Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #28, 12 pp.
Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

Poset topology of s-weak order via SB-labelings

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Abstract. Ceballos and Pons generalized weak order on permutations to a partial order on certain labeled trees, thereby introducing a new class of lattices called s-weak order. They also generalized the Tamari lattice by defining a particular sublattice of s-weak order called the s-Tamari lattice. We prove that the homotopy type of each open interval in s-weak order and in the s-Tamari lattice is either a ball or sphere. We do this by giving s-weak order and the s-Tamari lattice a type of edge labeling known as an SB-labeling. We characterize which intervals are homotopy equivalent to spheres and which are homotopy equivalent to balls; we also determine the dimension of the spheres for the intervals yielding spheres.

Keywords: Poset topology, s-weak order, s-Tamari lattice, SB-labelings

1 Introduction

In [3], Ceballos and Pons introduced a partial order called s**-weak order** on certain labeled trees known as s**-decreasing trees**. They observed this generalizes weak order on permutations. They proved s-weak order is a lattice. They also found a particular class of s-decreasing trees which play the role of 231-avoiding permutations. They thereby introduced a sublattice of s-weak order called the s-Tamari lattice which generalizes the Tamari lattice. We prove that the order complex of each open interval in s-weak order has the homotopy type of either a ball or sphere of some dimension. We prove the same statement for each open interval in the s-Tamari lattice. In both cases, we do this using the tool of SB-labelings developed by Hersh and Mészáros in [7]. Our result generalizes Hersh and Mészáros' result that weak order on permutations and the classical Tamari lattice admit SB-labelings, but our labelings are distinct from theirs in the classical case. In s-weak order and the s-Tamari lattice, these spheres are not always top dimensional which demonstrates that these posets are not always shellable. We intrinsically characterize which intervals in s-weak order and the s-Tamari lattice are homotopy equivalent to spheres and which are homotopy equivalent to balls. We also determine the dimension of the spheres for the intervals yielding homotopy spheres. As a corollary, we deduce that the Möbius functions of s-weak order and the s-Tamari

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lattice only take values in $\{-1, 0, 1\}$. Additionally, an SB-labeling implies that distinct sets of atoms in an interval have distinct joins. Further, topological understanding of a poset often strongly restricts the structure of chains in the poset. While one might wonder whether s-weak order always gives a Cambrian lattice of a finite Coxeter group, we show this is not the case.

Ceballos and González studied **s-increasing** trees which are equivalent to s-decreasing trees while studying Signature Catalan combinatorics in [1]. The authors also show that some instances of s-decreasing trees generalize certain pattern avoiding multiset permutations known as Stirling permutations which were introduced by Gessel and Stanley in [4]. Part of Ceballos and Pons' interest in s-weak order comes from geometry. They conjecture that the Hasse diagrams of s-weak order are the 1-skeleta of polytopal subdivisions of polytopes. They call these potential polytopal complexes **s-permutahedra**. They also conjecture that in particular cases the polytopes they are subdividing are classical permutahedra. Our result of an SB-labeling for s-weak order, though it considers these lattices from a topological perspective, seems to provide two pieces of evidence for Ceballos and Pons' conjecture. The first piece of evidence is that the Hasse diagrams of many lattices which admit SB-labelings can be realized as the 1-skeleta of polytopes. The second comes from the fact that Ceballos and Pons' geometric perspective is somewhat similar in flavor to one point of view in Hersh's work in [6]. Hersh studied posets which arise as the 1-skeleta of simple polytopes via directing edges by some cost vector. In particular, Hersh's Theorem 4.9 in [6] proves that all open intervals in lattices which are realizable as such 1-skeleta of simple polytopes are either homotopy balls or spheres.

Ceballos and Pons' interest in the s-Tamari lattice also stems from a geometric viewpoint. They showed that the s-Tamari lattice is isomorphic to another generalization of the classical Tamari lattice, namely the ν-Tamari lattice introduced by Préville-Ratelle and Viennot in [8]. The geometry of the ν-Tamari lattice was recently studied by Ceballos, Padrol, and Sarmiento in [2]. Similarly to how the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron, the Hasse diagram of the ν-Tamari lattice is the 1-skeleta of a polytopal subdivision of a polytope. Thus, the s-Tamari lattice also has such a geometric realization. In the context of the s-Tamari lattice, Ceballos and Pons call these polytopal complexes **s-associahedra**. Further, they conjecture that in particular cases s-associahedra can be obtained from the s-permutahedra by deleting certain facets. The fact that the s-Tamari lattice admits an SB-labeling and has a geometric realization as a polytopal complex seems to strengthen the evidence given by our result for Ceballos and Pons' conjecture of such realizations of s-permutahedra. Additionally, our result contributes two new classes of lattices which admit SB-labelings.

