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Counting directed acyclic and elementary digraphs

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Abstract. Directed acyclic graphs (DAGs) can be characterized as directed graphs whose strongly connected components are isolated vertices. Using this restriction on the strongly connected components, we discover that when $m = cn$, where m is the number of directed edges, n is the number of vertices, and $c < 1$, the asymptotic probability that a random digraph is acyclic is an explicit function $p(c) = e^{-c}(1-c)$. When $m = n(1 + \mu n^{-1/3})$, the asymptotic behaviour changes, and the probability that a digraph is acyclic becomes $n^{-1/3}C(\mu)$, where $C(\mu)$ is an explicit function of μ. In 2009, Łuczak and Seierstad showed that, as $\mu \to -\infty$, the strongly connected components of a random digraph with n vertices and $m = n(1 + \mu n^{-1/3})$ directed edges are, with high probability, only isolated vertices and cycles. We call such digraphs elementary digraphs. We express the probability that a random digraph is elementary as a function of μ. Those results are obtained using techniques from analytic combinatorics, developed in particular to study random graphs.

1 Introduction

Directed Acyclic Graphs (DAGs) appear naturally in the study of compacted trees, automaton for finite languages and partial orders. Until now, their asymptotics was known only for n vertices and $m = \Theta(n^2)$ edges (dense case). In this paper, we give a solution to the sparse case $m = n(1 + \mu n^{-1/3})$ with μ bounded or going to $-\infty$ (Theorem 3.3). The first case exhibits a phase transition reminiscent of directed graphs (see [12]). In the second case, when $0 < \lim \frac{m}{n} < 1$, we have $\mathbb{P}(\text{digraph is acyclic}) = e^{-m/n}(1 - \frac{m}{n})$.

ew exact enumeration results for

Exact and asymptotic enumeration. In 1973, Robinson [18] obtained his beautiful formula for the number $\mathcal{DAG}_{n,m}$ of labeled DAGs with n vertices and m edges

$$\mathcal{DAG}_{n,m} = n![z^n w^m] \frac{(1+w)^{\binom{n}{2}}}{\sum_{n \geqslant 0} (1+w)^{-\binom{n}{2}} \frac{(-z)^n}{n!}},$$

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and developed a framework for the enumeration of digraphs whose strongly connected components (called *strong components* in the following) belong to a given family of allowed strongly connected digraphs. This yielded the asymptotics of dense DAGs in [2]. The structure of random DAGs has been studied in [10, 13, 6].

We say that a digraph is *elementary* if all its strong components are either isolated vertices or cycles. In [11] and [12] it was shown that if the parameter $c = \frac{m}{n}$ is less than one, then a digraph is elementary asymptotically almost surely. More precisely, this happens when a digraph has n vertices and $m = n(1 + \mu n^{-1/3})$ edges, as $\mu \to -\infty$ with n. Other interesting structural results around the phase transition point are available in [16, 7]. The authors of [7] show that the strong components have asymptotically almost surely cubic kernels, *i.e.* the sum of the degrees of each of its nodes is at most three with high probability. This means that these cubic kernels play an analogous role as the classical cubic kernels in a random graphs, see [9].

A forthcoming independent approach of [17] in the analysis of asymptotics of DAGs (manuscript to appear), is similar in spirit to the tools used in [4] and relies on a bivariate singularity analysis of the generating function of DAGs. Their technique promises to unveil sparse DAGs asymptotics, covering as well the case where the ratio of the numbers of edges and vertices is bounded, but greater than 1 (the *supercritical case*).

Our contribution. Typically, the analysis of graphs is technically easier when loops and multiple edges are allowed, [9]: an adaptation of the symbolic techniques to the case of simple graphs becomes rather a technical, but not a conceptual obstacle. In [14] and [3] the dedicated *patchwork* concept is introduced allowing to handle this difficulty. The same principle concerns directed graphs. Nevertheless, in the current paper we consider the case of *simple digraphs* where loops and multiple edges are forbidden. In our model, however, the cycles of size 2 are allowed, because it is natural to suppose that for each two vertices i and j both directions are allowed. The analysis of simple digraphs is technically heavier than the analysis of multidigraphs, but we prefer to demonstrate explicitly that such an application is indeed possible.

