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The minimal excludant and colored partitions

Cristina Ballantine*[1] and Mircea Merca†[2]

[1] *Department of Mathematics, College of The Holy Cross, Worcester, MA, USA*
[2] *Department of Mathematics, University of Craiova, Craiova, Romania and Academy of Romanian Scientists, Bucharest, Romania*

Abstract. The minimal excludant of a partition λ, $\operatorname{mex}(\lambda)$, is the smallest positive integer that is not a part of λ. For a positive integer n, $\sigma \operatorname{mex}(n)$ denotes the sum of the minimal excludants of all partitions of n. Recently, Andrews and Newman obtained a new combinatorial interpretation for $\sigma \operatorname{mex}(n)$. They showed, using generating functions, that $\sigma \operatorname{mex}(n)$ equals the number of partitions of n into distinct parts using two colors. We give a purely combinatorial proof of this result and derive its generalization to the sum of least r-gaps. We introduce several new identities connecting the function $\sigma \operatorname{mex}(n)$ to the number of partitions with colored parts satisfying certain congruences.

Keywords: Partitions, minimal excludant, least gap in partitions, colored partitions.

1 Introduction

The minimal excludant or mex-function of a set S of positive integers is the least positive integer not in S. The history of this notion goes back to at least the 1930s when it was applied to combinatorial game theory [9, 8].

Recently, Andrews and Newman [3] considered the mex-function applied to integer partitions. They defined the minimal excludant of a partition λ, $\operatorname{mex}(\lambda)$, as the smallest positive integer that is not a part of λ. Then, for each positive integer n, they defined

$$\sigma \operatorname{mex}(n) := \sum_{\lambda \in \mathcal{P}(n)} \operatorname{mex}(\lambda),$$

where $\mathcal{P}(n)$ is the set of all partitions of n. Elsewhere in the literature, the minimal excludant of a partition λ is referred to as the least gap or smallest gap of λ. An exact and asymptotic formula for $\sigma \operatorname{mex}(n)$, as well as its generating function, is given in [7]. In [5] we studied a generalization of $\sigma \operatorname{mex}(n)$ and its connection to polygonal numbers.

Let $\mathcal{D}_2(n)$ be the set of partitions of n into distinct parts using two colors and let $D_2(n) = |\mathcal{D}_2(n)|$. We denote the colors of the parts of partitions in $\mathcal{D}_2(n)$ by 0 and 1. For example, $\mathcal{D}_2(4) = \{4_0, 4_1, 3_0 + 1_0, 3_0 + 1_1, 3_1 + 1_0, 3_1 + 1_1, 2_1 + 2_0, 2_1 + 1_1 + 1_0, 2_0 + 1_1 + 1_0\}$, and thus $D_2(4) = 9$. In [3], the authors give two proofs of the following theorem.

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*cballant@holycross.edu.
†mircea.merca@profinfo.edu.ro.

Theorem 1.1. *Given an integer $n \geqslant 0$, we have $\sigma\,\mathrm{mex}(n) = D_2(n)$.*

In Section 2, we provide a bijective proof of Theorem 1.1. We make use of the fact that

$$\sigma\,\mathrm{mex}(n) = \sum_{j\geqslant 0} p(n - j(j+1)/2), \tag{1.1}$$

where, as usual, $p(n)$ denotes the number of partitions of n. A combinatorial proof of (1.1) is given in [5, Theorem 1.1]. The same argument is also described in the second proof of [3, Theorem 1.1]. In fact, the result proven in [5] is a generalization of (1.1) to $\sigma_r\,\mathrm{mex}(n)$, the sum of r-gaps in all partitions of n. The r-gap of a partition λ is the least positive integer that does not appear r times as a part of λ. In Section 3, we give two generalizations of Theorem 1.1 to $\sigma_r\,\mathrm{mex}(n)$.

