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Recursions for rational q, t-Catalan numbers

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Abstract. We give a simple recursion labeled by binary sequences which computes rational q, t-Catalan power series, both in relatively prime and non relatively prime cases. It is inspired by, but not identical to recursions due to B. Elias, M. Hogancamp, and A. Mellit, obtained in their study of link homology. We also compare our recursion with that of Hogancamp–Mellit's and verify a connection between the Khovanov–Rozansky homology of (M, N) torus links and the rational q, t-Catalan power series for general positive M, N.

Keywords: rational Dyck paths, rational Catalan combinatorics, simultaneous core partitions, invariant integer subsets, semigroups

1 Introduction

In the last decade the rational q, t-Catalan numbers attracted a lot of interest in algebraic combinatorics. Given a pair of integers (M, N), we can consider the set of all partitions which are simultaneously M- and N-cores, that is, none of their hook lengths are divisible by M or N. It is easy to see (e.g. [9]) that such (M, N)-cores are in bijection with the subsets $\Delta \subset \mathbb{Z}_{\geq 0}$ such that $0 \in \Delta$, $\Delta + N \subset \Delta, \Delta + M \subset \Delta$ and $\overline{\Delta} := \mathbb{Z}_{\geq 0} \setminus \Delta$ is finite. We will relax the normalization condition $0 \in \Delta$ and call such subsets (M, N)–*invariant*.

If M and N are coprime, then Anderson [1] proved that the set of (M, N) cores is finite and, in fact, is in bijection with the set $\mathrm{Dyck}(M, N)$ of Dyck paths in the $M \times N$ rectangle. For such paths one can define two statistics `area` and `dinv` and define a bivariate polynomial

$$c_{M,N}(q, t) = \sum_{D \in \mathrm{Dyck}(M,N)} q^{\mathtt{area}(D)} t^{\mathtt{dinv}(D)}.$$

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This polynomial generalizes q,t-Catalan numbers of Garsia and Haiman [6] (which appear at $M = N + 1$) and has lots of remarkable properties, for example, it is symmetric in q and t. The latter follows from the so-called Rational Shuffle Conjecture [12, 4] recently proved by Mellit [16]. The statistic dinv has several equivalent definitions (see Definition 2.4 below); the most elegant one is obtained using the sweep map of Armstrong et. al. [3, 2]. Using the above bijections, one can translate dinv as a statistics on (M, N)-invariant subsets, which was explicitly defined in [8], see Section 2 for details. Thus,

$$c_{M,N}(q,t) = \sum_{\Delta \in I^0_{M,N}} q^{\mathtt{area}(\Delta)} t^{\mathtt{dinv}(\Delta)} = (1-q) \sum_{\Delta \in I_{M,N}} q^{\mathtt{area}(\Delta)} t^{\mathtt{dinv}(\Delta)}. \tag{1.1}$$

where $I_{M,N}$ ($I^0_{M,N}$) denotes the set of finite gap (M, N) invariant subsets (resp. with $0 \in \Delta$).

If M and N are not coprime, then the sets of (M, N) cores and invariant subsets are still in bijection and are infinite, but the relation between them and Dyck paths is more involved. Still, in [10] the authors defined a surjection from $I^0_{M,N}$ to $\mathrm{Dyck}(M, N)$, such that the dinv statistic is constant on the fibers, and the area statistic behaves in a natural and easily controlled way. In this case one can define $c_{M,N}(q,t)$ by the same equation (1.1). However, $c_{M,N}(q,t)$ is no longer a polynomial but a power series. In fact, we will show that it is a rational function with denominator $(1-q)^{d-1}$, where $d = \gcd(M, N)$.

The work of A. Mellit on the Shuffle Conjecture can be extended to show that the polynomial $(1-q)^{d-1} c_{M,N}(q,t)$ is symmetric in q and t in the non relatively prime case as well. However, to our knowledge, this did not appear in the literature yet. Note that in the non relatively prime case the coefficients of $(1-q)^{d-1} c_{M,N}(q,t)$ are not necessarily positive anymore (see Examples 3.1 to 3.3).

One of the most remarkable properties of $c_{M,N}(q,t)$ is its connection to *Khovanov–Rozansky homology* of (M, N) torus links conjectured in [7, 12, 13]. Elias, Hogancamp and Mellit proved this connection in various special cases (see [5, 14, 17]). In all those cases, the comparison between the power series $c_{M,N}(q,t)$ and the Poincaré power series of this homology is proved by obtaining certain recursions on the topological side and then verifying them on combinatorial side. In a recent preprint [15] Hogancamp and Mellit introduced a recursion for the Khovanov–Rozansky homology that works for any (M, N) torus link. The terms in these recursions are labeled by binary sequences of varying length.