Section 2 provides the necessary background on posets, s-decreasing trees, s-weak order, and the s-Tamari lattice. We largely follow the notation and definitions of [3]. We also observe that s-weak order is not generally a Cambrian lattice. Section 2 reviews the notion of SB-labeling as well. Section 3 is where we sketch the proofs of our main results, most notably giving SB-labelings for s-weak order and the s-Tamari lattice.

2 Background

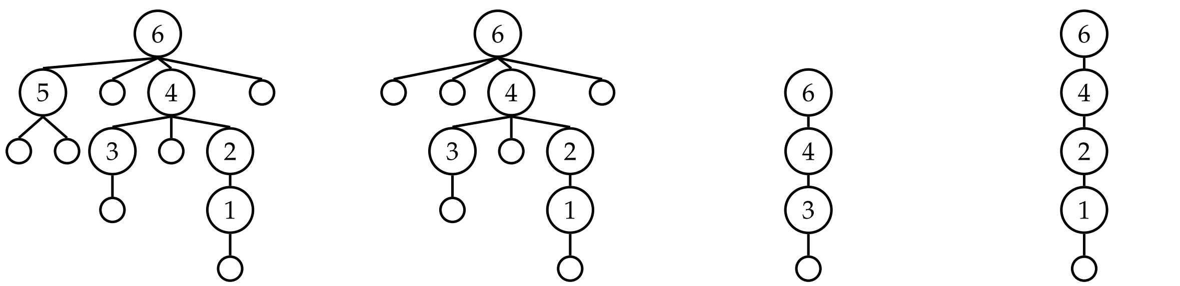
2.1 Background on posets

Let $(P, \leq)$ be a poset. For $x \leq y \in P$, the **closed interval** from x to y is the set $[x, y] = \left\{ z \in P \;\middle|\; x \leq z \leq y \right\}$. **The open interval from** x **to** y is defined analogously and denoted (x, y). We say that y **covers** x, denoted $x \lessdot y$, if $x \leq z \leq y$ implies $z = x$ or $z = y$. P is a **lattice** if each pair $x, y \in P$ has a unique least upper bound, denoted $x \vee y$, and a unique greatest lower bound, denoted $x \wedge y$. We denote by $\hat{0}$ (respectively $\hat{1}$) the unique minimal (respectively unique maximal) element of a finite lattice. The elements which cover $\hat{0}$ are called **atoms**. For $x, y \in P$ with $x < y$, a k**-chain from** x **to** y in P is a subset $C = \{x_0, x_1, \ldots, x_k\} \subset P$ such that $x = x_0 < x_1 < \cdots < x_k = y$. A chain C is said to be **saturated** if $x_i \lessdot x_{i+1}$ for all i. The **order complex** of P, denoted $\Delta(P)$, is the abstract simplicial complex with vertices the elements of P and i-dimensional faces the i-chains of P. For $x, y \in P$ with $x < y$, we denote by $\Delta(x, y)$ the order complex of the open interval (x, y) as an induced subposet of P. Thus, when we refer to topological properties of P, we mean the topological properties of a geometric realization of $\Delta(P)$. In particular, the homotopy type of P refers to the homotopy type of $\Delta(P)$. Hall's well known theorem shows that the Möbius function μ_P of P satisfies $\mu_P(x, y) = \tilde{\chi}(\Delta(x, y))$. Here, $\tilde{\chi}$ is the reduced Euler characteristic. This provides one of the important connections between the combinatorial and enumerative structure of a poset and its topology.

2.2 Background on s-weak order

We first define s-decreasing trees which are the elements of the partial order. Next we establish notation for working with these trees and various subtrees. Then, in analogy with weak order on permutations, we give the notion of inversion set for an s-decreasing tree containment of which gives s-weak order. Lastly, we give the notions of an ascent in an s-decreasing tree and of an s-tree rotation which together characterize the cover relations in s-weak order and provide us our SB-labeling of s-weak order.