Firstly, we transform the generating function of DAGs so that it can be decomposed into an infinite sum. Each of its summands is analyzed using a new bivariate semi-large powers lemma which is a generalization of [1]. We discover (in the above notations) that the first term of this infinite expansion is dominating in the *subcritical* case, *i.e.* when $\mu \to -\infty$; in the case when μ is bounded (the *critical* case), all the terms give contributions of the same order. Next, using the symbolic tools for directed graphs from [15], we express the generating function of elementary digraphs and apply similar tools to obtain explicitly the phase transition curve in digraphs, that is, the probability that a digraph is elementary, as a function of μ.

Related studies. Analytic techniques, largely covered in [5], are efficient for asymptotic analysis, because the coefficient extraction operation is naturally expressed through Cauchy formula. A recent study [8] is dealing with bivariate algebraic functions. In their case, a combination of two Hankel contours, necessary for careful analysis, can have a complicated mutual configuration in two-dimensional complex space, so a lot of details needs to be accounted for. Our approach is close to theirs, while we try to avoid the mentioned difficulty in our study. The principle idea behind our bivariate semi-large powers lemma is splitting a double complex integral into a product of two univariate ones.

Structure of the paper. In Section 2 we present new exact reformulations of the numbers of DAGs and elementary digraphs, which are later used in Section 3 to ease the asymptotic analysis.

2 Exact expressions using generating functions

Consider the following model of graphs and directed graphs. A graph G is characterized by its set $V(G)$ of labeled vertices and its set $E(G)$ of unoriented unlabeled edges. Loops and multiple edges are forbidden. The numbers of its vertices and edges are denoted by $n(G)$ and $m(G)$. An (n, m)-graph (or digraph) is a graph (or digraph) with n vertices and m edges.

We consider digraph without loops, such that from any vertex i to any vertex j there can be at most one directed edge. Therefore, two edges can link the same pair of vertices only if their orientations are different.

2.1 Exponential and graphic generating functions

Two helpful tools in the study of graphs and directed graphs are the *exponential generating function* (EGF) and *graphic generating function* (GGF). The EGF $\mathrm{F}(z, w)$ and the GGF $\mathbf{F}(z, w)$ associated to a graph or digraph family $\mathcal{F}$ are defined as

$$\mathrm{F}(z,w) = \sum_{G \in \mathcal{F}} w^{m(G)} \frac{z^{n(G)}}{n(G)!}, \quad \mathbf{F}(z,w) = \sum_{G \in \mathcal{F}} w^{m(G)} \frac{z^{n(G)}}{n(G)!(1+w)^{\binom{m(G)}{2}}}.$$

The total numbers of (n, m)-graphs and (n, m)-digraphs are $\binom{n(n-1)/2}{m}$ and $\binom{n(n-1)}{m}$. The classical counting expression for directed acyclic graphs is attributed to Robinson [18]. The EGF $\mathrm{G}(z, w)$ of all graphs and GGF of directed acyclic graphs $\mathbf{DAG}(z, w)$ are given by

$$\mathrm{G}(z,w) = \sum_{n \geqslant 0} (1+w)^{\binom{n}{2}} \frac{z^n}{n!}, \quad \mathbf{DAG}(z,w) = \frac{1}{\sum_{n \geqslant 0} (1+w)^{-\binom{n}{2}} \frac{(-z)^n}{n!}}. \tag{2.1}$$

We can reuse the EGF of graphs (2.1) to obtain an alternative expression for the number of (n, m)-DAGs $\mathcal{DAG}_{n,m}$:

$$\mathcal{DAG}_{n,m} = n![z^n w^m] \frac{(1+w)^{\binom{n}{2}}}{\mathrm{G}\left(-z, -\frac{w}{1+w}\right)}. \tag{2.2}$$

Before considering various digraph families, we need to recall the classical generating functions of simple graph families, namely the rooted and unrooted labeled trees and unicycles. A *unicycle* is a connected graph that has the same numbers of vertices and edges. Hence, it contains exactly one cycle.

Proposition 2.1 ([9]). *The EGFs $T(z)$ of rooted trees, $U(z)$ of trees and $V(z)$ of unicycles are characterized by the relations*

$$T(z) = ze^{T(z)}, \quad U(z) = T(z) - \frac{T(z)^2}{2}, \quad V(z) = \frac{1}{2}\log\left(\frac{1}{1-T(z)}\right) - \frac{T(z)}{2} - \frac{T(z)^2}{4}.$$

The *excess* of a graph (not necessarily connected) is defined as the difference between its numbers of edges and vertices. For example, trees have excess -1, while unicycles have excess 0. The bivariate EGFs of graphs of excess k can be obtained from their univariate EGFs by substituting $z \mapsto zw$ and multiplying by w^k. In particular, $T(z,w) = T(zw)/w$, $U(z,w) = U(zw)/w$, $V(z,w) = V(zw)$.