We introduce a recursion on the related power series $P_{\boldsymbol{u}}(q,t)$, which are labeled by binary sequences of fixed length $M + N$. Our main objective is to understand these recursions as clearly as possible in combinatorial terms, and the recursions become quite natural expressed in terms of the sequences $\boldsymbol{u}$. More precisely, given a finite gap (M, N) invariant subset Δ, we consider a length $M + N$ binary sequence $\boldsymbol{u} = \boldsymbol{u}(\Delta)$ recording the characteristic function of the intersection $\Delta \cap [0, M+N-1]$. We define

$$P_{\boldsymbol{u}}(q,t) := \sum_{\Delta \in I_{M,N},\, \boldsymbol{u}(\Delta)=\boldsymbol{u}} q^{\mathtt{area}(\Delta)} t^{\mathtt{codinv}(\Delta)},$$

where codinv is closely related to dinv (see Definition 2.4 below). Note that one gets

$$P_{0^{M+N}}(q,t) = \frac{q^{M+N} t^{\delta(M,N)}}{1-q} c_{M,N}(q,t).$$

We use it to prove the connection between $c_{M,N}(q,t)$ and the Poincaré power series of the Khovanov–Rozansky homology of (M,N) torus links for general (M,N). For a more detailed version of this extended abstract, see [11].

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2 The recursion

Let $(M,N) = (dm, dn)$ be a pair of positive integers, where m and n are relatively prime, so $d = \gcd(M,N)$.

Definition 2.1. The set $I_{M,N}$ of finite gap M,N-invariant subsets is defined by

$$I_{M,N} := \{\Delta \subset \mathbb{Z}_{\geq 0} : \Delta + N \subset \Delta, \Delta + M \subset \Delta, \sharp\overline{\Delta} < \infty\},$$

where $\Delta + N := \{k \in \mathbb{Z} : k - N \in \Delta\}$ denotes the shift of Δ by N, $\overline{\Delta} := \mathbb{Z}_{\geq 0} \setminus \Delta$ is the complement to Δ, and $\sharp\overline{\Delta}$ is the number of elements in $\overline{\Delta}$. The elements of $\overline{\Delta}$ are often called *gaps* in Δ. We will more simply refer to such Δ as invariant subsets.

We define statistics area and codinv on the invariant subsets. The area statistic simply counts the number of gaps in Δ. The statistics dinv and codinv are more involved.

Definition 2.2. We set $\mathtt{area}(\Delta) := \sharp\overline{\Delta}$.

Definition 2.3. Let $\Delta \in I_{M,N}$ be an invariant subset. The set $\mathtt{Ngen}(\Delta)$ of N-generators of Δ is defined by

$$\mathtt{Ngen}(\Delta) := \Delta \setminus (\Delta + N) = \{\mathtt{g} \in \Delta : \mathtt{g} - N \notin \Delta\}.$$

Definition 2.4. We set

$$\mathtt{codinv}(\Delta) = \sum_{\mathtt{g}\in\mathtt{Ngen}(\Delta)} [\mathtt{g}, \mathtt{g}+M-1] \cap \overline{\Delta}, \qquad \mathtt{dinv} = \delta(N,M) - \mathtt{codinv}(\Delta),$$

where $\delta(N,M) := \frac{1}{2}(NM - N - M + \gcd(M,N))$ and we use the *integer interval* notation:

$$[\mathtt{g}, \mathtt{g}+k] := \{\mathtt{g}, \mathtt{g}+1, \ldots, \mathtt{g}+k\}.$$

Definition 2.5. Let $\boldsymbol{u} = (u_0, \ldots, u_{N+M-1})$ be a sequence of 0's and 1's. We set

$$I_{\boldsymbol{u}} := \{\Delta \in I_{M,N} : \forall\, 0 \le i < N+M,\ i \in \Delta \Leftrightarrow u_i = 1\}.$$

We say that a sequence $\boldsymbol{u}$ is *admissible* if $I_{\boldsymbol{u}} \neq \emptyset$. Note that we number the entries of $\boldsymbol{u}$ starting at 0.

Definition 2.6. Let the power series $P_{\boldsymbol{u}}(q,t)$ be given by

$$P_{\boldsymbol{u}} = P_{\boldsymbol{u}}(q,t) := \sum_{\Delta\in I_{\boldsymbol{u}}} q^{\mathtt{area}(\Delta)} t^{\mathtt{codinv}(\Delta)}.$$

Observe $P_{1^{M+N}} = 1$, and this is the base case of our recursion for computing $P_{\boldsymbol{u}}$.

