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Crystal for stable Grothendieck polynomials

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Abstract. We introduce a type A crystal structure on decreasing factorizations on 321-avoiding elements in the 0-Hecke monoid which we call $\star$-crystal. This crystal is a K-theoretic generalization of the crystal on decreasing factorizations in the symmetric group of the first and last author. We prove that under the residue map the $\star$-crystal intertwines with the crystal on set-valued tableaux recently introduced by Monical, Pechenik and Scrimshaw. We also define a new insertion from decreasing factorization to pairs of semistandard Young tableaux and prove several properties, such as its relation to the Hecke insertion and the uncrowding algorithm. The new insertion also intertwines with the crystal operators.

Keywords: crystal bases, 0-Hecke monoid, stable Grothendieck polynomials

1 Introduction

The Grassmannian Grothendieck polynomials [8, 9, 6] represent Schubert classes in the K-theory of the Grassmannian. They can be expressed [2] as generating functions of semistandard set-valued tableaux

$$\mathfrak{G}_\lambda(x_1, \ldots, x_m; \beta) = \sum_{T \in \mathsf{SVT}^m(\lambda)} \beta^{\mathsf{ex}(T)} x^{\mathsf{wt}(T)}, \tag{1.1}$$

where $\mathsf{SVT}^m(\lambda)$ is the set of semistandard set-valued tableaux of shape λ in the alphabet $[m] := \{1, 2, \ldots, m\}$ and $\mathsf{ex}(T)$ is the excess of T. Recently, Monical, Pechenik and Scrimshaw [13] provided a type A_{m-1}-crystal structure on $\mathsf{SVT}^m(\lambda)$. In particular, this implies that the Grassmannian Grothendieck polynomial $\mathfrak{G}_\lambda(x_1, \ldots, x_m; \beta)$ is a positive sum of Schur polynomials $\sum_\mu \beta^{|\mu|-|\lambda|} M_\lambda^\mu \; s_\mu(x_1, \ldots, x_m)$, where M_λ^μ is the number of highest weight set-valued tableaux of weight μ in the crystal $\mathsf{SVT}^m(\lambda)$. A different combinatorial formula for the multiplicities M_λ^μ was given by Lenart [10, Theorem 2.2] in terms of flagged increasing tableaux.

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Stable Grothendieck polynomials [6], which are labeled by permutations $w \in \mathbb{S}_n$, generalize the Grassmannian Grothendieck polynomials. They are defined as

$$\mathfrak{G}_w(x_1,\ldots,x_m;\beta) = \sum_{(\mathbf{k},\mathbf{h})} \beta^{\ell(\mathbf{h})-\ell(w)} x^{\mathbf{k}},$$

where the sum is over decreasing factorizations $[\mathbf{k},\mathbf{h}]^t$ of w in the 0-Hecke algebra. When $\beta = 0$, $\mathfrak{G}_w$ specializes to the Stanley symmetric function F_w [18].

In this paper, we define a type A crystal structure on decreasing factorizations of w in the 0-Hecke algebra, when w is 321-avoiding. A permutation w is 321-avoiding if its reduced expressions do not contain any braids or equivalently w is fully commutative. The residue map (see Section 2.4) shows that 321-avoiding permutations correspond to skew shapes. We call our crystal $\star$-crystal. It is local in the sense that the crystal operators $f_i^\star$ and $e_i^\star$ only act on the i-th and $(i+1)$-th factors of the decreasing factorization. It generalizes the crystal of Morse and Schilling [15] for Stanley symmetric functions (or equivalently reduced decreasing factorizations of w) in the 321-avoiding case. We show that the $\star$-crystal and the crystal on set-valued tableaux intertwine under the residue map (see Theorem 2.13). We also show that the residue map and the Hecke insertion [3] are related (see Theorem 3.5), thereby resolving [13, Open Problem 5.8] in the 321-avoiding case. In addition, we provide a new insertion algorithm, which we call $\star$-insertion, from decreasing factorizations on 321-avoiding elements in the 0-Hecke monoid to pairs of (conjugates of) semistandard Young tableaux of the same shape (see Definition 3.7 and Theorem 3.10), which intertwines with crystal operators (see Theorem 4.3). This recovers the Schur expansion of $\mathfrak{G}_w$ of Fomin and Greene [5] when w is 321-avoiding, stating that