We say that a graph is *complex* if all its connected components have a positive excess. The EGF of complex graphs of excess k is

$$\mathrm{Complex}_k(z) = [y^k]\,\mathrm{Complex}(z/y, y).$$

It is known (see [9]) that a complex graph of excess r is reducible to a *kernel* (multigraph of minimal degree at least 3) of same excess, by recursively removing vertices of degree 0 and 1 and fusioning edges sharing a degree 2 vertex. The total weight of *cubic* kernels (all degrees equal to 3) of excess r is given by (2.3). They are central in the study of large critical graphs, because non-cubic kernels do not typically occur.

Proposition 2.2 ([9, Section 6]). *For each $r \geqslant 0$ there is a polynomial $P_r(T)$ such that*

$$\mathrm{Complex}_r(z) = e_r \frac{T(z)^{5r}}{(1-T(z))^{3r}} + \frac{P_r(T(z))}{(1-T(z))^{3r-1}}, \quad \textit{where} \quad e_r = \frac{(6r)!}{2^{5r}3^{2r}(2r)!(3r)!}. \tag{2.3}$$

Since any graph can be represented as a set of unrooted trees, unicycles and a complex component (whose excess is denoted by k below) the EGF of graphs is equal to

$$\mathrm{G}(z,w) = e^{U(zw)/w} e^{V(zw)} \sum_{k \geqslant 0} \mathrm{Complex}_k(zw) w^k. \tag{2.4}$$

2.2 Exact expression for directed acyclic graphs

In order to obtain the asymptotic number of DAGs, we need a decomposition different from (2.1). For comparison, in the expression (2.4) the first summand is asymptotically dominating in the case of subcritical graphs. Inside the critical window, all the summands of (2.4) give a contribution of the same asymptotic order.

Lemma 2.3. *The number $\mathcal{DAG}_{n,m}$ of (n,m)-DAGs is equal to*

$$n!^2 \sum_{t \geqslant 0} [z_0^n z_1^n] \frac{(U(z_0)+U(z_1))^{2n-m+t}}{(2n-m+t)!} \frac{e^{U(z_1)+V(z_0)}}{e^{V(z_1)}} [y^t] \frac{\sum_{j\geqslant 0} \operatorname{Complex}_j(z_0) y^j}{\sum_{k \geqslant 0} \operatorname{Complex}_k(z_1) \left(-\frac{y}{1+y}\right)^k} \frac{1}{(1+y)^n}.$$

Proof. Since $(1+w)^{\binom{n}{2}}$ is the generating function of graphs with n vertices, we can replace $(1+w)^{\binom{n}{2}}$ with $n![z^n]\,\mathrm{G}(z,w)$ in (2.2). Injecting the expression of $\mathrm{G}(z,w)$ from (2.4) in the resulting formula with $z \mapsto -z_1$ and $w \mapsto -\frac{w}{1+w}$, we obtain (see also Remark 2.4 for more intuitions)

$$\mathcal{DAG}_{n,m} = n!^2 [z_0^n z_1^n w^m] \frac{e^{U(z_0 w)/w + V(z_0 w)} \sum_{j \geqslant 0} \operatorname{Complex}_j(z_0 w) w^j}{e^{-U(\frac{z_1 w}{1+w})\frac{1+w}{w} + V(\frac{z_1 w}{w+1})} \sum_{k\geqslant 0} \operatorname{Complex}_k\left(\frac{z_1 w}{1+w}\right)\left(-\frac{w}{1+w}\right)^k}.$$

The change of variables $(z_0, z_1, w) \mapsto \left(\frac{z_0}{y}, \frac{1+y}{y} z_1, y\right)$ are applied, which results in

$$\mathcal{DAG}_{n,m} = n!^2 [z_0^n z_1^n y^{m-2n}] e^{(U(z_0)+U(z_1))/y} e^{U(z_1)} \frac{\sum_{j\geqslant 0} \operatorname{Complex}_j(z_0) y^j}{\sum_{k\geqslant 0} \operatorname{Complex}_k(z_1)\left(-\frac{y}{1+y}\right)^k} \frac{e^{V(z_0)-V(z_1)}}{(1+y)^n}.$$