In [1], the authors considered a restricted mex function. They defined $M_k(n)$ to be the number of partitions λ of n with $\mathrm{mex}(\lambda) = k$ and more parts $> k$ than parts $< k$. When $k = 1$, $M_1(n)$ is the number of partitions of n with smallest part greater than 1. Thus, if $n > 0$, we have $M_1(n) = p(n) - p(n-1)$, and from (1.1), we obtain

$$\sigma\,\mathrm{mex}(n) - \sigma\,\mathrm{mex}(n-1) - \delta(n) = \sum_{j=0}^{\infty} M_1\big(n - j(j+1)/2\big), \tag{1.2}$$

where δ is the characteristic function of the set of triangular numbers.

We generalize (1.2) in Section 4 where we give further connections between $\sigma\,\mathrm{mex}(n)$ and restricted mex functions or partitions and overpartitions. In Section 5 we present connections with partitions with colored odd parts.

2 Combinatorial Proof of Theorem 1.1

To prove the theorem, we adapt Sylvester's bijective proof of Jacobi's triple product identity [10]. Given $\lambda \in \mathcal{D}_2(n)$, let $\lambda^{(i)}$, $i = 0, 1$, be the (uncolored) partition whose parts are the parts of color i in λ. Then, $\lambda^{(1)}$ and $\lambda^{(2)}$ are partitions with distinct parts.

Example 2.1. If $\lambda = 4_1 + 3_0 + 3_1 + 2_0 + 1_0 \in \mathcal{D}_2(13)$, then $\lambda^{(0)} = 3 + 2 + 1$ and $\lambda^{(1)} = 4 + 3$.

Denote by $\eta(j)$ the staircase partition $\eta(j) = j + (j-1) + \cdots + 2 + 1$, with $\eta(0) = \emptyset$. We write $\ell(\lambda)$ for the number of parts in partition λ. The conjugate of a Ferrers diagram ν (not necessarily the diagram of a partition) is obtained by reflecting ν across the main diagonal. The sum, $\alpha + \beta$, of two composition $\alpha = (a_1, a_2, \ldots)$ and $\beta = (b_1, b_2, \ldots)$, is the composition whose parts are $a_i + b_i$ (appropriately using 0 as parts at the end of the shorter composition).

Definition 2.2. Given a diagram of left justified rows of boxes (not necessarily the Ferrers diagram of a partition), the *staircase profile* of the diagram is a zig-zag line starting in the upper left corner of the diagram with a right step and continuing in alternating down and right steps until the end of a row of the diagram is reached.

Example 2.3. Let α be the composition $\alpha = (1, 2, 3, 7, 7, 6, 6, 4, 2)$.

Figure 1: Staircase profile for α and the conjugate of α.

The *shifted Ferrers diagram* of a partition λ with distinct parts is the Ferrers diagram (with boxes of unit length) of λ with row i shifted $i-1$ units to the right.

We create a map

$$\varphi : \bigcup_{j\geqslant 0} \mathcal{P}(n - j(j+1)/2) \to \mathcal{D}_2(n)$$

as follows. Start with $\lambda \in \mathcal{P}(n - j(j+1)/2)$ for some $j \geqslant 0$. Append a diagram with rows of lengths $1, 2, \dots j$ (i.e., the diagram of $\eta(j)$ rotated by 90° counterclockwise) at the top of the diagram of λ. We obtain a diagram with n boxes. Draw the staircase profile of the new diagram. Let α be the partition whose parts are the length of the columns to the left of the staircase profile and β be the partition whose parts are the length of the rows to the right of the staircase profile. Then α and β are partitions with distinct parts. Moreover, $j \leqslant \ell(\alpha) - \ell(\beta) \leqslant j+1$. Color the parts of α with color $j \,(\text{mod } 2)$ and the parts of β with color $j+1 \,(\text{mod } 2)$. Then $\varphi(\lambda)$ is defined as the 2-color partition of n whose parts are the colored parts of α and β.