Lemma 2.7. *One has*

$$P_{0^{M+N}}(q,t) = q^{M+N} \sum_{\Delta\in I_{M,N}} q^{\mathtt{area}(\Delta)} t^{\mathtt{codinv}(\Delta)},$$

and

$$c_{M,N}(q,t) = q^{-N-M} t^{\delta(N,M)} (1-q) P_{0^{M+N}}(q, t^{-1}).$$

Proof. Indeed, for $\boldsymbol{u} = 0^{M+N}$ the set $I_{\boldsymbol{u}}$ consists of all (M,N)–invariant subsets which do not intersect with $[0, M+N-1]$. All such subsets are obtained from (M,N)–invariant subsets in $I_{M,N}$ by shift by $(M+N)$. It is easy to see that the shift does not change $\mathtt{codinv}$ and changes $\mathtt{area}$ by $(M+N)$. The second formula now follows from Equation (1.1) and the relation $\mathtt{dinv}(\Delta) + \mathtt{codinv}(\Delta) = \delta(N,M)$. □

Definition 2.8. Define $\rho : I_{M,N} \to I_{M,N}$ to be the shift map given by

$$\rho(\Delta) = \begin{cases} \Delta - 1 & \text{if } 0 \notin \Delta \\ (\Delta \setminus \{0\}) - 1 & \text{if } 0 \in \Delta. \end{cases}$$

Definition 2.9. Let $\boldsymbol{u} \in \{0,1\}^{M+N}$. We define

$$\lambda(\boldsymbol{u}) := \sum_{i=0}^{M-1} (u_{i+N} - u_i).$$

If $\Delta \in I_{\boldsymbol{u}}$ we set $\lambda(\Delta) := \lambda(\boldsymbol{u})$.

Theorem 2.10. *Let $\boldsymbol{u} = (u_0, \dots, u_{N+M-1})$ be an admissible sequence. Let also*

$$\boldsymbol{v} = (u_1, \dots, u_{N+M-1}, 1), \qquad \boldsymbol{v}' = (u_1, \dots, u_{N+M-1}, 0).$$

The power series $P_{\boldsymbol{u}}$ satisfy the following recurrence relation:

$$P_{\boldsymbol{u}} = \begin{cases} q(P_{\boldsymbol{v}} + P_{\boldsymbol{v}'}), & \text{if } u_0 = u_N = u_M = 0, \\ qP_{\boldsymbol{v}}, & \text{if } u_0 = 0 \text{ and } u_N + u_M > 0, \\ t^{\lambda(\boldsymbol{u})} P_{\boldsymbol{v}}, & \text{if } u_0 = u_N = u_M = 1. \end{cases} \tag{2.1}$$

Proof. With respect to the statistics above, the shift map ρ of Definition 2.8 has the following properties:

(a) If $0 \notin \Delta$, then $\mathtt{area}(\rho(\Delta)) = \mathtt{area}(\Delta) - 1$, while if $0 \in \Delta$, then $\mathtt{area}(\rho(\Delta)) = \mathtt{area}(\Delta)$.

(b) If at least one of the numbers N and M belongs to Δ, then $N + M - 1 \in \rho(\Delta)$, while if neither N nor M are in Δ, then either possibility $N + M - 1 \in \rho(\Delta)$ or $N + M - 1 \notin \rho(\Delta)$ may occur.

(c) If $0 \notin \Delta$, then $\mathtt{codinv}(\rho(\Delta)) = \mathtt{codinv}(\Delta)$.

(d) If $0 \in \Delta$, $\mathtt{codinv}(\rho(\Delta)) = \mathtt{codinv}(\Delta) - \sharp\left([0, M-1] \cap \overline{\Delta}\right) + \sharp\left([N, N+M-1] \cap \overline{\Delta}\right)$.

To prove part (d), one should observe that all the N-generators of Δ, except 0, are simply shifted down by one in $\rho(\Delta)$, while retaining the same contributions to $\mathtt{codinv}$. The N-generator $0 \in \Delta$ get replaced by $N - 1 \in \rho(\Delta)$. The contribution to $\mathtt{codinv}$ changes accordingly, and this change is measured by $\lambda(\boldsymbol{u})$ for $\Delta \in I_{\boldsymbol{u}}$. $\square$

Definition 2.11. We visualize the recursion (2.1) using the *decision tree*. Each node corresponds to a binary sequence $\boldsymbol{u}$ and the edges connect $\boldsymbol{u}$ with $\boldsymbol{v}$ and $\boldsymbol{v}'$ and are labeled by the corresponding coefficients:

case 1

$$\begin{tikzcd} & 0\boldsymbol{w} \arrow[dl, "q"'] \arrow[dr, "q"] & \\ \boldsymbol{w}1 & & \boldsymbol{w}0 \end{tikzcd}$$

case 2

$$\begin{tikzcd} 0\boldsymbol{w} \arrow[d, "q"] \\ \boldsymbol{w}1 \end{tikzcd}$$

case 3

$$\begin{tikzcd} 1\boldsymbol{w} \arrow[d, "t^{\lambda(1\boldsymbol{w})}"] \\ \boldsymbol{w}1 \end{tikzcd}$$

Here $\boldsymbol{w} \in \{0,1\}^{M+N-1}$, $\boldsymbol{u} = 0\boldsymbol{w}$ in cases 1 and 2 and $\boldsymbol{u} = 1\boldsymbol{w}$ in case 3, $\boldsymbol{v} = \boldsymbol{w}1$ and $\boldsymbol{v}' = \boldsymbol{w}0$. We color edges and labels in case 3 in red to emphasize that these carry powers of t while all other (black) edges are labeled by q.