We finish the proof by extracting the coefficient $[y^{m-2n}]$. ☐

Remark 2.4. The number of pairs (G_0, G_1) of graphs, each on n vertices, having a total of $m_0 + m_1 = m$ edges, is $n!^2 [z_0^n z_1^n w^m] G(z_0, w) G(z_1, w)$. Working as in the previous proof leads to

$$n!^2 \sum_{t\geqslant 0} [z_0^n z_1^n w^m] \frac{\left(U(z_0)+U(z_1)\right)^{2n-m+t}}{(2n-m+t)!} e^{V(z_0)+V(z_1)} \sum_{j \geqslant 0} \operatorname{Complex}_j(z_0) y^j \sum_{k\geqslant 0} \operatorname{Complex}_k(z_1) y^k.$$

which looks and behaves (when m/n stays smaller than or close to 1) like the expression for $\mathcal{DAG}_{n,m}$ from the last lemma. This motivates the following intuition. Typically, those two graphs should share the m edges more or less equally. Thus, when m/n is close to 1, m_0/n and m_1/n should be close to $1/2$, so G_0 and G_1 will exhibit critical graph structure. For a smaller ratio m/n, G_0 and G_1 will behave like subcritical graphs, containing only trees and unicycles. This heuristic explanation for the critical density for dags guides our analysis in the rest of the paper.

2.3 Exact expression for elementary digraphs

As we discovered in our previous paper [15], and which was also pointed in a different form in [18], the graphic generating function of the family of digraphs whose connected components belong to a given set $\mathcal{S}$ with the EGF $S(z,w)$, is given by

$$\mathbf{E}(z,w) = \frac{1}{e^{-\mathrm{S}(z,w)} \odot_z \mathbf{Set}(z,w)}, \quad \text{where} \quad \mathbf{Set}(z,w) = \sum_{n \geqslant 0} \frac{z^n}{n!(1+w)^{\binom{n}{2}}}, \tag{2.5}$$

and $\odot_z$ is the exponential Hadamard product, characterized by $\sum_n a_n \frac{z^n}{n!} \odot_z \sum_n b_n \frac{z^n}{n!} = \sum_n a_n b_n \frac{z^n}{n!}$. $\mathbf{Set}(z,w)$ is the GGF of sets of isolated vertices. In particular, for the case of *elementary digraphs*, *i.e.* the digraphs whose strong components are isolated vertices or cycles of length $\geqslant 2$ only, the EGF of $\mathcal{S}$ is given by

$$\mathrm{S}(z,w) = z + \ln \frac{1}{1-zw} - zw.$$

In order to expand the Hadamard product, we develop the exponent $e^{-\mathrm{S}(z,w)}$ and apply the simplification rule $aze^{az} \odot_z F(z) = z\frac{d}{dz}F(z)\Big|_{z \mapsto az}$. After developing the exponent and expanding the Hadamard product we obtain a very simple expression, namely

$$\mathbf{E}(z,w) = \frac{1}{(1-zw)e^{-z(1-w)} \odot_z \mathbf{Set}(z,w)} = \frac{1}{\mathbf{Set}(-z,w) + z\frac{w}{1-w}\frac{d}{dz}\mathbf{Set}(-z,w)}\Bigg|_{z \mapsto (1-w)z}. \tag{2.6}$$

The following lemma is a heavier version of this expression. One of the reasons behind its visual complexity is the choice of the simple digraphs instead of multidigraphs; however, during the asymptotic analysis, most of the decorations corresponding to simple digraphs are going to disappear.

Lemma 2.5. *The number $\mathcal{ED}_{n,m}$ of (n,m) elementary digraphs is equal to*

$$\mathcal{ED}_{n,m} = n!^2 \sum_{t \geqslant 0} [z_0^n z_1^n] \frac{(U(z_0)+U(z_1))^{2n-m+t}}{(2n-m+t)!} [y^t] e^{\frac{2}{1-y}U(z_1)+V(z_0)-V(z_1)} \left(\frac{1-y}{1+y}\right)^n$$
$$\times \frac{\sum_{j \geqslant 0} \mathrm{Complex}_j(z_0) y^j}{\sum_{k \geqslant 0} \left[\mathrm{Complex}_k(z_1)\left(1 - \frac{1+y}{1-y}(T(z_1) - V^\bullet(z_1))\right) + \frac{1+y}{1-y}\mathrm{Complex}_k^\bullet(z_1)\right]\left(-y\frac{1-y}{1+y}\right)^k},$$

where $\mathrm{Complex}_r^\bullet(z) = z\frac{d}{dz}\mathrm{Complex}_r^\bullet(z)$ *and* $V^\bullet(z) = z\frac{d}{dz}V(z)$.