$$\mathfrak{G}_w = \sum_{\mu} \beta^{|\mu|-\ell(w)} g_w^\mu s_\mu,$$

where $g_w^\mu = |\{T \in \mathsf{SSYT}^n(\mu') \mid w_C(T) \equiv w\}|$ and $w_C(T)$ is the column reading word of T (see Remark 4.4). We also show that the composition of the residue map with the $\star$-insertion is related to the uncrowding algorithm [2] (see Theorem 4.7).

The paper is organized as follows. In Section 2, we introduce the $\star$-crystal on decreasing factorizations in the 0-Hecke monoid and show that it intertwines with the crystal on semistandard set-valued tableaux [13] under the residue map. In Section 3, we discuss two insertion algorithms for decreasing factorizations. The first is the Hecke insertion introduced by Buch et al. [3] and the second is the $\star$-insertion. In Section 4, properties of the $\star$-insertion are discussed. In particular, we state that it intertwines with the crystal operators and that it relates to the uncrowding algorithm. Proofs of all statements appear in the long version of this extended abstract [14].

2 The $\star$-crystal

In this section, we define the K-theoretic generalization of the crystal on decreasing factorizations by Morse and Schilling [15] when the associated word is 321-avoiding. The underlying combinatorial objects are decreasing factorizations in the 0-Hecke monoid introduced in Section 2.1. The $\star$-crystal on these decreasing factorizations is defined in Section 2.2. We review the crystal structure on set-valued tableaux introduced by Monical, Pechenik and Scrimshaw [13] in Section 2.3. The residue map and the theorem that it intertwines the $\star$-crystal and the crystal on set-valued tableaux is given in Section 2.4.

2.1 Decreasing factorizations in the 0-Hecke monoid

The *symmetric group* $\mathbb{S}_n$ is generated by the simple transpositions s_i for $1 \leqslant i < n$ with relations $s_i s_j = s_j s_i$ for $|i-j| > 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i^2 = 1$. A *reduced expression* for an element $w \in \mathbb{S}_n$ is a word $a_1 a_2 \ldots a_\ell$ with $a_i \in [n-1] := \{1, 2, \ldots, n-1\}$ such that $w = s_{a_1} \cdots s_{a_\ell}$ and ℓ is minimal among all words representing w. In this case, ℓ is called the length of w.

Definition 2.1. The 0-*Hecke monoid* $\mathcal{H}_0(n)$, where $n \geqslant 1$ is an integer, is a monoid of finite words generated by positive integers in the alphabet $[n-1]$ subject to the relations

$$pq \equiv qp \quad \text{if } |p-q| > 1, \qquad pqp \equiv qpq \quad \text{for all } p, q, \qquad pp \equiv p \text{ for all } p. \tag{2.1}$$

We may form an equivalence relation $\equiv_{\mathcal{H}_0}$ on all words in the alphabet $[n-1]$ based on the relations (2.1). The equivalence classes are infinite since the last relation changes the length of the word. We say that a word $a = a_1 a_2 \ldots a_\ell$ is *reduced* if $\ell \geqslant 0$ is the smallest among all words in $\mathcal{H}_0(n)$ equivalent to a. In this case ℓ is the length of a. Note that $\mathcal{H}_0(n)$ is in bijection with $\mathbb{S}_n$ by identifying the reduced word $a_1 a_2 \ldots a_\ell$ in $\mathcal{H}_0(n)$ with $s_{a_1} s_{a_2} \cdots s_{a_\ell} \in \mathbb{S}_n$. We say $w \in \mathcal{H}_0(n)$ or $\mathbb{S}_n$ is 321-*avoiding* if none of the reduced words equivalent to w contain a consecutive subword of the form $i\; i+1\; i$ for any $i \in [n-1]$.

