, \ldots, x_6; \beta)$.

The Lascoux polynomials from Corollary 3.7 are are not the only ones equal to a Grothendieck polynomial; *e.g.*, $L_{s_2 \lambda}(\mathbf{x}; \beta) = \mathfrak{G}_{w_0 (s_2 s_4 s_3 s_4)}(x_1, \ldots, x_5; \beta)$ for $\lambda =$ . Yet, this is the only such coincidence when λ is a rectangle. T. Matsumura and S. Sugimoto have informed the authors [12, Theorem 3.3] can be extended to show every flagged Grothendieck polynomial is a Lascoux polynomial and will appear in their future work.

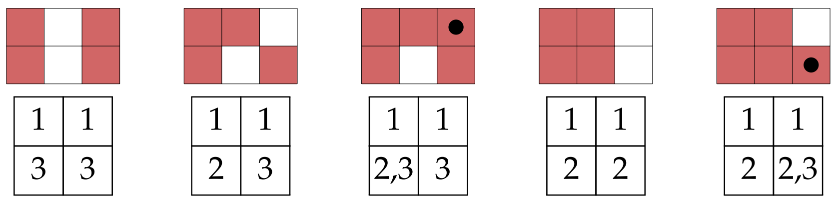
4 Bijection with K-Kohnert diagrams

Recall that there is a natural bijection between the set of semistandard Young tableaux of shape 1^r with entries at most n and the collection of subsets of $\{1, \ldots, n\}$ of size r. For row i (starting from the bottom row and going up) of a K-Kohnert diagram D, consider the subset of $\{1, \ldots, n\}$ given by the horizontal coordinates of the unmarked boxes. Construct column i (from right to left) of a tableau T by applying the natural bijection given above to this subset. Now, for every marked box in position (x, i) of D, there is a rightmost unmarked box (x', i) to the left of (x, i). Insert x into the cell of column i containing x'. In other words, we insert x into the topmost possible cell of column i such that the column is semistandard. Write $\phi(D)$ for the resulting tableau T.

It is straightforward to see that the map ϕ is invertible and β-weight preserving. We will show below that $\phi(D)$ is in fact always a semistandard set-valued tableau.

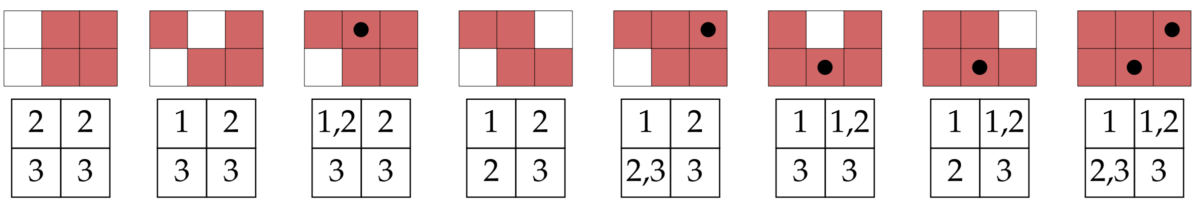
Proposition 4.1. *Let λ be an $r \times s$ rectangle. For any $w \in S_n^{\lambda}$, ϕ restricts to a β-weight preserving bijection $\phi \colon \mathcal{D}_{w\lambda} \to \mathsf{SV}_w^n(\lambda)$.*

Example 4.2. Consider λ be a 2×2 square and $w = s_2$. Under ϕ described above,



where we have shaded in the selected boxes and put a $\bullet$ in the marked boxes.

We continue to $w' = s_1 s_2$ to obtain all of $\mathsf{SV}_{w'}^3(\lambda) = \mathsf{SV}^3(\lambda)$ under ϕ:



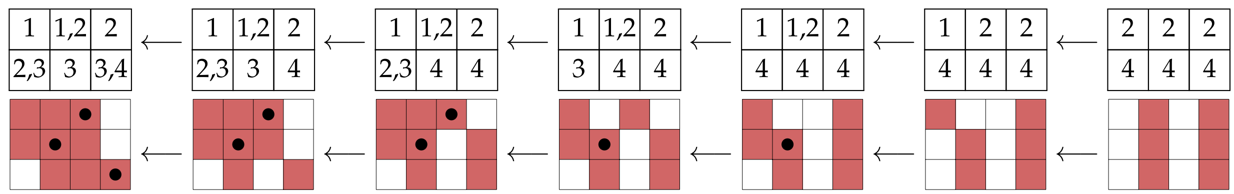
To prove Proposition 4.1, we construct (K-)Kohnert moves on set-valued tableaux.