Proof. Let us denote $v = \frac{w}{1+w}$. Using the already mentioned representation

$$\mathbf{Set}(-z,w) = \mathrm{G}\left(-z, -\frac{w}{1+w}\right) = e^{-U(zv)/v + V(zv)} \sum_{r \geqslant 0} \mathrm{Complex}_r(zv)\,(-v)^r,$$

and by replacing $(1+w)^{\binom{n}{2}}$ with the generating function of graphs with n vertices as in the proof of Lemma 2.3, we can write the denominator of (2.6) prior to substitution $z \mapsto (1-w)z$ as

$$\mathbf{Set}(-z,w) + z\frac{w}{1-w}\frac{d}{dz}\mathbf{Set}(-z,w) = e^{-U(zv)/v+V(zv)}\Bigg[\sum_{r\geqslant 0}\mathrm{Complex}_r(zv)(-v)^r \\ -\frac{1+w}{1-w}\left([T(zv)-V^{\bullet}(zv)]\sum_{r\geqslant 0}\mathrm{Complex}_r(zv)(-v)^r + \sum_{r\geqslant 0}\mathrm{Complex}^{\bullet}_r(zv)(-v)^r\right)\Bigg],$$

Next, the change of variables $(z_0, z_1, w) \mapsto \left(\frac{z_0}{y}, \frac{1+y}{1-y}\frac{z_1}{y}, y\right)$ yields

$$\mathcal{ED}_{n,m} = n!^2[z_0^n z_1^n y^{m-2n}]e^{(U(z_0)+U(z_1))/y}e^{\frac{2}{1-y}U(z_1)+V(z_0)-V(z_1)}\left(\frac{1-y}{1+y}\right)^n \\ \times \frac{\sum_{j\geqslant 0}\mathrm{Complex}_j(z_0)y^j}{\sum_{k\geqslant 0}\left[\mathrm{Complex}_k(z_1)\left(1-\frac{1+y}{1-y}(T(z_1)-V^{\bullet}(z_1))\right)+\frac{1+y}{1-y}\mathrm{Complex}^{\bullet}_k(z_1)\right]\left(-y\frac{1-y}{1+y}\right)^k}.$$

The proof is finished by extracting the coefficient $[y^{m-2n}]$. □

3 Asymptotic analysis

3.1 Bivariate semi-large powers lemma

The typical structure of critical random graphs can be obtained by application of the *semi-large powers theorem* [5, Theorem IX.16, Case (ii)]. Since DAGs behave like a superposition of two graphs (see Remark 2.4), we design a bivariate variant of this theorem.

Lemma 3.1. *Consider two integers n and m going to infinity, such that $m = n(1+\mu n^{-1/3})$ with μ either staying in a bounded real interval, or $\mu \to -\infty$ while $\liminf_{n\to\infty} m/n > 0$; let the function $F(z_0, z_1)$ be analytic on the open torus of radii $(1,1)$ $\{z_0, z_1 \in \mathbb{C}\colon |z_0| < 1, |z_1| < 1\}$ and continuous on its closure, and let r_0 and r_1 be two real values, then the following asymptotics holds as $n \to \infty$*

$$[z_0^n z_1^n]\,(U(z_0)+U(z_1))^{2n-m}\frac{F(T(z_0),T(z_1))}{(1-T(z_0))^{r_0}(1-T(z_1))^{r_1}} \\ \sim \frac{e^{2n}}{4}\left(\frac{3}{n}\right)^{(4-r_0-r_1)/3} F\left(\frac{m}{n},\frac{m}{n}\right)H\left(\frac{3}{2},\frac{r_0}{2},-\frac{3^{2/3}}{2}\mu\right)H\left(\frac{3}{2},\frac{r_1}{2},-\frac{3^{2/3}}{2}\mu\right), \quad (3.1)$$

where the function $H(\lambda, r, x)$ is defined as $\frac{1}{\lambda}\sum_{k\geqslant 0}\Gamma\left(\frac{\lambda+r-k-1}{\lambda}\right)^{-1}\frac{(-x)^k}{k!}$.