A **weak composition** is a sequence of non-negative integers $s = (s(1), \ldots, s(n))$ with $s(i) \in \mathbb{N}$ for all $i \in [n]$. We say the **length** of a weak composition s is $l(s) := n$. For a weak composition s, an s**-decreasing tree** is a planar rooted tree T with n internal vertices which are labeled 1 to n (leaves are not labeled and are the only unlabeled vertices) such that internal vertex i has $s(i) + 1$ children and all labeled descendants of i have labels less than i. The $s(i) + 1$ children of i are indexed by 0 to $s(i)$. We denote the full subtree of T rooted at i by T^i, and denote the full subtrees rooted at the $s(i)+1$ children of i by $T^i_0, \ldots, T^i_{s(i)}$, respectively. For i and $0 \leq j \leq s(i)$, we denote by $T^i \setminus j$ the subtree of T obtained from T^i by replacing T^i_j with a leaf. Let k be the jth child of i in T. We define the **jth left subtree of i in** T, denoted ${}_LT^i_j$, to be the subtree of T with root i obtained by walking from i to k and then down the left most subtree possible until reaching a leaf. Similarly, we define the **jth right subtree of i in** T, denoted ${}_RT^i_j$, to be the subtree of T with root i obtained by walking from i to k and then down the right most subtree possible until reaching a leaf. Figure 1 is an example of an s-decreasing tree with $s = (0,0,0,2,1,3)$, along with some examples of the subtrees just defined.



(a) T (b) $T^6 \setminus 0$ (c) ${}_LT^6_2$ (d) ${}_RT^6_2$

Figure 1: An s-decreasing tree T with $s = (0,0,0,2,1,3)$ and examples of the defined subtrees.

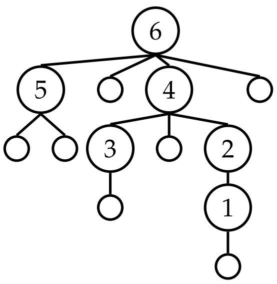
For $1 \leq x < y \leq n$, the **cardinality** of (y,x) in T, denoted $\#_T(y,x)$, is defined as follows: if x is left of y in T or $x \in T^y_0$ then $\#_T(y,x) = 0$; if $x \in T^y_i$ with $0 < i < s(y)$, then $\#_T(y,x) = i$; and if $x \in T^y_{s(y)}$ or x is right of y in T, then $\#_T(y,x) = s(y)$. If $\#_T(y,x) > 0$, then (y,x) is said to be a **tree inversion** of T. We denote by $\mathrm{inv}\,(T)$ the multi-set of tree inversions of T counted with multiplicity their cardinality. Figure 2 is the s-decreasing tree from Figure 1 with its cardinalities listed. Now we can also formally describe the jth left and right subtrees of i in T.

$$ {}_LT^i_j = \left\{ d \in T^i \;\middle|\; d = i, \text{ or } d \in T^i_j \text{ and } \#_T(e,d) = 0 \;\forall e \in T^i_j \text{ such that } d < e \right\}. $$

$$ {}_RT^i_j = \left\{ d \in T^i \;\middle|\; d = i, \text{ or } d \in T^i_j \text{ and } \#_T(e,d) = s(e) \;\forall e \in T^i_j \text{ such that } d < e \right\}. $$

Remark 2.1. For $s = (1, \ldots, 1)$, s-decreasing trees are in by bijection with permutations in $S_{l(s)}$ and tree inversions biject with inversions of the corresponding permutation.

A **multi-inversion set** on $[n]$ is a multi-set I of pairs (y,x) such that $1 \leq x < y \leq n$. We write $\#_I(y,x)$ for the multiplicity of (y,x) in I so if (y,x) does not appear in I,



$$\begin{array}{ccccc}
\#_T(6,5)=0 & \#_T(6,4)=2 & \#_T(6,3)=2 & \#_T(6,2)=2 & \#_T(6,1)=2 \\
 & \#_T(5,4)=1 & \#_T(5,3)=1 & \#_T(5,2)=1 & \#_T(5,1)=1 \\
 & & \#_T(4,3)=0 & \#_T(4,2)=2 & \#_T(4,1)=2 \\
 & & & \#_T(3,2)=0 & \#_T(3,1)=0 \\
 & & & & \#_T(2,1)=0
\end{array}$$

Figure 2: An s-decreasing tree and its cardinalities for $s = (0,0,0,2,1,3)$.