Definition 4.3. Let $T \in \mathsf{SV}^n(\lambda)$. Consider $x \in \mathbb{Z}$ such that $x \in T$. Let $\mathcal{C}$ be the leftmost column of T containing an x in box b. Let x' be minimal such that $x'+1, x'+2, \dots, x \in \mathcal{C}$, and let b' be the box in $\mathcal{C}$ containing $x'+1$. If $\{x'+1\}, \dots, \{x-1\}$ are not in $\mathcal{C}$ (*i.e.*, the corresponding boxes only have 1 entry), $x' = 0$, or $x \neq \min \mathsf{b}$, then we do not have a (K-)Kohnert move. Otherwise, define the *Kohnert move* on T to remove x from b, moving all entries $x'+1, \dots, x-1$ down one row (which inserts $x-1$ into b), and inserting x' into b'. A *K-Kohnert move* is the same as before except we leave $x \in \mathsf{b}$.

Lemma 4.4. *Let $T \in \mathsf{SV}^n(\lambda)$. Applying any (K-)Kohnert move to T results in $T' \in \mathsf{SV}^n(\lambda)$.*

Now we prove Proposition 4.1 by using our flagging characterization of K-Demazure crystals from Lemma 3.4 and showing that ϕ intertwines the (K-)Kohnert moves on K-Kohnert diagrams with the (K-)Kohnert moves on set-valued tableaux.

Example 4.5. Let $\lambda = 3^2$ be a 2×3 rectangle and consider $n = 4$. We exhibit the sequence of (K-)Kohnert moves described in the proof of Proposition 4.1 to obtain the element $\begin{array}{|c|c|c|}\hline 1 & 1,2 & 2 \\ \hline 2,3 & 3 & 3,4 \\ \hline \end{array} \in \mathrm{SV}^4_{s_1 s_3 s_2}(\lambda)$ from the initial tableau $\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 4 & 4 & 4 \\ \hline \end{array}$ (below is the diagram under ϕ^{-1}):



Remark 4.6. The proof of the intertwining of (K-)Kohnert moves did not require λ to be a rectangle, but ϕ does require it as otherwise the image might not be a partition.

Theorem 4.7. *The Ross–Yong–Kirillov Conjecture (Conjeture 2.2) holds for L_a when a is any weak composition with a unique nonzero part size; i.e., $L_a(\mathbf{x}; \beta) = \sum_{D \in \mathcal{D}_a} \mathrm{wt}_\beta(D)$.*

5 Bijection with set-valued skyline tableaux

Conjeture 2.3 is equivalent to showing that

$$\overline{L}_{w\lambda} = \mathrm{ch}_\beta \left(\overline{\mathrm{SV}}^n_w(\lambda) \right), \quad \text{where } \overline{\mathrm{SV}}^n_w(\lambda) := \mathrm{SV}^n_w(\lambda) \setminus \bigcup_{v < w} \mathrm{SV}^n_v(\lambda), \tag{5.1}$$

with the union taken over all v strictly less that w in Bruhat order, by inclusion-exclusion, applying Möbius inversion on (strong) Bruhat order, and Equation (2.2)

Proposition 5.1. *Let λ be an $r \times s$ rectangle. For any $w \in S^\lambda_n$, there exists a β-weight preserving bijection $\psi\colon \mathrm{SLT}_{w\lambda} \to \overline{\mathrm{SV}}^n_w(\lambda)$.*

We prove Proposition 5.1 by explicitly defining the bijection as follows. Consider some $S \in \mathrm{SLT}_{w\lambda}$ and define $T := \psi(S)$ by (1) sorting the anchor entries in each row in increasing order left to right; (2) placing each free entry f in the leftmost box of its row such that f is less than the anchor entry; (3) constructing the i-th column of T from the $(r + 1 - i)$-th row of the result from the previous step, as in Section 4.

Example 5.2. Let λ be a 2×2 rectangle and $n = 3$. Then the set-valued skyline tableaux $\mathrm{SLT}_{s_2\lambda}$ and their corresponding elements in $\overline{\mathrm{SV}}^3_{s_2}(\lambda)$ under ψ are given by

$$\begin{array}{|c|c|c|}\hline 1 & \cdot & 3 \\ \hline 1 & \cdot & 3 \\ \hline \end{array} \longmapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & \cdot & 2 \\ \hline 1 & \cdot & 3 \\ \hline \end{array} \longmapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & \cdot & 2,3 \\ \hline 1 & \cdot & 3 \\ \hline \end{array} \longmapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2,3 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & \cdot & 2 \\ \hline 1 & \cdot & 2,3 \\ \hline \end{array} \longmapsto \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2,3 \\ \hline \end{array}.$$

Theorem 5.3. *Monical's Skyline Conjecture (Conjeture 2.3) holds for L_a when a is any weak composition with a unique nonzero part size; i.e., $\overline{L}_a = \sum_{S \in \mathrm{SLT}_a} \mathrm{wt}_\beta(S)$.*

6 K-key tableaux

A *key tableau* K is a semistandard tableau such that the entries in the j-th column of K are a subset of those in the $(j-1)$-st column of K. Every semistandard tableau T has a unique *(right) key tableau* $k(T)$ associated with it, and a Demazure atom can be computed as a generating function for all semistandard tableaux T with $k(T) = K_{w\lambda}$ [10]. Let $\prec$ denote the partial order on semistandard tableaux of shape λ such that $T \preceq T'$ if and only if every entry of T is at most the corresponding entry in T'. A Demazure character $\kappa_{w\lambda}$ can be given by summing over all semistandard Young tableaux T of shape λ such that $k(T) \preceq K_{w\lambda}$, where $K_{w\lambda}$ is the unique key tableau of shape λ and weight $w\lambda$ [10].