Conversely, start with $\mu \in \mathcal{D}_2(n)$. Let $\ell_i(\mu)$, $i = 0, 1$, be the number of parts of color i in μ and set $r = \ell_0(\mu) - \ell_1(\mu)$. Let

$$\varepsilon = \begin{cases} 0 & \text{if } r \geq 0 \\ 1 & \text{if } r < 0, \end{cases} \quad \text{and} \quad j = |r| + \frac{(-1)^{|r|+\varepsilon} - 1}{2}.$$

Remove the top j rows (i.e., the rotated diagram of $\eta(j)$) from the conjugate of the shifted diagram of $\mu^{(\varepsilon)}$ to obtain a composition γ. Define $\varphi^{-1}(\mu) = \gamma + \mu^{(s)}$ where $s \neq \varepsilon$. Then, $\varphi^{-1}(\mu) \in \mathcal{P}(n - j(j+1)/2)$.

Example 2.4. Let $n = 38, j = 3$, and let $\lambda = 7+7+6+6+4+2$ be a partition of $n - j(j+1)/2 = 32$. We add the rotated diagram of $\eta(3)$ to the top of the diagram of λ and draw the staircase profile (see Figure 1). Then $\alpha = 9+8+6+5+3+2$ and $\beta = 3+2$. Since j is odd, we have $\varphi(\lambda) = 9_1 + 8_1 + 6_1 + 5_1 + 3_1 + 3_0 + 2_1 + 2_0$.

Conversely, suppose $\mu = 9_1 + 8_1 + 6_1 + 5_1 + 3_1 + 3_0 + 2_1 + 2_0 \in \mathcal{D}(38)$. Then $\ell_0(\mu) = 2$ and $\ell_1(\mu) = 6$. We have $r = \ell_0(\mu) - \ell_1(\mu) = -4$ and $j = 3$. We remove the first 3 rows from the conjugate of the shifted diagram of $\mu^{(1)}$ (which is precisely the diagram below the staircase profile in Figure 1) and add the resulting composition γ to $\mu^{(0)} = (3,2)$. We obtain $\varphi^{-1}(\mu) = 7+7+6+6+4+2 \in \mathcal{P}(32)$.

3 Generalizations of Theorem 1.1 to r-gaps

Recall that the r-gap of a partition λ is the least positive integer that does not appear r times as a part of λ. In [5], we proved combinatorially that

$$\sigma_r \operatorname{mex}(n) = \sum_{j \geqslant 0} p(n - rj(j+1)/2). \tag{3.1}$$

We can employ a transformation similar to that in the combinatorial proof of Theorem 1.1 to prove its generalization to sums of r-gaps.

Let $\widetilde{D}_3^{(r)}(n)$ be the number of partitions λ of n into distinct parts using three colors, $0, 1$, and 2, such that:

(i) The set of parts of color 2 is either empty or $\{t(r-1) \mid 1 \leqslant t \leqslant j\}$ for some $j \geqslant 1$.

(ii) $\ell_{j \,(\mathrm{mod}\ 2)}(\lambda) - \ell_{j+1\,(\mathrm{mod}\ 2)}(\lambda) \in \{j, j+1\}$, where $j = 0$ if $\lambda^{(2)} = \emptyset$.

Theorem 3.1. *Let n, r be integers with $r > 0$ and $n \geqslant 0$. Then $\sigma_r \operatorname{mex}(n) = \widetilde{D}_3^{(r)}(n)$.*

Proof. For a sketch of the proof see [6]. □

In [5] we give the generating function for $\sigma_r \operatorname{mex}(n)$, namely

$$\sum_{n=0}^{\infty} \sigma_r \operatorname{mex}(n) q^n = \frac{(q^{2r}; q^{2r})_\infty}{(q;q)_\infty (q^r; q^{2r})_\infty}, \tag{3.2}$$

where $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$ if $n > 0$, $(a;q)_n = 1$ if $n = 0$, and $(a;q)_\infty = \lim_{n \to \infty} (a;q)_n$.