$\#_I(y,x) = 0$. Given multi-inversion sets I and J, we say I is **included** in J and write $I \subseteq J$ if $\#_I(y,x) \leq \#_J(y,x)$ for all $1 \leq x < y \leq n$. We also define the **multi-inversion set complement**, denoted $J - I$, to be the multi-inversion set with $\#_{J-I}(y,x) = \#_J(y,x) - \#_I(y,x)$ whenever this difference is non-negative and 0 otherwise. We say I is **transitive** if for each $x < y < z$, $\#_I(y,x) = 0$ or $\#_I(z,y) \leq \#_I(z,x)$. For I and J with $\#_I(y,x) \leq s(y)$ and $\#_J(y,x) \leq s(y)$ for all $1 \leq x < y \leq n$, the **transitive closure** of $I \cup J$, denoted $(I \cup J)^{tc}$, is the transitive multi-inversion set satisfying $\#_{(I\cup J)^{tc}}(y,x) \leq s(y)$ and $\min\{\#_I(y,x) + \#_J(y,x), s(y)\} \leq \#_{(I\cup J)^{tc}}(y,x)$ for all $x < y$ which is smallest by inclusion. Transitivity is easily verified on the multi-inversion set for the s-decreasing tree in Figure 2. Using a subscript to indicate the multiplicity,

$$\mathrm{inv}\,(T) = \{(4,1)_2, (4,2)_2, (5,1)_1, (5,2)_1, (5,3)_1, (5,4)_1, (6,1)_2, (6,2)_2, (6,3)_2, (6,4)_2\}.$$

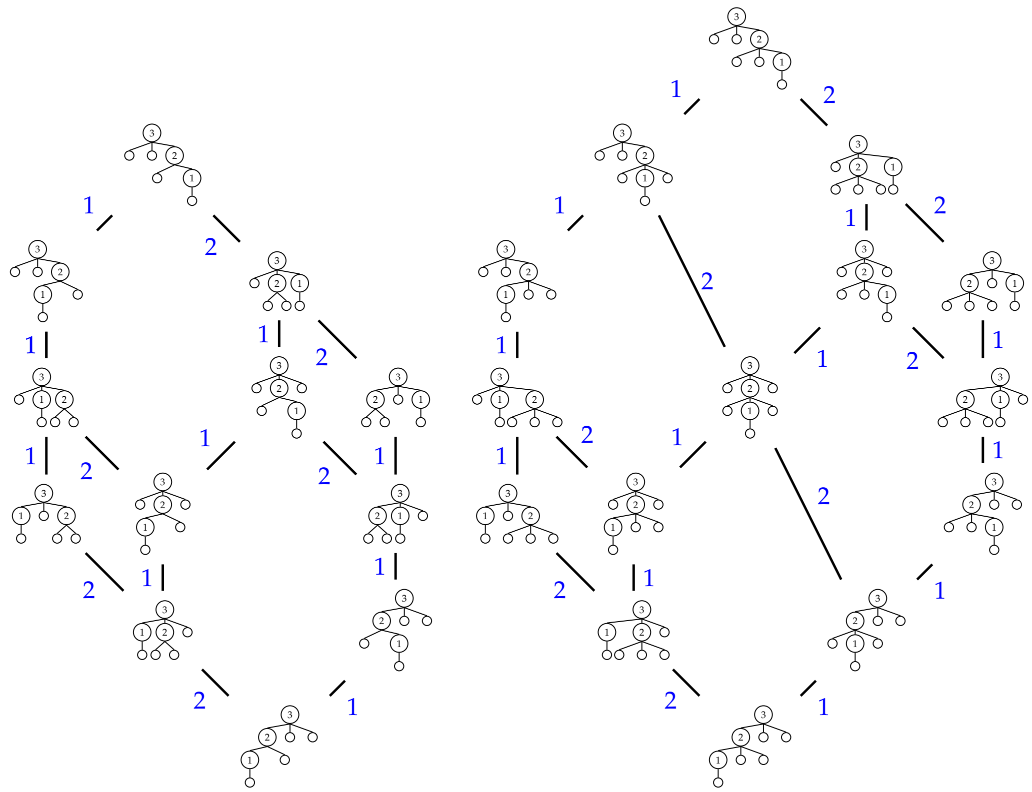
Definition 2.2. *[3, Definition 2.5] Let T and Z be two s-decreasing trees. We define the relation $T \preceq Z$ if and only if $\mathrm{inv}\,(T) \subseteq \mathrm{inv}\,(Z)$. We call the relation $\preceq$ the s-**weak order**.*

It turns out s-weak order is a lattice. For s-decreasing trees T and Z, their join is defined by $\mathrm{inv}\,(T \vee Z) = (\mathrm{inv}\,(T) \cup \mathrm{inv}\,(Z))^{tc}$. Further, for $s = (1,\ldots,1)$, s-weak order is isomorphic to weak order on $S_{l(s)}$. Figure 3 shows examples of s-weak order. The labeling of the examples is our SB-labeling and will be defined shortly.