Remark 3.2. A direct computation shows that $H(\cdot,\cdot,\cdot)$ from (3.1) can be expressed as

$$H\left(\frac{3}{2}, r, -\frac{3^{2/3}}{2}\mu\right) = \frac{2}{3}e^{\mu^3/6}3^{(2r+1)/3}A(2r,\mu),$$

where the function $A(y,\mu)$ is defined in [9] as

$$A(y,\mu) = \frac{e^{-\mu^3/6}}{3^{(y+1)/3}}\sum_{k\geqslant 0}\frac{\left(\frac{1}{2}3^{2/3}\mu\right)^k}{k!\Gamma\left(\frac{y+1-2k}{3}\right)} \text{ and satisfies } \lim_{\mu\to-\infty} A(y,\mu)|\mu|^{y-1/2} = \frac{1}{\sqrt{2\pi}}.$$

We provide only the proof of the harder case when μ is bounded. In order to adapt the proof of Lemma 3.1 to the case $\mu \to -\infty$, a simpler saddle-point bound can be used.

Proof of Lemma 3.1. The first step is to represent the coefficient extraction operation from (3.1) as a double complex integral, using Cauchy formula, and to approximate this double integral with a product of two complex integrals. We start with the Puiseux expansion of the EGF of rooted labeled trees $T(z)$ and unrooted labeled trees $U(z) = T(z) - \frac{T^2(z)}{2}$:

$$T(z) = 1 - \sqrt{2}\sqrt{1-ez} + \frac{2}{3}(1-ez) + \mathcal{O}(1-ez)^{3/2}, \tag{3.2}$$

$$U(z) = \frac{1}{2} - (1-ez) + \frac{2^{3/2}}{3}(1-ez)^{3/2} + \mathcal{O}(1-ez)^2. \tag{3.3}$$

Applying Cauchy's integral theorem, we rewrite the coefficient extraction (3.1) in the form

$$\frac{1}{(2i\pi)^2}\oint\oint (U(z_0)+U(z_1))^{2n-m}\frac{F(T(z_0),T(z_1))}{(1-T(z_0))^{r_0}(1-T(z_1))^{r_1}}\frac{dz_0}{z_0^{n+1}}\frac{dz_1}{z_1^{n+1}}.$$

In order to accomplish the separation of the integrals, we represent the term $(U(z_0)+U(z_1))^{2n-m}$ as the exponent of the logarithm, and evaluate the leading terms in Newton–Puiseux expansion of the logarithm

$$(U(z_0)+U(z_1))^{2n-m} = e^{(2n-m)\log(U(z_0)+U(z_1))}.$$

By plugging the leading terms of $U(z_0)$ and $U(z_1)$ from (3.3) into the previous expression and developing the logarithm around $z_0 = z_1 = e^{-1}$, we notice that the leading powers of $(1-ez_0)$ and $(1-ez_1)$ contain only the exponents $\{0, 1, \frac{3}{2}\}$, and thus, asymptotically, no products need to be taken into account, see Table 1.

	0	1	$\frac{3}{2}$
0	✓	✓	✓
1	✓	–	–
$\frac{3}{2}$	✓	–	–

Table 1: Contributing exponents of $(1-ez_0)$ and $(1-ez_1)$ for bivariate semi-large powers lemma

A further step is to inject $m = n + \mu n^{2/3}$, $1 - ez_0 = \alpha_0 n^{-2/3}$, and $1 - ez_1 = \alpha_1 n^{-2/3}$, where $\alpha_0, \alpha_1 \in \mathbb{C}$. By using expansion (3.2) in order to approximate the terms $(1 - T(z_0))$ and $(1 - T(z_1))$, we rewrite the answer in the form

$$\left(\frac{n}{2^{3/2}}\right)^{(r_0+r_1-4)/3} \frac{e^{2n}}{4(2i\pi)^2} F(1,1) \times \\ \oint \oint e^{\mu(\alpha_0+\alpha_1)+\frac{2^{3/2}}{3}(\alpha_0^{3/2}+\alpha_1^{3/2})+\mathcal{O}\left(n^{-1/3}\right)} \left(1 + \mathcal{O}(n^{-1/3})\right) \frac{d\alpha_0}{\alpha_0^{r_0/2}} \frac{d\alpha_1}{\alpha_1^{r_1/2}}.$$

After removal of the negligible terms, a product of integrals is obtained

$$F(1,1) \left(\frac{n}{2^{3/2}}\right)^{(r_0+r_1-4)/3} \frac{e^{2n}}{4} \frac{1}{2i\pi} \oint e^{\mu\alpha_0+\frac{2^{3/2}}{3}\alpha_0^{3/2}} \frac{d\alpha_0}{\alpha_0^{r_0/2}} \times \frac{1}{2i\pi} \oint e^{\mu\alpha_1+\frac{2^{3/2}}{3}\alpha_1^{3/2}} \frac{d\alpha_1}{\alpha_1^{r_1/2}}.$$