Denote $\widetilde{D}_2^{(r)}(n)$ the number of partitions λ of n using two colors, 0 and 1, such that:

(i) $\lambda^{(0)}$ is a partition into distinct parts divisible by r.

(ii) $\lambda^{(1)}$ is a partition with parts repeated at most $2r - 1$ times.

The following generalization of Theorem 1.1 is immediate from (3.2).

Theorem 3.2. *Let n, r be integers with $r > 0$ and $n \geq 0$. Then $\sigma_r \operatorname{mex}(n) = \widetilde{D}_2^{(r)}(n)$.*

4 Identities involving restricted mex-functions

In this section we introduce identities relating $\sigma\,\mathrm{mex}(n)$ and restricted mex functions for partitions and overpartitions.

4.1 σ mex(n) and $M_k(n)$

We have the follwing generalization of (1.2).

Theorem 4.1. *Let k, n be integers with $k \geqslant 1$ and $n \geqslant 0$. Then,*

$$(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k}(-1)^j\sigma\,\mathrm{mex}\big(n-j(3j-1)/2\big)-\delta(n)\right)=\sum_{j=0}^{\infty}M_k\big(n-j(j+1)/2\big).$$

The following infinite family of linear inequalities involving σ mex is immediate.

Corollary 4.2. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k}(-1)^j\sigma\,\mathrm{mex}\big(n-j(3j-1)/2\big)-\delta(n)\right)\geqslant 0,$$

with strict inequality if $n \geqslant k(3k+1)/2$.

Analytic proof of Theorem 4.1. In [1], the authors gave the following truncated Euler's pentagonal number theorem.

$$\frac{(-1)^{k-1}}{(q;q)_\infty}\sum_{n=-(k-1)}^{k}(-1)^j q^{n(3n-1)/2}=(-1)^{k-1}+\sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix},\tag{4.1}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leqslant k\leqslant n,\\ 0, & \text{otherwise.}\end{cases}$$

Multiplying both sides of (4.1) by

$$\frac{(q^2,q^2)_\infty}{(q,q^2)_\infty}=\sum_{n=0}^{\infty}q^{n(n+1)/2},$$

and using (3.2) with $r=1$ and the generating function for $M_k(n)$ [1],

$$\sum_{n=0}^{\infty}M_k(n)q^n=\sum_{n=k}^{\infty}\frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix},$$

we obtain

$$(-1)^{k-1}\left(\left(\sum_{n=0}^{\infty}\sigma\,\mathrm{mex}(n)q^n\right)\left(\sum_{n=-(k-1)}^{k}(-1)^j q^{n(3n-1)/2}\right)-\sum_{n=0}^{\infty}q^{n(n+1)/2}\right)$$
$$=\left(\sum_{n=0}^{\infty}q^{n(n+1)/2}\right)\left(\sum_{n=0}^{\infty}M_k(n)q^n\right).$$

The proof follows easily using Cauchy's multiplication of two power series. □

Combinatorial proof of Theorem 4.1. The statement of Theorem 4.1 is equivalent to identity (1.2) together with

$$\sigma\,\mathrm{mex}\left(n-\frac{k(3k+1)}{2}\right)-\sigma\,\mathrm{mex}\left(n-\frac{k(3k+5)}{2}-1\right)$$
$$=\sum_{j=0}^{\infty}\Big(M_k\big(n-j(j+1)/2\big)+M_{k+1}\big(n-j(j+1)/2\big)\Big). \tag{4.2}$$

