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Higher order Turán inequalities for *k*-regular partitions

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Abstract. Nicolas (in 1978) and DeSalvo and Pak (in 2015) proved that the partition function $p(n)$ is log concave for $n \geq 25$. In 2019, Chen, Jia and Wang proved that $p(n)$ satisfies the third order Turán inequality, and that the associated degree 3 Jensen polynomials are hyperbolic for $n \geq 94$. More recently, Griffin, Ono, Rolen and Zagier proved more generally that for all d, the degree d Jensen polynomials associated to $p(n)$ are hyperbolic for sufficiently large n. In this paper, we prove that the same result holds for the k-regular partition function $p_k(n)$ for $k \geq 2$. In particular, for any positive integers d and k, the order d Turán inequalities hold for $p_k(n)$ for sufficiently large n. The case when $d = k = 2$ proves a conjecture by Neil Sloane that $p_2(n)$ is log concave.

Keywords: Higher order Turán inequalities, k-regular partitions, Jensen polynomials, Hermite polynomials

1 Introduction and Statement of results

The Turán inequality (or sometimes called the Newton inequality) arises in the study of real entire functions in the Laguerre-Pólya class which is closely related to the study of the Riemann hypothesis [4], [12]. It is well known that the Riemann hypothesis is true if and only if the Riemann Xi function is in the Laguerre-Pólya class, where the Riemann Xi function is the entire order 1 function defined by

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

A necessary condition for the Riemann Xi function being in the Laguerre-Pólya class is that the Maclaurin coefficients of the Xi function satisfy both the Turán and higher order Turán inequalities [11, 4].

A sequence $\{a_m\}_{m=0}^{\infty}$ is *log concave* if it satisfies the (second order) Turán inequality $a_m^2 \geq a_{m-1}a_{m+1}$ for all $m \geq 1$. The sequence $\{a_m\}_{m=0}^{\infty}$ satisfies the third order Turán inequality if for $m \geq 1$, we have

$$4(a_m^2 - a_{m-1}a_{m+1})(a_{m+1}^2 - a_m a_{m+2}) \geq (a_m a_{m+1} - a_{m-1}a_{m+2})^2.$$

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Nicolas [8] and DeSalvo and Pak [3] proved that the partition function $p(n)$ is log concave for $n \geq 25$, where $p(n)$ is the number of partitions of n. Recall that a partition of a positive integer n with r parts is a weakly decreasing sequence of r positive integers that sums to n. We set $p(0) = 1$. Chen [1] conjectured that $p(n)$ satisfies the third order Turán inequality for $n \geq 95$ which is proved by Chen, Jia and Wang [2]. Their result also shows that the cubic polynomial

$$\sum_{k=0}^{3} \binom{3}{k} p(n+k)x^k$$

has only real simple roots for $n \geq 95$. They also conjectured that for $d \geq 4$, there exists a positive integer $N(d)$ such that the Jensen polynomials $J_p^{d,n}(X)$ for $p(n)$ as defined in (1.1) have only real roots for all $n \geq N(d)$. Note that the case for $d = 1$ is trivial with $N(1) = 1$, and the log concavity of $p(n)$ for $n \geq 25$ proves the case for $d = 2$ with $N(2) = 25$. Chen, Jia and Wang [2] proved the case for $d = 3$ with $N(3) = 94$. Larson and Wagner [7] proved the minimum value of $N(d)$ for $d \leq 5$ and gave an upper bound for all other $N(d)$. The conjecture was proven for all $d \geq 1$ in a recent paper by Griffin, Ono, Rolen and Zagier [5] where they proved the hyperbolicity of the Jensen polynomials associated to a large family of sequences. Given an arbitrary sequence $\alpha = (\alpha(0), \alpha(1), \alpha(2), \cdots)$ of real numbers, the associated *Jensen polynomial* $J_\alpha^{d,n}(X)$ *of degree* d *and shift* n is defined by

$$J_\alpha^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j) X^j. \tag{1.1}$$

The Jensen polynomials also have a close relationship to the Riemann Hypothesis. Indeed, Pólya [10] proved that the Riemann Hypothesis is equivalent to the hyperbolicity of all of the Jensen polynomials $J_\gamma^{d,n}(X)$ associated to the Taylor coefficients $\{\gamma(j)\}_{j=0}^{\infty}$ of $\frac{1}{8}\Xi\left(\frac{i\sqrt{x}}{2}\right)$. Griffin, Ono, Rolen and Zagier [5] proved that for each $d \geq 1$, all but finitely many $J_\gamma^{d,n}(X)$ are hyperbolic, which provides new evidence supporting the Riemann Hypothesis.