Each of the integrals can be evaluated similarly as in [5, Theorem IX.16, Case (ii)]: in order to evaluate such integral, a variable change $u = -\frac{2^{3/2}}{3}\alpha^{3/2}$ is applied, and the integral is expressed as an infinite sum using a Hankel contour formula for the Gamma function:

$$\frac{1}{2i\pi} \oint e^{\frac{3^{2/3}}{2}\mu u^{2/3}} e^{-u} \left(\frac{3^{2/3}u^{2/3}}{2}\right)^{-\frac{1+r}{2}} \frac{du}{\sqrt{2}} = \frac{2^{r/2}}{3^{(1+r)/3}} \sum_{k \geqslant 0} \frac{\left(\frac{3^{2/3}\mu}{2}\right)^k}{k!} \frac{1}{2i\pi} \oint u^{\frac{2k-1-r}{3}} e^{-u} du.$$

□

3.2 Asymptotic analysis of directed acyclic graphs

Since we are going to apply Lemma 3.1 to each of the terms of the infinite sum of Lemma 2.3, it is useful to introduce the following notation

$$s_r^+(\mu) = H\left(\frac{3}{2}, \frac{3r}{2} + \frac{1}{4}, -\frac{3^{2/3}}{2}\mu\right), \quad s_r^-(\mu) = H\left(\frac{3}{2}, \frac{3r}{2} - \frac{1}{4}, -\frac{3^{2/3}}{2}\mu\right);$$

$$S^+(y;\mu) = \sum_{r \geqslant 0} s_r^+(\mu) y^r, \;\; S^-(y;\mu) = \sum_{r \geqslant 0} s_r^-(\mu) y^r, \;\; E(y) = \sum_{r \geqslant 0} e_r y^r, \;\; e_r^{(-1)} = [y^r]\frac{1}{E(-y)},$$

where e_r is given by Proposition 2.2. This notation will be used throughout the next two sections.

Theorem 3.3. *When $m = n(1 + \mu n^{-1/3})$ and μ either stays in a bounded real interval, or $\mu \to -\infty$ while* $\liminf m/n > 0$ *as $n \to \infty$,*

$$\mathbb{P}((n,m)\text{-}digraph\ is\ acyclic) \sim \frac{3^{5/6}}{2} \frac{e^{\frac{m}{n}-\frac{\mu^3}{6}}\sqrt{2\pi}}{n^{1/3}} \sum_{q \geqslant 0} 3^{-q} e_q^{(-1)} s_q^-(\mu).$$

In particular, for the sparse case where $\mu \to -\infty$ (which covers $\limsup m/n < 1$*),*

$$\mathbb{P}((n,m)\text{-digraph is acyclic}) \sim e^{m/n}(1 - m/n).$$

Proof. In order to apply Lemma 3.1 (bivariate semi-large powers), we develop the coefficient operator $[y^t]$ in Lemma 2.3 using the approximation of $\mathrm{Complex}_k(T)$ from Proposition 2.2 and drop the terms that give negligible contribution:

$$[y^t] \frac{\sum_{j\geqslant 0} \mathrm{Complex}_j(z_0) y^j}{\sum\limits_{k\geqslant 0} \mathrm{Complex}_k(z_1) \left(-\frac{y}{1+y}\right)^k} \frac{1}{(1+y)^n}$$

$$= \sum_{p+q+r=t} \frac{e_p T(z_0)^{2p}}{(1-T(z_0))^{3p}} \frac{e_q^{(-1)} T(z_1)^{2q}}{(1-T(z_1))^{3q}} [y^r] \frac{1}{(1+y)^n} (1 + \mathcal{O}(1-T(z_0)))(1 + \mathcal{O}(1 - T(z_1))).$$