Using (1.1), identity (4.2) becomes

$$\sum_{j=0}^{\infty}\left(p\left(n-\frac{j(j+1)}{2}-\frac{k(3k+1)}{2}\right)-p\left(n-\frac{j(j+1)}{2}-\frac{k(3k+5)}{2}-1\right)\right)$$
$$=\sum_{j=0}^{\infty}\Big(M_k\big(n-j(j+1)/2\big)+M_{k+1}\big(n-j(j+1)/2\big)\Big). \tag{4.3}$$

Identity (4.3) was proved combinatorially in [11]. Together with the combinatorial proof of (1.1), this gives a combinatorial proof of Theorem 4.1. □

Next, we give a combinatorial interpretation for $\sum_{t=0}^{\infty}M_k\big(n-t(t+1)/2\big)$. For integers k,n such that $k\geqslant 1$ and $n\geqslant 0$, we denote by $D_3^{(k)}(n)$ the number of partitions μ of n into distinct parts using three colors and satisfying the following conditions:

(i) μ has exactly k parts of color 2 and, if $k>1$, twice the smallest part of color 2 is greater than largest part of color 2.

(ii) With r and j as in the combinatorial proof of Theorem 1.1, the largest part of color $j \pmod 2$ must equal j more that the smallest part of color 2.

Proposition 4.3. *For integers k,n such that $k\geqslant 1$ and $n\geqslant 0$, we have*

$$\sum_{t=0}^{\infty}M_k\big(n-t(t+1)/2\big)=D_3^{(k)}(n). \tag{4.4}$$

Proof. See [6]. □

Combining Theorems 1.1 and 4.1, and Proposition 4.3 we obtain the following corollary which, by the discussion above, has both analytic and combinatorial proofs.

Corollary 4.4. *For integers k, n such that $k \geqslant 1$ and $n \geqslant 0$, we have*

$$(-1)^{\max(0,k-1)} \left(\sum_{j=-\max(0,k-1)}^{k} (-1)^j \sigma\,\mathrm{mex}\big(n - j(3j-1)/2\big) - \delta(n) \right) = D_3^{(k)}(n).$$

Note that, if $k = 0$, the statement of the corollary reduces to Theorem 1.1.

4.2 $\sigma\,\mathrm{mex}(n)$ and overpartitions

Overpartitions are ordinary partitions with the added condition that the first appearance of any part may be overlined. There are eight overpartitions of 3:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

As usual, we denote by $\overline{p}(n)$ the number of overpartitions of n. The generating function for $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

We have the following identity relating $\sigma\,\mathrm{mex}(n), \overline{p}(n)$ and $M_k(n)$.

Theorem 4.5. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1} \left(\sum_{j=-(k-1)}^{k} (-1)^j \overline{p}\big(n - j(3j-1)\big) - \sigma\,\mathrm{mex}(n) \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} M_k(j) \sigma\,\mathrm{mex}(n-2j).$$

Proof. By (4.1), with q replaced by q^2, we obtain

$$\frac{(-1)^{k-1}}{(q^2;q^2)_\infty} \left(\sum_{n=-(k-1)}^{k} (-1)^j q^{n(3n-1)} - 1 \right) = \sum_{n=k}^{\infty} M_k(n) q^{2n}. \tag{4.5}$$

Multiplying both sides of (4.5) by the generating function for $\sigma\,\mathrm{mex}(n)$, we obtain

$$(-1)^{k-1} \left(\left(\sum_{n=0}^{\infty} \overline{p}(n) q^n \right) \left(\sum_{n=-(k-1)}^{k} (-1)^j q^{n(3n-1)} \right) - \sum_{n=0}^{\infty} \sigma\,\mathrm{mex}(n) q^n \right)$$
$$= \left(\sum_{n=0}^{\infty} \sigma\,\mathrm{mex}(n) q^n \right) \left(\sum_{n=0}^{\infty} M_k(n) q^{2n} \right).$$

The proof follows by equating the coefficients of q^n in this identity. □

The limiting case $k \to \infty$ of Theorem 4.5 reads as follows.