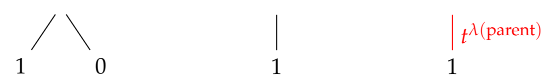
Remark 2.12. If we never identify vertices with the same label, we will indeed get an infinite tree. However, it is convenient to make the graph finite by keeping each 1^{M+N} as a terminal vertex, and identifying the pairs of vertices with the same label, whenever one vertex is a predecessor of another. This leads to directed cycles, which we analyze below. See also the examples in Section 3.

Theorem 2.13. *The recursion in Theorem 2.10 has a unique solution given the initial condition $P_{1^{M+N}}(q,t) = 1$. Moreover, for any sequence $\boldsymbol{u}$ the power series $P_{\boldsymbol{u}}(q,t)$ can be expressed as a rational function with the denominator $\prod_{i=1}^{d}(1-q^{\ell_i})$, where $0 < \ell_i < d$ for all i.*

We will see in Section 4 that the denominator of $P_{0^{N+M}}$ can be reduced to $(1-q)^d$.

3 Examples

In this section we present some examples of decision trees defined in Definition 2.11 and Remark 2.12. As all black edges have weight q, we can drop this label. Further, it is sometimes convenient to just record the (new) rightmost entry at each node, simplifying the picture in Definition 2.11 as follows:



Also, we will replace all branches with a single terminal vertex by the corresponding monomial. We will refer to the result as to "compact decision tree".

Example 3.1. The decision tree for $(M,N) = (2,2)$ is shown in Figure 1. We compute

$$P_{1011} = qt, \quad P_{0101} = q(P_{1011} + P_{0101}), \quad P_{0001} = q^3 + q^2 P_{0101}, \quad P_{0000} = qP_{0001} + qP_{0000}.$$

Finally,

$$P_{0101} = \frac{qP_{1011}}{1-q}, \quad P_{0000} = \frac{qP_{0001}}{1-q} = \frac{q^4}{1-q} + \frac{q^5 t}{(1-q)^2}.$$

Observe the q,t-symmetry of $(1-q)c_{2,2}(q,t) = q^{-4}t(1-q)^2 P_{0000}(q,t^{-1}) = q + t - qt$.

Example 3.2. For $(M,N) = (3,3)$ we compute

$$q^{-6}P_{000000} = \frac{1+qt}{1-q} + \frac{qt^2 + 2q^2t^2}{(1-q)^2} + \frac{q^3t^3}{(1-q)^3}.$$

Observe the q,t-symmetry of

$$\begin{aligned}(1-q)^2 c_{3,3}(q,t) &= t^3 q^{-6}(1-q)^3 P_{000000}(q,t^{-1}) \\ &= q^3t^2 + q^2t^3 - 2q^3t - 2qt^3 + q^3 + t^3 + q^2t + qt^2 - 2q^2t^2 + qt\end{aligned}$$

Example 3.3. The decision tree for $(M,N) = (4,6)$ is shown in Figure 2. We will use the shorthand notation $P_{0^{10}} := P_{0000000000}$ and $P_{(01)^5} = P_{0101010101}$ and so on. We compute

$$P_{(01)^5} = qP_{(01)^5} + q^2t(q^3t + q^4t^6) \qquad P_{0^{10}} = qP_{0^{10}} + qP_{0^91}.$$

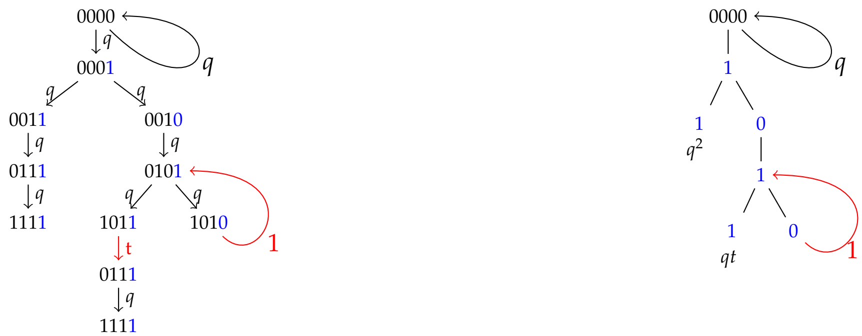


Figure 1: Decision tree for $(M,N) = (2,2)$ and the corresponding compact decision tree on the right.