Definition 2.3. *[3, Section 2.2] Let T be an s-decreasing tree and $1 \leq a < b \leq n$. The pair (a,b) is a **tree ascent** of T if the following hold: (i) $a \in T_i^b$ for some $0 \leq i < s(b)$, (ii) if $a \in T_j^e$ for any $a < e < b$, then $j = s(e)$, (iii) if $s(a) > 0$, then $T_{s(a)}^a$ is a leaf.*

The tree ascents of the s-decreasing tree in Figure 2 are $(1,6), (2,6), (3,4), (5,6)$. However, in the examples in Figure 3, $(1,3)$ is not a tree ascent of either minimal element since $1 \in T_0^2$ and $0 \neq s(2)$ in both cases. When $s = (1,\ldots,1)$, this notion of ascent corresponds to the definition of ascents for permutations as illustrated in Figure 4.

Remark 2.4. An s-decreasing tree, T, cannot have tree ascents (a,b) and (a,c) with $b \neq c$. This would contradict condition (ii) of Definition 2.3. This implies for $c \in [n]$ there is at most one $d \in [n]$ such that (c,d) is a tree ascent of T. Thus, whenever (a,b) and (c,d) are distinct tree ascents of T, we may assume $a < c$.



(a) $s = (0,1,2)$ (b) $s = (0,2,2)$

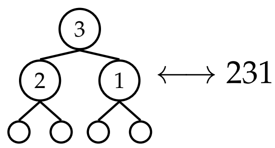
Figure 3: Examples of s-weak order.

Remark 2.5. We observe that conditions (i) and (ii) of Definition 2.3 together are equivalent to $a \in {}_RT_i^b$ for some $0 \leq i < s(b)$.

Definition 2.6. *[3, Section 2.2] Let T be an s-decreasing tree with tree ascent (a,b). Let $a \in {}_RT_j^b$ for some $j < s(b)$. Let g be the parent of a. Either $g = b$, or $a \in T_{s(g)}^g$ with $g \in T^b$. Let m be the smallest element of ${}_LT_{j+1}^b$ which is still larger than a. Define an s-decreasing tree Z to be the same as T except for the following changes: $Z_{s(g)}^g = T_0^a$ instead of T^a, $Z_i^a = T_i^a$ for $0 < i < s(a)$ and Z_0^a is a leaf (if $s(a) > 0$), $Z_{s(a)}^a = T_0^m$, and $Z_0^m = Z^a$. We call Z the **s-tree rotation of** T **along** (a,b), and denote this $T \xrightarrow{(a,b)} Z$.*

Figure 5 illustrates an s-tree rotation. Intuitively, we move T^a along b to the next subtree of T^b leaving T_0^a behind. This gives the characterization of cover relations, from which we derive our labeling.

Theorem 2.7. *[3, Theorem 2.7] Let T and Z be s-decreasing trees. Then $T \prec Z$ if and only if there is a unique pair (a,b) which is a tree ascent of T such that $T \xrightarrow{(a,b)} Z$.*



Tree ascents of T: $(2,3)$

Ascents of the permutation 231: $(2,3)$

Figure 4: An s-decreasing tree with $s = (1,1,1)$ and its tree ascents, as well as, the corresponding permutation in S_3 and its ascents.

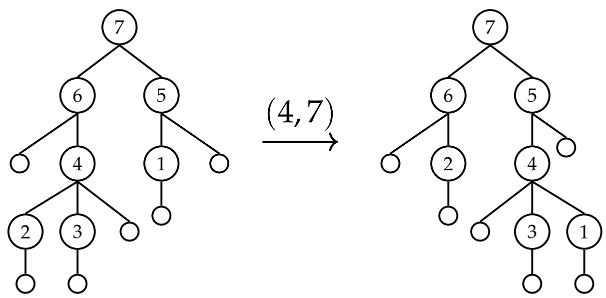


Figure 5: Illustration of the s-tree rotation along the tree ascent $(4,7)$.