Definition 2.2. A *decreasing factorization* of $w \in \mathcal{H}_0(n)$ into m *factors* is a product of the form $\mathbf{h} = h^m \ldots h^2 h^1$, where the sequence in each factor $h^i = h^i_1 h^i_2 \ldots h^i_{\ell_i}$ is either empty (meaning $\ell_i = 0$) or strictly decreasing (meaning $h^i_1 > h^i_2 > \cdots > h^i_{\ell_i}$) for each $1 \leqslant i \leqslant m$ and $\mathbf{h} \equiv_{\mathcal{H}_0} w$ in $\mathcal{H}_0(n)$. The set of all possible decreasing factorizations into m factors is denoted by $\mathcal{H}^m$. We call $\mathsf{ex}(\mathbf{h}) = \mathsf{len}(\mathbf{h}) - \ell$ the *excess* of $\mathbf{h}$, where $\mathsf{len}(\mathbf{h})$ is the number of letters in $\mathbf{h}$ and ℓ is the length of w. We say $\mathbf{h}$ is 321-avoiding if w is 321-avoiding.

2.2 The $\star$-crystal

Let $\mathcal{H}^{m,\star}$ be the set of 321-avoiding decreasing factorizations in $\mathcal{H}^m$. We introduce a type A_{m-1} crystal structure on $\mathcal{H}^{m,\star}$, which we call the $\star$-crystal. This generalizes the crystal for Stanley symmetric functions [15] (see also [11]).

Definition 2.3. For any $\mathbf{h} = h^m \dots h^2 h^1 \in \mathcal{H}^{m,\star}$, we define *crystal operators* $e_i^\star$ and $f_i^\star$ for $i \in [m-1]$ and a *weight function* $\mathsf{wt}(\mathbf{h}) = (\mathsf{len}(h^1), \mathsf{len}(h^2), \dots, \mathsf{len}(h^m))$. We begin with a pairing process:

- Start with the largest letter b in h^{i+1}, pair it with the smallest $a \geqslant b$ in h^i. If there is no such a, then b is unpaired.
- The pairing proceeds in decreasing order on elements of h^{i+1} and with each iteration, previously paired letters of h^i are ignored.

If all letters in h^i are paired, then $f_i^\star$ annihilates $\mathbf{h}$. Otherwise, let x be the largest unpaired letter in h^i. The crystal operator $f_i^\star$ acts on $\mathbf{h}$ in either of the following ways:

1. If $x+1 \in h^i \cap h^{i+1}$, then remove $x+1$ from h^i, add x to h^{i+1}.
2. Otherwise, remove x from h^i and add x to h^{i+1}.

The operator $e_i^\star$ is defined as the partial inverse of $f_i^\star$, that is, $e_i^\star(\mathbf{h})$ is the unique $\mathbf{h}'$ such that $f_i^\star(\mathbf{h}') = \mathbf{h}$ if it exists and zero otherwise.

Example 2.4. Let $\mathbf{h} = (7532)(621)(6)$, then $f_1^\star(\mathbf{h}) = 0$ and $f_2^\star(\mathbf{h}) = (75321)(61)(6)$.

Proposition 2.5. *Let* $\mathbf{h} = h^m \dots h^1 \in \mathcal{H}^{m,\star}$ *such that* $e_i^\star(\mathbf{h}) \neq 0$. *Then* $e_i^\star(\mathbf{h}) \in \mathcal{H}^{m,\star}$, $e_i^\star(\mathbf{h}) \equiv_{\mathcal{H}_0} \mathbf{h}$, *and* $\mathsf{ex}(e_i^\star(\mathbf{h})) = \mathsf{ex}(\mathbf{h})$. *Furthermore, the* j*-th factor in* $e_i^\star(\mathbf{h})$ *and* $\mathbf{h}$ *agrees for* $j \notin \{i, i+1\}$. *Analogous statements hold for* $f_i^\star$.

It will be shown in Section 2.4 that $\mathcal{H}^{m,\star}$ is indeed a Stembridge crystal of type A_{m-1} (for an introduction to crystal and terminology, see [4]).