There is a classical result by Hermite that generalizes the Turán inequalities using Jensen polynomials. Let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with real coefficients. Let $\beta_1, \beta_2, \cdots, \beta_n$ be the roots of f and denote $S_0 = n$ and

$$S_m = \beta_1^m + \beta_2^m + \cdots + \beta_n^m, \qquad m = 1, 2, 3, \cdots$$

their Newton sums. Let $M(f)$ be the Hankel matrix of $S_0, \cdots S_{2n-2}$, i.e.

$$M(f) := \begin{pmatrix} S_0 & S_1 & S_2 & \cdots & S_{n-1} \\ S_1 & S_2 & S_3 & \cdots & S_n \\ S_2 & S_3 & S_4 & \cdots & S_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{n-1} & S_n & S_{n+1} & \cdots & S_{2n-2} \end{pmatrix}.$$

Hermite's theorem [9] states that f is hyperbolic if and only if $M(f)$ is positive semi-definite. Recall that a polynomial with real coefficients is called hyperbolic if all of its roots are real. It is well known that S_m can be expressed in terms of the coefficients $a_0, \cdots, a_{n-1}$ of f for $m \geq 1$, and a matrix is positive semi-definite if and only if all its principle minors are non-negative. Thus Hermite's theorem provides a set of inequality conditions on the coefficients of a hyperbolic polynomial f:

$$\Delta_1 = S_0 = n, \Delta_2 = \begin{vmatrix} S_0 & S_1 \\ S_1 & S_2 \end{vmatrix} \geq 0, \cdots, \Delta_n = \begin{vmatrix} S_0 & S_1 & S_2 & \cdots & S_{n-1} \\ S_1 & S_2 & S_3 & \cdots & S_n \\ S_2 & S_3 & S_4 & \cdots & S_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{n-1} & S_n & S_{n+1} & \cdots & S_{2n-2} \end{vmatrix} \geq 0.$$

For a given sequence $\alpha(n)$, when Hermite's theorem is applied to $J_\alpha^{d,n}(X)$ then the condition that all minors Δ_k of the Hankel matrix $M(J_\alpha^{d,n}(X))$ are non-negative gives a set of inequalities on the sequence $\alpha(n)$, and we call them the *order k Turán inequalities*. In other words, $J_\alpha^{d,n}(X)$ is hyperbolic if and only if the subsequence $\{\alpha(n+j)\}_{j=0}^{\infty}$ satisfies all the order k Turán inequalities for all $1 \leq k \leq d$. In particular, the result in [5] shows that for any $d \geq 1$, the partition function $\{p(n)\}$ satisfies the order d Turán inequality for sufficiently large n.

For a positive integer $k \geq 2$, let the *k-regular partition function* $p_k(n)$ be defined as the number of partitions of n in which none of the parts are multiples of k. For any fixed integer $k \geq 2$, it is well known that the generating function for the sequence $\{p_k(n)\}_{n=0}^{\infty}$ is given by

$$\sum_{n \geq 0} p_k(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{kn})}{(1-q^n)}.$$

Sloane conjectured that the sequence of partitions of n with distinct parts, is log concave for sufficiently large n. It is well known that the number of k-regular partitions is equal to the number of partition with parts appearing at most $k-1$ times. In particular, 2-regular partitions are equinumerous with partitions in which all parts are distinct is also famously known as Euler's Odd-Distinct theorem. Therefore, Sloane's conjecture is equivalent to the log-concavity of $p_2(n)$ for sufficiently large n. In this paper, we prove Sloane's conjecture, and in fact we prove the more general result that for any d, the k-regular partition functions $p_k(n)$ satisfy the order d Turán inequality for sufficiently large n.

As has been done for other sequences, we can define the Jensen polynomials $J_{p_k}^{d,n}(X)$ of degree d and shift n for the sequence $p_k(n)$ by setting $\alpha = p_k$ in (1.1). In the spirit of much other work on the Turán inequalities, we may reframe the conjecture in terms of the hyperbolicity of the associated Jensen polynomials:

Conjecture. *Let $d \geq 1, k \geq 2$ be integers. There exists a positive integer $N(d)$ such that the degree d Jensen polynomial $J_{p_k}^{d,n}(X)$ associated to $p_k(n)$ are hyperbolic for all $n \geq N(d)$.*

For a natural number d, we define the Hermite polynomials $H_d(X)$ by the generating function

$$e^{-t^2+Xt} = \sum_{d=0}^{\infty} H_d(X)\frac{t^d}{d!} = 1 + Xt + (X^2-2)\frac{t^2}{2} + (X^3-6X)\frac{t^3}{6} + \cdots,$$

orthogonal with respect to the given measure, which is a slightly different normalization than is standard (see Remark 1.3). In view of these definitions, we can now state our main results.