Corollary 4.6. *For* $n \geqslant 0$, $\sigma\,\mathrm{mex}(n) = \sum_{j=-\infty}^{\infty} (-1)^j \overline{p}\big(n - j(3j-1)\big)$.

Remark 4.7. Since it is known that $\overline{p}(n)$ is odd if and only if $n = 0$, it follows that $\sigma\,\mathrm{mex}(n)$ is odd if and only if $12n+1$ is a square.

In [2], the authors denoted by $\overline{M}_k(n)$ the number of overpartitions of n in which the first part larger than k appears at least $k+1$ times. We have the following identity.

Theorem 4.8. *For integers* $k, n > 0$, *we have*

$$(-1)^k \left(\sigma\,\mathrm{mex}(n) + 2\sum_{j=1}^{k} (-1)^j \sigma\,\mathrm{mex}(n-j^2) - \delta'(n) \right) = \sum_{j=-\infty}^{\infty} (-1)^j \overline{M}_k\big(n - j(3j-1)\big),$$

where $\delta'(n) = (-1)^m$ *if* $n = m(3m-1)$, $m \in \mathbb{Z}$ *and* $\delta'(n) = 0$ *otherwise.*

Proof. The proof, given in [6], follows from a truncated theta series identity [2]. □

There is a substantial amount of numerical evidence to conjecture the following inequality.

Conjecture 4.9. *For* $k, n > 0$,

$$\sum_{j=-\infty}^{\infty} (-1)^j \overline{M}_k\big(n - j(3j-1)\big) \geqslant 0,$$

with strict inequality if $n \geqslant (k+1)^2$.

A combinatorial interpretation for the sum in this conjecture would be interesting.

4.3 $\sigma\,\mathrm{mex}(n)$ and partitions into distinct parts

To keep notation uniform, let $D_1(n)$ be the number of partitions of n into distinct parts. Set $D_1(x) = 0$ if x is not a positive integer. For proof of the next theorem see [6].

Theorem 4.10. *For any integer* $n \geqslant 0$, *we have*

$$\sum_{j=0}^{\infty} (-1)^{j(j+1)/2} \sigma\,\mathrm{mex}\big(n - j(j+1)/2\big) = \sum_{j=0}^{\infty} D_1\left(\frac{n - j(j+1)/2}{2}\right). \tag{4.6}$$

Let $D_2^*(n)$ be the number of partitions of n with distinct parts using two colors such that: (i) parts of color 0 form a gap-free partition (staircase) and (ii) only even parts can have color 1. Then, we have the following identity of Watson type [4] which gives a combinatorial interpretation for the right hand side of (4.6). For its proof see [6].

Proposition 4.11. *For $n \geqslant 0$,*

$$\sum_{j=0}^{\infty} D_1\left(\frac{n-j(j+1)/2}{2}\right) = D_2^*(n).$$

In [2], the authors denoted by $MP_k(n)$ the number of partitions of n in which the first part larger than $2k-1$ is odd and appears exactly k times. All other odd parts appear at most once. We have the following truncated form of Theorem 4.10.

Theorem 4.12. *For integers $n, k > 0$,*

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}\sigma\,\mathrm{mex}\big(n-j(j+1)/2\big) - D_2^*(n)\right) = \sum_{j=0}^{n} MP_k(j) D_2^*(n-j).$$

Proof. The proof, given in [6], follows from the truncated theta series identity of [2]. □

A combinatorial interpretation for $\sum_{j=0}^{n} MP_k(j) D_2^*(n-j)$ would be very welcome.

The following corollary of Theorem 4.12 is immediate.

Corollary 4.13. *For integers $n, k > 0$,*

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}\sigma\,\mathrm{mex}\big(n-j(j+1)/2\big) - D_2^*(n)\right) \geqslant 0,$$

with strict inequality if $n \geqslant k(2k+1)$.