Hence

$$q^{-10}P_{0^{10}} = \frac{1}{1-q}\Big(q^7t^7 + q^6t^6 + q^6t^7 + q^5t^5 + q^5t^6 + q^4t^4 + q^4t^5 + q^4t^6 + q^3t^3 + q^3t^4 + 2q^3t^5 + q^2t^2 + q^2t^3 + 2q^2t^4 + qt + qt^2 + qt^3 + 1\Big) + \frac{1}{(1-q)^2}(q^4t^6 + q^5t^7 + q^7t^7 + q^8t^8).$$

Observe the q,t-symmetry of

$$\begin{aligned}(1-q)c_{4,6}(q,t) &= t^8(1-q)^2q^{-10}P_{0^{10}}(q,t^{-1}) \\ &= -q^8t - qt^8 + q^8 + t^8 - q^7t^2 - q^2t^7 + q^7t + qt^7 - q^6t^3 - q^3t^6 + q^6t + \\ &\quad qt^6 + q^5t + qt^5 - q^5t^4 - q^4t^5 - q^4t^3 - q^3t^4 + 2q^4t^2 + 2q^2t^4 + 2q^3t^3.\end{aligned}$$

4 Comparison with the work of Hogancamp and Mellit.

Our next goal is match this recursion with the $a = 0$ specialization of the following recursion due to Hogancamp and Mellit [15].

Definition 4.1 ([15])**.** The power series $R_{\boldsymbol{x},\boldsymbol{y}}(q,t,a)$ in variables q,t and a depend on a pair of words $\boldsymbol{x}$ and $\boldsymbol{y}$ in the alphabet $\{0,\times\}$. These power series satisfy the following recursive relations:

$$R_{0\boldsymbol{x},0\boldsymbol{y}} = t^{-|\boldsymbol{x}|}R_{\boldsymbol{x}\times,\boldsymbol{y}\times} + qt^{-|\boldsymbol{x}|}R_{\boldsymbol{x}0,\boldsymbol{y}0},$$
$$R_{\times\boldsymbol{x},0\boldsymbol{y}} = R_{\boldsymbol{x}\times,\boldsymbol{y}},$$
$$R_{0\boldsymbol{x},\times\boldsymbol{y}} = R_{\boldsymbol{x},\boldsymbol{y}\times},$$
$$R_{\times\boldsymbol{x},\times\boldsymbol{y}} = (t^{|\boldsymbol{x}|} + a)R_{\boldsymbol{x},\boldsymbol{y}},$$
$$R_{\varnothing,\varnothing} = 1,$$

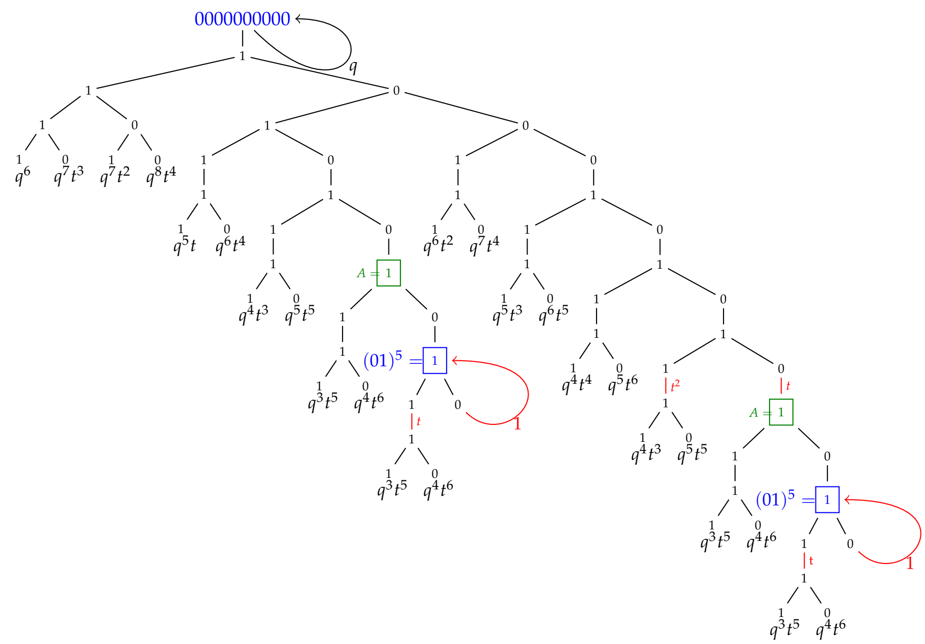


Figure 2: Compact decision tree for $(M, N) = (4, 6)$.

where $|\boldsymbol{x}|$ denotes the number of $\times$'s in $\boldsymbol{x}$.

Remark 4.2. Our recursion differs from the one in [15] by reversing the order in both sequences $\boldsymbol{x}, \boldsymbol{y}$.