Definition 2.8. *Let $T \prec Z$ be a cover relation in s-weak order given by $T \xrightarrow{(a,b)} Z$ where (a,b) is a tree ascent of T. Define an edge labeling of s-weak order by $\lambda(T,Z) = a$.*

Remark 2.9. One might wonder if s-weak order is a Cambrian lattice of some finite Coxeter group. Cambrian lattices were defined by Reading in [9] as certain lattice quotients of weak order. However, from s-weak order with $s = (0,0,2)$ which has order 9 we may conclude that s-weak order is not generally a Cambrian lattice of a finite Coxeter group. The Cambrian lattices of a finite Coxeter group W all have order the Coxeter Catalan number $Cat(W)$. The only W with $Cat(W) = 9$ is the dihedral group $I_2(7)$ see [5]. However, s-weak order with $s = (0,0,2)$ has largest anti-chain of cardinality 3 while the largest anti-chain in a Cambrian lattice of $I_2(7)$ has cardinality at most 2.

2.3 Background on the s-Tamari lattice

The Tamari lattice is the sublattice of weak order on permutations generated by the 231-avoiding permutations. Similarly, the s-Tamari lattice is the sublattice of s-weak order generated by certain s-decreasing trees. An s-decreasing tree T is called an **s-Tamari tree** if for any $a < b < c$, $\#_T(c,a) \leq \#_T(c,b)$. That is, all the labels in T_i^c are smaller than all the labels in T_j^c for $i < j$. We denote the partial order on s-Tamari trees induced by s-weak order by $\preceq_{Tam}$. A subscript Tam will be used to denote objects in the s-Tamari lattice. For instance, $\prec_{Tam}$ denotes cover relations in the s-Tamari lattice. Taking $s = (1,\ldots,1)$, the s-Tamari lattice is isomorphic to the Tamari lattice on $l(s)$.

Theorem 2.10. *[3, Theorem 3.2] The collection of s-Tamari trees forms a sublattice of s-weak order, called the s-Tamari lattice.*

Similarly to s-weak order, the cover relations in the s-Tamari lattice can be characterized as certain tree rotations. For $a < b$, we say that (a, b) is a **Tamari tree ascent** of T if a is a non-right most child of b, that is, a is a direct descendant of b and $\#_T(b, a) < s(b)$. Note that it no longer matters whether or not $T^a_{s(a)}$ is a leaf. Then the **s-Tamari rotation of T along** (a, b) is essentially the same as an s-tree rotation except that the smaller element of the Tamari tree ascent may have right descendants and those right descendants are moved along with a if $s(a) > 0$. We denote this rotation by $T \overset{Tam(a,b)}{\longrightarrow} Z$ where Z is the resulting s-Tamari tree. Figure 6 illustrates such a rotation. Then we have that $T \prec_{Tam} Z$ if and only if $T \overset{Tam(a,b)}{\longrightarrow} Z$.

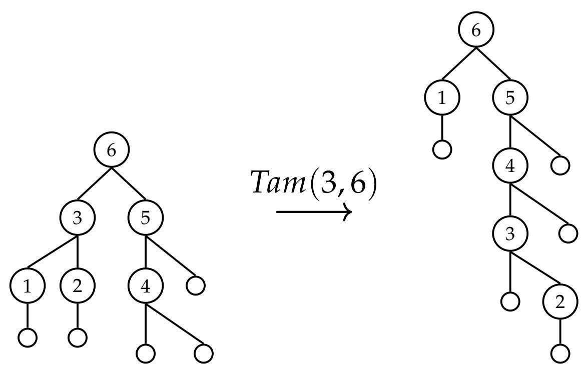


Figure 6: s-Tamari rotation along the Tamari tree ascent $(3, 6)$.

Remark 2.11. An s-Tamari tree T cannot have Tamari tree ascents (a, b) and (a, c) with $b \neq c$.

2.4 Background on SB-labelings

Hersh and Mészáros developed the notion of an SB-labeling in [7] to show when certain lattices have open intervals which are homotopy balls or spheres.

Definition 2.12. *[7, Definition 3.4] An **SB-labeling** is an edge labeling on a finite lattice L satisfying the following conditions for each $u, v, w \in L$ such that v and w are distinct elements which each cover u: (i) $\lambda(u, v) \neq \lambda(u, w)$ (ii) Each saturated chain from u to $v \vee w$ uses both labels $\lambda(u, v)$ and $\lambda(u, w)$ a positive number of times. (iii) None of the saturated chains from u to $v \vee w$ use any labels besides $\lambda(u, v)$ and $\lambda(u, w)$.*

Figure 7 is an SB-labeling of weak order on S_3 which is actually our labeling of s-weak order with $s = (1, 1, 1)$. We will use the following theorem of Hersh and Mészáros.