A second corollary involves the function $\mathrm{pod}(n)$, the number of partitions of n in which odd parts are not repeated, i.e.,

Corollary 4.14. *For $n \geqslant 0$, $\sigma\,\mathrm{mex}(n) = \sum_{j=0}^{n} \mathrm{pod}(j) D_2^*(n-j)$.*

5 $\sigma\,\mathrm{mex}(n)$ and partitions with colored odd parts

In this section we present several identities relating $\sigma\,\mathrm{mex}(n)$ with the number of partitions of n in which odd parts are colored in with j colors, $j = 2, 3, 4$. Elsewhere in the literature, colored partitions are referred to as vector partitions. Due to space restrictions, we will present the proofs of all theorems in this section in a future article.

5.1 Three colors for the odd parts

Let $C_3(n)$ be the number of partitions of n using 3 colors for the odd parts and let $C_3'(n)$ be the number of partitions of n into parts not congruent to 2 mod 4 using 3 colors for the odd parts. The generating functions for $C_3(n)$ and $C_3'(n)$ are respectively

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{1}{(q^2;q^2)_\infty (q;q^2)_\infty^3} \quad \text{and} \quad \sum_{n=0}^{\infty} C_3'(n)q^n = \frac{1}{(q^4;q^4)_\infty (q;q^2)_\infty^3}.$$

Using the truncated Euler's pentagonal number theorem [1], we prove the following identity which relates $C_3(n)$ and the function $M_k(n)$ defined in Section 1.

Theorem 5.1. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k} (-1)^j C_3\big(n-j(3j-1)/2\big) - \sigma\operatorname{mex}(n)\right) = \sum_{j=0}^{n} \sigma\operatorname{mex}(j) M_k(n-j).$$

A combinatorial interpretation of $\sum_{j=0}^{n} \sigma\operatorname{mex}(j) M_k(n-j)$ would be appealing.

The limiting case $k \to \infty$ of Theorem 5.1 gives the following decomposition of $\sigma\operatorname{mex}(n)$.

Corollary 5.2. *For $n \geqslant 0$, we have $\sigma\operatorname{mex}(n) = \sum_{j=-\infty}^{\infty} (-1)^j C_3\big(n-j(3j-1)/2\big)$.*

Using the truncated theta series identity of [2], we prove the following identity which relates $C_3'(n)$ and the function $MP_k(n)$ of Section 4.3.

Theorem 5.3. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1} (-1)^{j(j+1)/2} C_3'\big(n-j(j+1)/2\big) - \sigma\operatorname{mex}(n)\right) = \sum_{j=0}^{n} \sigma\operatorname{mex}(j) MP_k(n-j).$$

Corollary 5.4. *For $n \geqslant 0$, $\sigma\operatorname{mex}(n) = \sum_{j=0}^{\infty} (-1)^{j(j+1)/2} C_3'\big(n-j(j+1)/2\big)$.*

5.2 Four colors for the odd parts

Let $C_4(n)$ be the number of partitions of n using 4 colors for the odd parts. The generating function for $C_4(n)$ is

$$\sum_{n=0}^{\infty} C_4(n)q^n = \frac{1}{(q^2;q^2)_\infty (q;q^2)_\infty^4}.$$

Then, $C_4(n)$ and the function $M_k(n)$ of Section 4.2 are related by the next theorem and its corollary.

Theorem 5.5. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^k \left(C_4(n) + 2\sum_{j=1}^{k} (-1)^j C_4(n-j^2) - \sigma\,\mathrm{mex}(n) \right) = \sum_{j=0}^{n} C_4(j)\overline{M}_k(n-j).$$

Corollary 5.6. *For $n \geqslant 0$, $\sigma\,\mathrm{mex}(n) = C_4(n) + 2\sum_{j=1}^{\infty}(-1)^j C_4(n-j^2)$.*

Note that the partition functions $\sigma\,\mathrm{mex}(n)$ and $C_4(n)$ have the same parity.