In order to compare recursions, we will need to go through certain reformulations and also adjust both the `area` and `codinv` statistics. First, we will need to replace the binary sequence $\boldsymbol{u}$ of length $N + M$ by two sequences $(\boldsymbol{v}, \boldsymbol{w})$ in the alphabet $\{0, \bullet, \times\}$ of lengths M and N respectively. Sequence $\boldsymbol{v}$ records gaps (encoded by 0), N-generators (encoded by $\times$), and the rest of the elements of Δ (encoded by $\bullet$) on the interval $[N, N + M - 1]$. Similarly, sequence $\boldsymbol{w}$ records gaps, M-generators, and the rest of the elements of Δ on the interval $[M, N + M - 1]$. More formally, one gets the following definition:

Definition 4.3. Let $\boldsymbol{u} = (u_0, \ldots, u_{N+M-1}) \in \{0, 1\}^{N+M}$ be an admissible binary sequence. Define $\boldsymbol{v} = (v_0, \ldots, v_{M-1}) \in \{0, \bullet, \times\}^M$ and $\boldsymbol{w} = (w_0, \ldots, w_{N-1}) \in \{0, \bullet, \times\}^N$ as follows:

1. $v_i = 0$ whenever $u_{N+i} = 0$,
2. $v_i = \bullet$ whenever $u_{N+i} = u_i = 1$,
3. $v_i = \times$ whenever $u_{N+i} = 1$ $u_i = 0$.

1. $w_i = 0$ whenever $u_{M+i} = 0$,
2. $w_i = \bullet$ whenever $u_{M+i} = u_i = 1$,
3. $w_i = \times$ whenever $u_{M+i} = 1,\ u_i = 0$.

We say that a pair of sequences $(\boldsymbol{v}, \boldsymbol{w})$ is admissible if $\boldsymbol{v}$ and $\boldsymbol{w}$ are obtained from an admissible binary sequence $\boldsymbol{u}$ according to the rule above.

Clearly, the pair of sequences $\boldsymbol{v}, \boldsymbol{w}$ fully determine the binary sequence $\boldsymbol{u}$. In other words, the above Definition 4.3 describes a map

$$\mathbf{b} : \{0,1\}^{M+N} \to \{0, \bullet, \times\}^M \times \{0, \bullet, \times\}^N$$

and it is injective when restricted to the domain of admissible sequences.

Definition 4.4. By abusing notation, we set $I_{\boldsymbol{v},\boldsymbol{w}} := I_{\boldsymbol{u}}$ and $P_{\boldsymbol{v},\boldsymbol{w}}(q,t) := P_{\boldsymbol{u}}(q,t)$.

$P_{\boldsymbol{v},\boldsymbol{w}}(q,t)$ satisfies a recursion, given in Theorem 4.7 below, that looks very similar to the $a = 0$ version of the Hogancamp–Mellit [15] recursion, but it is not exactly the same. In order to get an exact match, let us make the following adjustments to the statistics:

Definition 4.5. Define statistics $\mathtt{area}'$ and $\mathtt{codinv}'$ on the set $I_{M,N}$ of (M,N)-invariant subsets in $\mathbb{Z}_{\geq 0}$ by

$$\mathtt{area}'(\Delta) = \sharp(\overline{\Delta} \cap \mathbb{Z}_{\geq N+M}), \qquad \text{and}$$

$$\mathtt{codinv}'(\Delta) = \sum_{\mathtt{g} \in \mathtt{Ngen}(\Delta)} \sharp\left([\mathtt{g}, \mathtt{g} + M - 1] \cap \overline{\Delta} \cap \mathbb{Z}_{\geq N+M}\right) - \frac{\lambda(\Delta)(\lambda(\Delta)-1)}{2}$$

Definition 4.6. As before, let $\boldsymbol{u} = (u_0, \ldots, u_{N+M-1}) \in \{0,1\}^{N+M}$ be an admissible binary sequence. The generating series $Q_{\boldsymbol{u}}(q,t)$ is defined by

$$Q_{\boldsymbol{u}}(q,t) := \sum_{\Delta \in I_{\boldsymbol{u}}} t^{-\mathtt{codinv}'(\Delta)} q^{\mathtt{area}'(\Delta)}.$$

When $(\boldsymbol{v}, \boldsymbol{w}) = \mathbf{b}(\boldsymbol{u})$, we also set

$$Q_{\boldsymbol{v},\boldsymbol{w}}(q,t) := Q_{\boldsymbol{u}}(q,t).$$

Note that for any $\Delta \in I_{0^{M+N}}$ one gets that $\mathtt{area}'(\Delta) = -N - M + \mathtt{area}(\Delta)$ and that $\mathtt{codinv}'(\Delta) = \mathtt{codinv}(\Delta)$. Therefore,