Theorem 2.13. *[7, Theorem 3.7] If L is a finite lattice which admits an SB-labeling, then each open interval in L is homotopy equivalent to a ball or a sphere of some dimension. Moreover, $\Delta(u,v)$ is homotopy equivalent to a sphere if and only if v is a join of atoms of the interval, in which case it is homotopy equivalent to a sphere S^{d-2} where d is the number of atoms in $[u,v]$.*

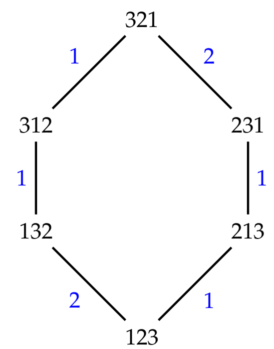


Figure 7: An SB-labeling of weak order on S_3.

3 An SB-labeling of s-weak order and the s-Tamari lattice

We begin with showing the labeling of Definition 2.8 is an SB-labeling of s-weak order. Then we proceed to the SB-labeling of the s-Tamari lattice. While we mention the main ideas of our proofs here, the proofs themselves are rather long and technical.

Remark 3.1. In the case $s = (1, \dots, 1)$, the labeling of Definition 2.8 gives an SB-labeling of weak order on $S_{l(s)}$. Our labeling is distinct from the labeling for finite Coxeter groups given by Hersh and Mészáros in [7].

The examples in Figure 3 illustrate the labeling of Definition 2.8. In both cases, it is easily verified that this is an SB-labeling. As Figure 3 suggests, the main point of our proof is proving that for $T \prec Z, Q$, the interval $[T, Z \vee Q]$ is a diamond, a pentagon, or a hexagon. Then we verify that, in any case, the labeling on the two maximal chains satisfies Definition 2.12. For the remainder of this section let (a,b) and (c,d) be tree ascents of T with $a < c$. Also, let Z and Q be the s-decreasing trees such that $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$. The following definitions and proposition allow us to describe the join of two atoms in an interval in terms of tree inversion sets.

Definition 3.2. *Let T and Z be as above. The **tree inversions added by the s-tree rotation along** (a,b) is the set*

$$A_T(a,b) = \left\{ (f,e) \;\middle|\; \#_Z(f,e) > \#_T(f,e) \right\}.$$

Definition 3.3. *We recall that (a,b) and (c,d) are tree ascents of T with $a < c$. The* ***secondary tree inversions added*** *is the set valued function*

$$F_T(a,c) = \begin{cases} \left\{ (d,e) \;\middle|\; e \in T^a \setminus 0 \right\}, & \text{if } b = c \text{ and } a \in T_0^c \\ \\ \varnothing, & \text{otherwise} \end{cases}$$

Proposition 3.4. *We recall that $T \xrightarrow{(a,b)} Z$. Then $(f,e) \in A_T(a,b)$ if and only if $f = b$ and $e \in T^a \setminus 0$, in which case $\#_Z(f,e) = \#_T(f,e) + 1$.*

This proposition can essentially be read off from Definition 2.6. Using it we show the following characterization of inv $(Z \vee Q)$.

Lemma 3.5. *For $T \prec Z, Q$ as before, inv $(Z \vee Q) -$ inv $(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$ and the three sets in this union are pairwise disjoint.*

Proposition 3.6. *Let T be an s-decreasing tree and let $1 \leq a < b \leq n$ be such that (a,b) is a tree ascent of T with $s(a) > 0$. Then no pair of the form (e,b) such that $e \in T^a$ and $e < a$ is a tree ascent of T.*

This proposition follows by contradiction from (ii) of Definition 2.3. It combines with Lemma 3.5 to show there are no other saturated chains in the relevant intervals besides the two forming the diamond, pentagon or hexagon. The situation precluded by Proposition 3.6 may occur if $s(a) = 0$. To prove that we actually have the two desired chains in the interval $[T, Z \vee Q]$ and to characterize which intervals are diamonds, pentagons, and hexagons, we need the following lemma.

Lemma 3.7. *If (a,b) is not a tree ascent of Q or (c,d) is not a tree ascent of Z, then $b = c$ and $s(c) > 0$. Moreover, if (a,c) is not a tree ascent of Q, then $a \in T_0^c$. If (c,d) is not a tree ascent of Z, then $a \in T_{s(c)-1}^c$.*

We show this lemma by considering the ways that Definition 2.3 can be violated, then using Definition 2.6 and some other observations to show these are the only two possibilities. This leads to the following characterization of the saturated chains in $[T, Z \vee Q]$.