Theorem 1.1. *If $k \geq 2$ and $d \in \mathbb{Z}^+$, then*

$$\lim_{n\to\infty} \widehat{J}_{p_k}^{d,n}(X) = H_d(X),$$

uniformly for X in compact subsets of $\mathbb{R}$, where $\widehat{J}_{p_k}^{d,n}(X)$ are renormalized Jensen polynomials for $p_k(n)$ as defined in (2.2).

Corollary 1.2. *If $k \geq 2$ and $d \in \mathbb{Z}^+$, then $J_{p_k}^{d,n}(X)$ is hyperbolic for sufficiently large n.*

Remark 1.3. There are two widely used definitions of Hermite polynomials (the so-called "physicists" and "probabilists" Hermite polynomials):

The "physicists' Hermite polynomials" are defined as

$$He_n(X) := (-1)^n e^{\frac{X^2}{2}} \frac{d^n}{dX^n} e^{-\frac{X^2}{2}}.$$

The "probabilists' Hermite polynomials" are defined as

$$H_n^{pr}(X) := (-1)^n e^{x^2} \frac{d^n}{dX^n} e^{-x^2}.$$

The relation between these definitions and ours is as follows:

$$H_n(2X) = H_n^{pr}(X) = 2^{\frac{n}{2}} He_n(\sqrt{X}).$$

2 Sketch of Proof of Theorem 1.1

2.1 Polya-Jensen method

In 1927, Pólya [10] demonstrated that the Riemann hypothesis is equivalent to the hyperbolicity of the Jensen polynomials for $\zeta(s)$ at its point of symmetry. This approach to the Riemann hypothesis had not received much significant work until the recent work of Griffin, Ono, Rolen and Zagier [5], in which the hyperbolicity of these polynomials is proved for all $d \geq 1$ for sufficiently large n. The primary method utilized by this paper can be stated as follows:

Theorem 2.1 (Theorem 3 & Corollary 4, [5])**.** *Let $\{\alpha(n)\}$, $\{A(n)\}$, and $\{\delta(n)\}$ be sequences of positive real numbers such that $\delta(n) \to 0$ as $n \to \infty$. Suppose further that for a fixed $d \geq 1$ and for all $0 \leq j \leq d$, we have*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + o\big(\delta(n)^d\big) \qquad \text{as } n \to \infty.$$

Then the renormalized Jensen polynomials

$$\widehat{J}_{\alpha}^{d,n}(X) = \frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X - 1}{\exp(A(n))}\right)$$

satisfy $\lim_{n\to\infty} \widehat{J}_{\alpha}^{d,n}(X) = H_d(X)$ uniformly for X in any compact subset of $\mathbb{R}$. Furthermore, this implies that the polynomials $J_{\alpha}^{d,n}(X)$ are hyperbolic for all but finitely many values of n.

Because the conditions for this result are so general, the method can be utilized in a wide variety of circumstances. For instance, it is shown in Theorem 7 of [5] that if $a_f(n)$ are the (real) Fourier coefficients of a modular form f on $\mathrm{SL}_2(\mathbb{Z})$ holomorphic apart from a pole at infinity, then there are sequences $A_f(n)$ and $\delta_f(n)$ such that $\alpha(n) = a_f(n)$ satisfies the required conditions. What we prove can then be regarded as a first instance of the Polya-Jensen method for modular forms on congruence subgroups, since the sequences $p_k(n)$ are coefficients of weight zero weakly holomorphic modular forms on proper subgroups of $SL_2(\mathbb{Z})$.

2.2 Explicit Formula of $p_k(n)$

Hagis [6] proved an explicit formula for $p_k(n)$ similar to Rademacher's explicit formula for $p(n)$, and this explicit formula as well as the asymptotic behavior of this formula will play an important role in our proof. In order to state Hagis' results, we must first make preliminary definitions, most important the quantities $A(m,t,n,s,D)$ and $L(m,t,n,s,D)$, which are Kloosterman-type sums and multiples of Bessel functions respectively.