$$Q_{0^M,0^N}(q,t) = q^{-N-M} P_{0^M,0^N}(q,t^{-1}). \tag{4.1}$$

Theorem 4.7. *The following recursions hold:*

$$\begin{aligned} P_{0\boldsymbol{v},0\boldsymbol{w}} &= q(P_{\boldsymbol{v}\times,\boldsymbol{w}\times} + P_{\boldsymbol{v}0,\boldsymbol{w}0}), \\ P_{\times\boldsymbol{v},0\boldsymbol{w}} &= qP_{\boldsymbol{v}\times,\boldsymbol{w}\bullet}, \\ P_{0\boldsymbol{v},\times\boldsymbol{w}} &= qP_{\boldsymbol{v}\bullet,\boldsymbol{w}\times}, \\ P_{\times\boldsymbol{v},\times\boldsymbol{w}} &= qP_{\boldsymbol{v}\bullet,\boldsymbol{w}\bullet}, \\ P_{\bullet\boldsymbol{v},\bullet\boldsymbol{w}} &= t^{|\boldsymbol{v}|}P_{\boldsymbol{v}\bullet,\boldsymbol{w}\bullet}. \end{aligned} \qquad \begin{aligned} Q_{0\boldsymbol{v},0\boldsymbol{w}} &= t^{-|\boldsymbol{v}|}Q_{\boldsymbol{v}\times,\boldsymbol{w}\times} + qt^{-|\boldsymbol{v}|}Q_{\boldsymbol{v}0,\boldsymbol{w}0}, \\ Q_{\times\boldsymbol{v},0\boldsymbol{w}} &= Q_{\boldsymbol{v}\times,\boldsymbol{w}\bullet}, \\ Q_{0\boldsymbol{v},\times\boldsymbol{w}} &= Q_{\boldsymbol{v}\bullet,\boldsymbol{w}\times}, \\ Q_{\times\boldsymbol{v},\times\boldsymbol{w}} &= t^{|\boldsymbol{v}|}Q_{\boldsymbol{v}\bullet,\boldsymbol{w}\bullet}, \\ Q_{\bullet\boldsymbol{v},\bullet\boldsymbol{w}} &= Q_{\boldsymbol{v}\bullet,\boldsymbol{w}\bullet}, \end{aligned}$$

where the recursions on the left are equivalent to those in (2.1).

The final observation is that in the recursion for Q one can completely ignore the $\bullet$'s.

Definition 4.8. Let ϕ be the map from the words in the alphabet $\{0, \bullet, \times\}$ to the words in the alphabet $\{0, \times\}$ given by simply forgetting all $\bullet$s.

Theorem 4.9. *Let $\boldsymbol{x} = \phi(\boldsymbol{v})$ and $\boldsymbol{y} = \phi(\boldsymbol{w})$. Then*

$$R_{\boldsymbol{x},\boldsymbol{y}}(q, t, 0) = Q_{\boldsymbol{v},\boldsymbol{w}}(q, t).$$

By Theorem 4.9 and Lemma 2.7 we immediately get the following.

Corollary 4.10. *We have*

$$R_{0^M,0^N}(q, t, 0) = Q_{0^M,0^N}(q, t) = q^{-N-M}P_{0^{M+N}}(q, t^{-1}) = \frac{t^{-\delta(N,M)}}{1-q}c_{M,N}(q, t).$$

The Hogancamp–Mellit recursion for the series $R_{\boldsymbol{x},\boldsymbol{y}}(q, t, 0)$ as well as for the full series $R_{\boldsymbol{x},\boldsymbol{y}}(q, t, a)$ has an important advantage over the recursion for $P_{\boldsymbol{u}}(q, t)$: one can use it to show that the denominator of $R_{\boldsymbol{x},\boldsymbol{y}}(q, t, a)$ can be simplified to a power of $(1 - q)$ as compared to the denominator $\prod_{i=1}^{d}(1 - q^{\ell_i}), 0 < \ell_i < d$ predicted by Theorem 2.13 before reducing expressions. Indeed, the decision tree for the recursion for $R_{\boldsymbol{x},\boldsymbol{y}}(q, t, a)$ can only have loops of length one, producing $(1 - q)$ in the denominator, while the decision trees for $P_{\boldsymbol{u}}(q, t)$ might include loops of lengths equal to any divisor of d, producing factors $(1 - q^l)$, $l < d$ in the denominator.

Corollary 4.11. *The power series $P_{0^{M+N}}(q, t)$ can be expressed as a rational function with denominator equal to $(1 - q)^d$, where $d = \gcd(N, M)$.*

5 Higher a-degrees

The Hogancamp–Mellit construction involves an extra variable a. One can enhance the generating series and the recurrence relations described in Section 2 to recover the full three variable functions in the following way.

Definition 5.1. Let $\Delta \in I_{M,N}$ be an invariant subset. A number $k \in \overline{\Delta}$ is called a *double cogenerator* of Δ if $k + N \in \Delta$ and $k + M \in \Delta$. Let $\texttt{Cogen}(\Delta) \subset \overline{\Delta}$ denote the set of all double cogenerators of Δ.

Remark 5.2. Note that we only consider **non-negative** double cogenerators. If $\Delta \in I_{0\ldots0}$, then all cogenerators are positive, so it does not matter for such Δ's. However, this choice will matter for the recurrence relations.

We will also need the following statistic:

Definition 5.3. Let $\Delta \in I_{N,M}$ be an invariant subset, and $k \in \mathbb{Z}$ be an integer. We set

$$\lambda_k(\Delta) := \sharp(\mathtt{Ngen}(\Delta) \cap [k+N+1, k+N+M]).$$

Now we are ready to define the enhancement of the counting function from Section 2.