Lemma 3.8. *If (c,d) is a tree ascent of Z, then there is a saturated chain $T \xrightarrow{(a,b)} Z \xrightarrow{(c,d)} Z \vee Q$. Similarly, if (a,b) is a tree ascent of Q, there is a saturated chain $T \xrightarrow{(c,d)} Q \xrightarrow{(a,b)} Z \vee Q$. If (c,d) is not a tree ascent of Z, then there is a saturated chain $T \xrightarrow{(a,c)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} Z \vee Q$. If (a,b) is not a tree ascent of Q, then there is a saturated chain $T \xrightarrow{(c,d)} Q \xrightarrow{(a,d)} P \xrightarrow{(a,c)} Z \vee Q$.*

This lemma can be seen intuitively by drawing similar but slightly more complicated diagrams than those in Figure 5 and tracking what happens using Definition 2.6.

Theorem 3.9. *The edge labeling of Definition 2.8 is an SB-labeling of s-weak order.*

Proof sketch. First, we note that condition (i) of Definition 2.12 is satisfied by Remark 2.4. We then use Proposition 3.6 and Lemma 3.5 to show there are no saturated chains from T to $Z \vee Q$ besides the corresponding pair of chains from Lemma 3.8. Then we simply read off the label sequences from those two chains and verify conditions (ii) and (iii) of Definition 2.12. □

Theorem 3.9 and Theorem 2.13 combine to give the following corollary.

Corollary 3.10. *For $T \preceq Z$, $\Delta(T,Z)$ is homotopy equivalent to a sphere or ball. Moreover, $\mu(T,Z) \in \{-1,0,1\}$.*

Then we characterize the intervals which give homotopy spheres and the dimensions of those spheres. This simply follows from the characterization of the join in s-weak order and Theorem 2.13.

Theorem 3.11. *For $T \preceq Z$, $\Delta(T,Z)$ is homotopy equivalent to a sphere if and only if*

$$inv\,(Z) = (inv\,(T) \cup A_T(a_1,b_1) \cup \cdots \cup A_T(a_l,b_l))^{tc}$$

where $(a_1,b_1),\ldots,(a_l,b_l)$ are the tree ascents of T such that $(b_i,a_i) \in inv\,(Z) - inv\,(T)$. In this case, the dimension of the sphere is $l-2$.

The SB-labeling for the s-Tamari lattice is nearly identical. The proofs are also nearly identical. For $T \prec_{Tam} Z$ given by $T \xrightarrow{Tam(a,b)} Z$, we define and characterize $A_T^{Tam}(a,b) = \mathrm{inv}\,(Z) - \mathrm{inv}\,(T)$. We use this to show the following labeling is an SB-labeling on the s-Tamari lattice. We also characterize the intervals yielding homotopy spheres.

Theorem 3.12. *Let $T \prec_{Tam} Z$ be a cover relation in the s-Tamari lattice given by $T \xrightarrow{Tam(a,b)} Z$ where (a,b) is a Tamari tree ascent of T. Define an edge labeling of the s-Tamari lattice by $\lambda(T,Z) = a$. Then λ is an SB-labeling of the s-Tamari lattice.*

Proof sketch. Remark 2.11 implies (i) of Definition 2.12 is satisfied. We show lemmas for the s-Tamari lattice corresponding to Proposition 3.6 and Lemma 3.8. These lemmas imply that if $T \prec_{Tam} Z, Q$, the interval $[T, Z \vee Q]_{Tam}$ has precisely two saturated chains. We then verify (ii) and (iii) of Definition 2.12 for these two chains. □

Corollary 3.13. *For $T \preceq_{Tam} Z$, $\Delta(T,Z)_{Tam}$ is homotopy equivalent to a sphere or ball. Moreover, $\mu_{Tam}(T,Z) \in \{-1,0,1\}$.*

Theorem 3.14. *For $T \preceq_{Tam} Z$, $\Delta(T,Z)_{Tam}$ is homotopy equivalent to a sphere if and only if*

$$inv\,(Z) = \left(inv\,(T) \cup A_T^{Tam}(a_1,b_1) \cup \cdots \cup A_T^{Tam}(a_l,b_l)\right)^{tc}$$

where $(a_1,b_1),\ldots,(a_l,b_l)$ are the Tamari tree ascents of T such that $(b_i,a_i) \in inv\,(Z) - inv\,(T)$. Moreover, the dimension of the sphere is $l-2$.

Acknowledgements

The author is grateful to Patricia Hersh for guidance and many helpful discussions throughout the course of this work. The author also thanks Joseph Doolittle for helpful discussions at FPSAC 2019 about the beginning of this project and the FPSAC referees for thoughtful revisions which improved this paper.

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