Based on the bijection from Proposition 5.1 and the (K-)Kohnert moves on set-valued tableaux (Definition 4.3), the following is a natural possible extension of key tableaux to the K-theory setting. For $T \in \mathrm{SV}^n(\lambda)$, define $\mathcal{K}(T) := k\big(\max(T)\big)$, where $\max(T)$ is semistandard tableau obtained by taking the greatest entry in each box of T. Thus Theorem 3.5 and Lemma 3.4 imply that for λ and $r \times s$ rectangle

$$L_{w\lambda}(\mathbf{x};\beta) = \sum_{\substack{T \in \mathrm{SV}^n(\lambda) \\ \mathcal{K}(T) \preceq K_{w\lambda}}} \mathrm{wt}_\beta(T), \qquad \overline{L}_{w\lambda}(\mathbf{x};\beta) = \sum_{\substack{T \in \mathrm{SV}^n(\lambda) \\ \mathcal{K}(T) = K_{w\lambda}}} \mathrm{wt}_\beta(T), \tag{6.1}$$

or equivalently summed over $\mathrm{SV}^n_w(\lambda)$ and $\overline{\mathrm{SV}}^n_w(\lambda)$ respectively. However, these formulas do not work for general λ as, for example, $\mathcal{K}\left(\begin{array}{|c|c|}\hline 1 & 1,2,3 \\ \hline 2,3 & \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}$, but it can only contribute to the Lascoux polynomial/atom corresponding to $w_0\lambda$, where $\lambda = 21$, as it has an excess of 3. Moreover, the weak K-crystal in [14, Figure 7] does *not* decompose the K-crystal into atoms, as $\begin{array}{|c|c|}\hline 1 & 2,3 \\ \hline 3 & \\ \cline{1-1}\end{array}$ should not be in the atom for to w_0.

Instead, we conjecture that (6.1) modifies as follows. Recall that the Lusztig involution on the highest weight crystal $B(\mu)$ is defined by sending the highest weight element U to the lowest weight element U^* and extending to $B(\mu)$ by

$$e_i(T^*) = (f_{n-i}T)^*, \qquad f_i(T^*) = (e_{n-i}T)^*, \qquad \mathrm{wt}(T^*) = w_0\,\mathrm{wt}(T). \tag{6.2}$$

We can extend this naively to $\mathrm{SV}^n(\lambda)$ by acting on each irreducible component $B(\mu)$. Define the *(right) K-key tableau* of a set-valued tableau $T \in \mathrm{SV}^n(\lambda)$ by

$$K(T) := k(\min(T^*)^*),$$

where $\min(T)$ is obtained from T by taking the least entry in each box of T.

Conjecture 6.1. *Let λ be a partition. Define the sets*

$$\mathrm{SV}^n_w(\lambda) := \{T \in \mathrm{SV}^n(\lambda) \mid K(T) \preceq K_{w\lambda}\}, \qquad \overline{\mathrm{SV}}^n_w(\lambda) := \{T \in \mathrm{SV}^n(\lambda) \mid K(T) = K_{w\lambda}\}.$$

Then we have $L_{w\lambda}(\mathbf{x};\beta) = \sum_{T\in\mathrm{SV}^n_w(\lambda)} \mathrm{wt}_\beta(T)$, and $\overline{L}_{w\lambda}(\mathbf{x};\beta) = \sum_{T\in\overline{\mathrm{SV}}^n_w(\lambda)} \mathrm{wt}_\beta(T)$.

We show (6.1) establishes Conjeture 6.1 when λ is a rectangle by constructing a *K-Lusztig involution* $\star\colon \mathrm{SV}^n(\lambda) \to \mathrm{SV}^n(\lambda)$ that also satisfies (6.2). However, it is a twist of the Lusztig involution by permuting the connected $U_q(\mathfrak{sl}_n)$-components of $\mathrm{SV}^n(\lambda)$. Let λ be a rectangle and $T \in \mathrm{SV}^n(\lambda)$. Define $T^\star$ to be the set-valued tableau obtained by rotating the tableau 180° and then replacing each $i \mapsto n+1-i$. We note this is a well-known description of the Lusztig involution on semistandard tableaux of shape λ.