Let D divide $t+1$, $J = J(t,D) := \frac{\frac{t}{D} - D}{24D}$, and $a = a(t) = \frac{t}{24}$, and denote by I_1 the order one modified Bessel function. Then the quantity $L(m,t,n,s,D)$ is defined by

$$L(m,t,n,s,D) := \frac{D^{3/2}}{m} \sqrt{\frac{J-s}{n+a}} \cdot I_1\left(\frac{4\pi D}{m} \sqrt{\frac{(J-s)(n+a)}{t+1}} \right).$$

The quantity $A(m,t,n,s,D)$ is a modified Kloosterman sum, and requires additional definitions. We denote by $g = g(m)$ the value $\gcd(3,m)$ if m is odd and $8\gcd(3,m)$ when m is even. We define the ratios $M = M(m,D) := \frac{m}{D}$ and $f = f(m) := \frac{24}{g}$. Define $r = r(m)$ to be any integer satisfying $fr \equiv 1 \pmod{gm}$. In a manner analogous to g, we define $G = G(m,D) := \gcd(3,M)$ for odd M and $8\gcd(3,M)$ for M even. Set $B = B(m,D) := \frac{g}{G}$ and define A as any integer satisfying $AB \equiv 1 \pmod{GM}$. We also define $T = T(t,D) := \frac{t+1}{D}$, choosing $T' = T'(t,D)$ satisfying $TT' \equiv 1 \pmod{GM}$. The quantities $U = U(t,m,D)$ and $V = V(t,m,D)$ are defined by $V := ABT'D - 1$ and $U := 1-(t+1)AB$. Hagis also defines particular roots of unity, $w(h,t,m,D)$, which satisfy

$$w(h,t,m,D) = C(h,t,m,D)\exp(2\pi i(rUh + rVh')/gm),$$

where the $C(h,t,m,D)$ satisfy $|C(h,t,m,D)| = 1$ and are defined independently of h if m is odd or if m is even and $h \equiv d \pmod 8$ for some odd d. Then the quantity $A(m,t,n,s,D)$ to be the Kloosterman sum with multiplier system given by the formula

$$A(m,t,n,s,D) := \sum_{\substack{h \pmod m \\ \gcd(h,m)=1}} w(h,t,m,D)\exp(-2\pi i(nh - DT'sh')/m),$$

where $hh' \equiv 1 \pmod{gm}$. Define also the value $P(s)$ as in Euler's pentagonal number theory to have $P(s) = (-1)^j$ if s is a pentagonal number, that is, s is in the form $\frac{3j^2 \pm j}{2}$, and 0 otherwise. Using the notation so far developed, Hagis proved [6, Theorem 3] that

$$p_k(n) = \frac{2\pi}{k} \sum_{\substack{D|k \\ D<\sqrt{k}}} \sum_{\substack{m \\ \gcd(k,m)=D}}^{\infty} \sum_{s<J(k,D)} P(s)A(m,k-1,n,s,D)L(m,k-1,n,s,D).$$

Hagis found this formula by the Hardy-Ramanujan circle method, though the same formula may also be derived by computing a generating function for $p_k(n)$ by quotients of the Dedekind eta function and expanding this by Poincare series. Using this explicit formula, Hagis also proved [6, Corollary 4.1] that as $n \to \infty$ we have

$$p_k(n) = 2\pi\sqrt{\frac{m_k}{k(n+km_k)}} \cdot I_1\left(4\pi\sqrt{m_k(n+km_k)}\right)(1+O(-cn^{1/2})), \tag{2.1}$$

where I_1 is a modified Bessel function, $c = c(k)$ is a constant, and $m_k = (k-1)/24k$.

Fix $d \geq 1$ and $k \geq 2$, and let the sequences $A_k(n)$, $\delta_k(n)$ be defined by

$$A_k(n) = 2\pi\sqrt{m_k/n} + \frac{3}{4}\sum_{r=1}^{\lfloor 3d/4 \rfloor} \frac{(-1)^r}{rn^r} \quad \text{and} \quad \delta_k(n) = \left(-\sum_{r=2}^{\infty} \frac{4\pi\sqrt{m_k}\binom{1/2}{r}}{n^{r-1/2}}\right)^{1/2}.$$

Define the renormalized Jensen polynomials $\widehat{J}_{p_k}^{d,n}(X)$ by

$$\widehat{J}_{p_k}^{d,n}(X) := \frac{\delta_k(n)^{-d}}{p_k(n)} J_{p_k}^{d,n}\left(\frac{\delta_k(n)X - 1}{\exp(A_k(n))}\right). \tag{2.2}$$

By application of the Pólya-Jensen method, in order to prove Theorem 1.1, it suffices to show that for any fixed d and all $0 \leq j \leq d$,

$$\log\left(\frac{p_k(n+j)}{p_k(n)}\right) = A_k(n)j - \delta_k(n)^2 j^2 + o\left(\delta_k(n)^d\right) \qquad \text{as } n \to \infty \tag{2.3}$$

which follows by using (2.1).

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