5.3 Two colors for parts $\not\equiv 0$ mod 4

Let $C_2(n)$ be the number of partitions of n using two colors for the parts not congruent to 0 mod 4. The generating function for $C_2(n)$ is

$$\sum_{n=0}^{\infty} C_2(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty^2}.$$

The following identity relating $C_2(n)$ and $\overline{M}_k(n)$ follows from the truncated theta identity of [2].

Theorem 5.7. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^k \left(C_2(n) + 2\sum_{j=1}^{k} (-1)^j C_2(n-2j^2) - \sigma\,\mathrm{mex}(n) \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} \overline{M}_k(j)\sigma\,\mathrm{mex}(n-2j).$$

Corollary 5.8. *For $n \geqslant 0$, $\sigma\,\mathrm{mex}(n) = C_2(n) + 2\sum_{j=1}^{\infty}(-1)^j C_2(n-2j^2)$.*

We see that the partition functions $\sigma\,\mathrm{mex}(n)$ and $C_2(n)$ have the same parity.

5.4 Two colors for the odd parts in partitions into parts $\not\equiv 4$ mod 8

We denote by $C_2^*(n)$ the number of partitions of n into parts not congruent to 4 mod 8 using two colors for the odd parts. The generating function for $C_2^*(n)$ is given by

$$\sum_{n=0}^{\infty} C_2^*(n)q^n = \frac{1}{(q^2,q^6,q^8;q^8)_\infty (q;q^2)_\infty^2}.$$

The proof of following theorem relating $C_2^*(n)$ and MP_k again uses results from [2].

Theorem 5.9. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}C_2^*\big(n-j(j+1)\big)-\sigma\,\mathrm{mex}(n)\right)=\sum_{j=0}^{\lfloor n/2\rfloor}MP_k(j)\sigma\,\mathrm{mex}(n-2j).$$

Corollary 5.10. *For $n \geqslant 0$, $\sigma\,\mathrm{mex}(n)=\sum_{j=0}^{\infty}(-1)^{j(j+1)/2}C_2^*\big(n-j(j+1)\big)$.*

References

[1] G. Andrews and M. Merca. "The truncated pentagonal number theorem". *J. Combin. Theory Ser. A* **119**.8 (2012), pp. 1639–1643. Link.

[2] G. Andrews and M. Merca. "Truncated theta series and a problem of Guo and Zeng". *J. Combin. Theory Ser. A* **154** (2018), pp. 610–619. Link.

[3] G. Andrews and D. Newman. "Partitions and the minimal excludant". *Ann. Comb.* **23**.2 (2019), pp. 249–254. Link.

[4] C. Ballantine and M. Merca. "On identities of Watson type". *Ars Math. Contemp.* **17**.1 (2019), pp. 277–290. Link.

[5] C. Ballantine and M. Merca. "Bisected theta series, least r-gaps in partitions, and polygonal numbers". *Ramanujan J.* **52**.2 (2020), pp. 433–444. Link.

[6] C. Ballantine and M. Merca. "Combinatorial Proof of the Minimal Excludant Theorem". 2019. arXiv:1908.06789.

[7] P. Grabner and A. Knopfmacher. "Analysis of some new partition statistics". *Ramanujan J.* **12**.3 (2006), pp. 439–454. Link.

[8] P. Grundy. "Mathematics and games". *Eureka* **2** (1939), pp. 6–9.

[9] R. Sprague. "Über mathematische Kampfspiele". *Tohoku Mathematical Journal, First Series* **41** (1935), pp. 438–444.

[10] J. Sylvester and F. Franklin. "A Constructive Theory of Partitions, Arranged in Three Acts, an Interact and an Exodion". *Amer. J. Math.* **5**.1-4 (1882), pp. 251–330. Link.

[11] A. Yee. "A truncated Jacobi triple product theorem". *J. Combin. Theory Ser. A* **130** (2015), pp. 1–14. Link.