Proposition 6.2. *Let λ be a rectangle. The K-Lusztig involution $\star$ satisfies (6.2). For $T \in \mathrm{SV}^n(\lambda)$ as a tensor product of rows $T = R_1 \otimes \cdots \otimes R_k$, we have $T^\star = R_k^* \otimes \cdots \otimes R_1^*$.*

This also suggests that Conjeture 6.1 holds for a definition of a (right) K-key tableau $K'(T) := k(\min(T^\dagger)^*)$, where $T^\dagger$ is constructed from T according to *any* automorphism of $\mathrm{SV}^n(\lambda)$ such that $\mathrm{wt}(T^\dagger) = w_0\,\mathrm{wt}(T)$. However, given a (weak) K-crystal structure on $\mathrm{SV}^n(\lambda)$, it would be preferable to have a $T^\dagger$ construction that matches the labeling of tableaux T by K-keys $K'(T)$ with the decomposition of the K-crystal by K-Demazure subcrystals, as is the case with our K-Lusztig involution $T^\star$. Furthermore, it is likely that in general we want $T^\star = R_k^* \otimes \cdots \otimes R_1^*$ as in Proposition 6.2, but this would require an appropriate K-rectification or insertion scheme in order to obtain a result back in $\mathrm{SV}^n(\lambda)$.

Acknowledgements

OP is grateful for interesting conversations with B. Proctor. TS thanks T. Ikeda, T. Matsumura, and S. Sugimoto for stimulating discussions. The authors thank C. Monical for useful discussions. This work benefited from computations using SageMath [18].

References

[1] A. Buch. "A Littlewood-Richardson rule for the *K*-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78. Link.

[2] D. Bump and A. Schilling. *Crystal bases*. Representations and combinatorics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017, pp. xii+279. Link.

[3] S. Fomin and A. Kirillov. "Grothendieck polynomials and the Yang-Baxter equation". *Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique*. DIMACS, Piscataway, NJ, 1994, pp. 183–189. Link.

[4] M. Kashiwara. "The crystal base and Littelmann's refined Demazure character formula". *Duke Math. J.* **71**.3 (1993), pp. 839–858. Link.

[5] A. Kirillov. "Notes on Schubert, Grothendieck and key polynomials". *SIGMA Symmetry Integrability Geom. Methods Appl.* **12** (2016), Paper No. 034, 1–56. Link.

[6] A. Lascoux. "Anneau de Grothendieck de la variété de drapeaux". *The Grothendieck Festschrift, Vol. III*. Vol. 88. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 1–34. Link.

[7] A. Lascoux. "Transition on Grothendieck polynomials". *Physics and combinatorics, 2000 (Nagoya)*. World Sci. Publ., River Edge, NJ, 2001, pp. 164–179. Link.

[8] A. Lascoux and M. Schützenberger. "Structure de Hopf de l'Anneau de cohomologie et de l'Anneau de Grothendieck d'une variété de drapeaux". *C. R. Acad. Sci. Paris Sér. I Math.* **295**.11 (1982), pp. 629–633.

[9] A. Lascoux and M. Schützenberger. "Symmetry and flag manifolds". *Invariant theory (Montecatini, 1982)*. Vol. 996. Lecture Notes in Math. Springer, Berlin, 1983, pp. 118–144. Link.

[10] A. Lascoux and M. Schützenberger. "Keys & standard bases". *Invariant theory and tableaux (Minneapolis, MN, 1988)*. Vol. 19. IMA Vol. Math. Appl. Springer, New York, 1990, pp. 125–144.

[11] C. Lenart and K. Zainoulline. "Towards generalized cohomology Schubert calculus via formal root polynomials". *Math. Res. Lett.* **24**.3 (2017), pp. 839–877. Link.

[12] T. Matsumura. "Flagged Grothendieck polynomials". *J. Algebraic Combin.* (2018). Link.

[13] C. Monical. "Set-valued skyline fillings". 2016. arXiv:1611.08777.

[14] C. Monical, O. Pechenik, and T. Scrimshaw. "Crystal structures for symmetric Grothendieck polynomials". 2018. arXiv:1807.03294.

[15] C. Monical, O. Pechenik, and D. Searles. "Polynomials from combinatorial K-theory" (2020). To appear in *Canad. J. Math.* Link.

[16] O. Pechenik and T. Scrimshaw. "K-theoretic crystals for set-valued tableaux of rectangular shape". 2019. arXiv:1904.09674.

[17] C. Ross and A. Yong. "Combinatorial rules for three bases of polynomials". *Sém. Lothar. Combin.* **74** (2015), Art. B74a, 11 pages.

[18] *Sage Mathematics Software (Version 8.6)*. http://www.sagemath.org. The Sage Developers. 2019.