Then we apply Lemma 3.1 and the approximation $(2n-m+t)! \sim (2n-m)! n^t$ to obtain

$$\begin{aligned} \mathcal{DAG}_{n,m} &= n!^2 \sum_{t\geqslant 0} [z_0^n z_1^n] \sum_{p+q+r=t} \frac{(U(z_0)+U(z_1))^{2n-m+t}}{(2n-m+t)!} e^{U(z_1)} \frac{e_p T(z_0)^{2p}}{(1-T(z_0))^{3p+1/2}} \\ &\quad \times \frac{e_q^{(-1)} T(z_1)^{2q}}{(1-T(z_1))^{3q-1/2}} [y^r] \frac{1}{(1+y)^n} (1+\mathcal{O}(1-T(z_0)))(1+\mathcal{O}(1-T(z_1))) \\ &= \frac{n!^2}{(2n-m)!} \sum_{t\geqslant 0} \sum_{p+q+t=t} \frac{e^{m/2n}}{n^t} e_p e_q^{(-1)} \frac{e^{2n}}{4} \left(\frac{3}{n}\right)^{(4-(3p+1/2)-(3q-1/2))/3} \\ &\quad \times s_p^+(\mu) s_q^-(\mu) [y^r] \frac{1}{(1+y)^n}. \end{aligned}$$

The power of n in the sum is $n^{-4/3-r}$, and the sum over r of $n^{-r}[y^r](1+y)^{-n}$ is equal to $(1+1/n)^{-n}$ and converges to $e^{-m/n}$. Finally, the sums over p and q are decoupled and we obtain

$$\mathcal{DAG}_{n,m} \sim \frac{n!^2}{(2n-m)!} \frac{e^{2n-m/2n}}{n^{4/3}} \frac{3^{4/3}}{4} \sum_{p\geqslant 0} 3^{-p} e_p s_p^+(\mu) \sum_{q\geqslant 0} 3^{-q} e_q^{(-1)} s_q^-(\mu).$$

The sum over p admits a closed expression $\sqrt{\frac{2}{3\pi}} e^{\mu^3/6}$ (see Remark 3.2 and [9, Section 14]). Applying Stirling's formula, we can rescale the asymptotic number of DAGs by the total number of digraphs:

$$\mathcal{DAG}_{n,m} \sim \binom{n(n-1)}{m} \frac{3^{5/6}}{2} \frac{e^{\frac{m}{n} - \frac{\mu^3}{6}} \sqrt{2\pi}}{n^{1/3}} \sum_{q\geqslant 0} 3^{-q} e_q^{(-1)} s_q^-(\mu).$$

This gives the main statement. To obtain the sparse case, we need to use the fact that when $\mu \to -\infty$, the first summand of the sum over q is dominating, and therefore, this sum is asymptotically equivalent to $\sqrt{\frac{2}{\pi}\frac{e^{\mu^3/6}}{|\mu|^{-1}}} 3^{-5/6}$ (see [9, Equation (10.3)]). $\square$

3.3 Asymptotic analysis of elementary digraphs

Theorem 3.4. *When $m = n(1 + \mu n^{-1/3})$ and μ either stays in a bounded real interval, or $\mu \to -\infty$ while $\liminf m/n > 0$ as $n \to \infty$,*

$$\mathbb{P}((n,m)\text{-digraph is elementary}) \sim e^{-\mu^3/6}\sqrt{\frac{3\pi}{2}}\sum_{q\geqslant 0} 3^{-q}\,\widehat{e}_q^{(-1)} s_q^{+}(\mu),$$

where the coefficients $\widehat{e}_q^{(-1)}$ are given by

$$\sum_{q\geqslant 0}\widehat{e}_q^{(-1)} y^q := \frac{1}{\frac{y}{2} + E(y) + 3y^2E'(y)}.$$

In particular, when $\mu \to -\infty$, $|\mu| \ll n^{-1/3}$,

$$\mathbb{P}((n,m)\text{-digraph is elementary}) \sim 1 - \frac{1}{2|\mu|^3}.$$

Proof. The key ingredient is the exact expression from Lemma 2.5. As in the proof of Theorem 3.3, we can drop the terms that give negligible contributions and develop the coefficient operator $[y^t]$ accordingly. The key difference between the proofs is the form of the denominator: after taking out a common multiple $(1 - T(z_1))$ (ignoring higher powers in variable y), the denominator can be again regarded as a formal power series in $\frac{y}{(1-T(z_1))^3}$. In order to obtain the asymptotics, the transformed expression should be developed, then Lemma 3.1 (bivariate semi-large powers) is applied, and finally the sums are decoupled. For the sum corresponding to variable z_0, we apply again the hypergeometric summationformula from [9]. In order to settle the subcritical case $\mu \to -\infty$, we apply the asymptotic approximation of $s_q^+(\mu)$ from Remark 3.2. □

Remark 3.5. Curiously enough, the coefficient $1/2$ in the subcritical probability can be given the same interpretation as a similar coefficient $5/24$ arising in the probability that a random graph does not contain a complex component: namely the compensation factor of the simplest cubic forbidden multigraph.

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