Definition 5.4. Let the power series $\hat{P}_{\boldsymbol{u}}(q,t,a)$ be given by

$$\hat{P}_{\boldsymbol{u}} := \sum_{\Delta \in I_{\boldsymbol{u}}} q^{\mathtt{area}(\Delta)} t^{\mathtt{codinv}(\Delta)} \prod_{k \in \mathtt{Cogen}(\Delta)} \left(1 + a t^{\lambda_k(\Delta)}\right).$$

Theorem 5.5. *Let $\boldsymbol{u} = (u_0, \ldots, u_{N+M-1})$ be an admissible sequence. Let also*

$$\begin{aligned} \boldsymbol{v} &= (u_1, \ldots, u_{N+M-1}, 1), \\ \boldsymbol{v}' &= (u_1, \ldots, u_{N+M-1}, 0). \end{aligned}$$

The power series $\hat{P}_{\boldsymbol{u}}$ satisfy the following recurrence relation:

$$\hat{P}_{\boldsymbol{u}} = \begin{cases} q(P_{\boldsymbol{v}} + P_{\boldsymbol{v}'}), & \text{if } u_0 = u_N = u_M = 0, \\ qP_{\boldsymbol{v}}, & \text{if } u_0 = 0 \text{ and } u_N + u_M = 1, \\ q\left(1 + a t^{\lambda(\boldsymbol{v})}\right) P_{\boldsymbol{v}}, & \text{if } u_0 = 0 \text{ and } u_N = u_M = 1, \\ t^{\lambda(\boldsymbol{u})} P_{\boldsymbol{v}}, & \text{if } u_0 = u_N = u_M = 1. \end{cases}$$

Similar to Section 4, in order to match with the recursion of Hogancamp and Mellit, we switch to the adjusted statistics $\mathtt{area}'$, $\mathtt{codinv}'$, and $\mathtt{dinv}'$.

Definition 5.6. As before, let $\boldsymbol{u} = (u_0, \ldots, u_{N+M-1}) \in \{0,1\}^{N+M}$ be an admissible binary sequence. The generating series $\hat{Q}_{\boldsymbol{u}}(q,t,a)$ is defined by

$$\hat{Q}_{\boldsymbol{u}}(q,t,a) := \sum_{\Delta \in I_{\boldsymbol{u}}} t^{-\mathtt{codinv}'(\Delta)} q^{\mathtt{area}'(\Delta)} \prod_{k \in \mathtt{Cogen}(\Delta)} \left(1 + a t^{-\lambda_k(\Delta)}\right).$$

We also set $\hat{Q}_{\boldsymbol{v},\boldsymbol{w}}(q,t,a) := \hat{Q}_{\boldsymbol{u}}(q,t,a)$, where the sequences $\boldsymbol{v} = (v_0, \ldots, v_{M-1}) \in \{0, \bullet, \times\}^M$ and $\boldsymbol{w} = (w_0, \ldots, w_{N-1}) \in \{0, \bullet, \times\}^N$ are determined in the same way as in Definition 4.3, i.e. $(\boldsymbol{v}, \boldsymbol{w}) = \mathbf{b}(\boldsymbol{u})$.

Note that similar to Section 4, one gets $\hat{Q}_{0^M,0^N}(q,t,a) = q^{-N-M}\hat{P}_{0^M,0^N}(q,t^{-1},a)$.

Theorem 5.7. *The following recursion holds:*

$$\begin{aligned} \hat{Q}_{0\boldsymbol{v},0\boldsymbol{w}} &= t^{-|\boldsymbol{v}|}\hat{Q}_{\boldsymbol{v}\times,\boldsymbol{w}\times} + qt^{-|\boldsymbol{v}|}\hat{Q}_{\boldsymbol{v}0,\boldsymbol{w}0}, & \hat{Q}_{\times\boldsymbol{v},\times\boldsymbol{w}} &= (t^{|\boldsymbol{v}|} + a)\hat{Q}_{\boldsymbol{v}\bullet,\boldsymbol{w}\bullet}, \\ \hat{Q}_{\times\boldsymbol{v},0\boldsymbol{w}} &= \hat{Q}_{\boldsymbol{v}\times,\boldsymbol{w}\bullet}, & \hat{Q}_{\bullet\boldsymbol{v},\bullet\boldsymbol{w}} &= \hat{Q}_{\boldsymbol{v}\bullet,\boldsymbol{w}\bullet}. \\ \hat{Q}_{0\boldsymbol{v},\times\boldsymbol{w}} &= \hat{Q}_{\boldsymbol{v}\bullet,\boldsymbol{w}\times}, \end{aligned}$$

Corollary 5.8. *Let $\boldsymbol{x} = \phi(\boldsymbol{v})$ and $\boldsymbol{y} = \phi(\boldsymbol{w})$. Then $R_{\boldsymbol{x},\boldsymbol{y}}(q,t,a) = \hat{Q}_{\boldsymbol{v},\boldsymbol{w}}(q,t,a)$.*

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