2.3 The crystal on set-valued tableaux

In this section, we review the type A crystal structure on set-valued tableaux introduced in [13]. In fact, in [13] the authors only considered the crystal structure on straight-shaped set-valued tableaux. Here we consider the slight generalization to skew shapes, see Theorem 2.8. We use French notation for partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant 0$, that is, in the Ferrers diagram for λ, the largest part λ_1 is at the bottom.

Definition 2.6 ([2])**.** A *semistandard set-valued tableau* T is the filling of a skew shape λ/μ with nonempty subsets of positive integers such that (1) for all adjacent cells A, B in the same row with A to the left of B, $\max(A) \leqslant \min(B)$, (2) for all adjacent cells A, C in the same column with A below C, $\max(A) < \min(C)$. We denote the set of all semistandard set-valued tableaux of shape λ/μ by $\mathsf{SVT}(\lambda/\mu)$. If the maximum entry in the tableaux is m, the set is denoted by $\mathsf{SVT}^m(\lambda/\mu)$.

We now review the crystal structure on semistandard set-valued tableaux given in [13]. We state the definition on skew shapes rather than just straight shapes.

Definition 2.7. Let $T \in \mathsf{SVT}^m(\lambda/\mu)$. We employ the following pairing rule for letters i and $i+1$. Assign $-$ to every column of T containing an i but not an $i+1$. Similarly, assign $+$ to every column of T containing an $i+1$ but not an i. Then, successively pair each $+$ that is adjacent to a $-$, removing all paired signs until nothing can be paired.

The *operator* f_i changes the i in the rightmost column with an unpaired $-$ (if this exists) to $i+1$, except if the cell b containing that i has a cell to its right, denoted $b^{\rightarrow}$, that contains both i and $i+1$. In that case, f_i removes i from $b^{\rightarrow}$ and adds $i+1$ to b. Finally, if no unpaired $-$ exists, then f_i annihilates T. The *operator* e_i is the partial inverse of f_i.

The *weight* $\mathrm{wt}(T)$ of T is the integer vector whose i-th component counts the number of i's that occur in T.

The above described operators e_i and f_i define a type A_{m-1} crystal structure on $\mathsf{SVT}^m(\lambda)$ [13, Theorem 3.9]. Their proof also holds for skew shapes.

Theorem 2.8. *The crystal* $\mathsf{SVT}^m(\lambda/\mu)$ *of Definition 2.7 is a Stembridge crystal of type* A_{m-1}.

2.4 The residue map

In this section, we define the residue map from set-valued tableaux of skew shape to 321-avoiding decreasing factorizations in the 0-Hecke monoid. We then show in Theorem 2.13 that the residue map intertwines with the crystal operators, proving that $\mathcal{H}^{m,\star}$ is indeed a crystal of type A_{m-1} (see Corollary 2.14).

Definition 2.9. Given $T \in \mathsf{SVT}^m(\lambda/\mu)$, we define the *residue map* $\mathsf{res} : \mathsf{SVT}^m(\lambda/\mu) \to \mathcal{H}^m$ as follows. Label all cells (i,j) in λ/μ with $\ell(\lambda) + j - i$, where $\ell(\lambda)$ is the number of parts in λ. Produce a decreasing factorization $\mathbf{h} = h^m h^{m-1} \ldots h^2 h^1$ by declaring h^i to be the (possibly empty) sequence formed by taking the labels of all cells in T containing i and then arranging these labels in decreasing order. This defines $\mathsf{res}(T) := \mathbf{h}$.

Example 2.10. Let T be the set-valued tableau of shape $(2,2)/(1)$

$$T = \begin{array}{|c|c|} \hline 23 & 3 \\ \hline \multicolumn{1}{c|}{} & 12 \\ \cline{2-2} \end{array}\,. \qquad \text{We label the cells of } T \text{ as follows:} \qquad \begin{array}{|c|c|} \hline 23_1 & 3_2 \\ \hline \multicolumn{1}{c|}{} & 12_3 \\ \cline{2-2} \end{array}\,.$$

To read off the third factor, we search for all cells containing 3; these cells have labels 1 and 2, so we have 21 in the third factor. Altogether, we obtain $\mathsf{res}(T) = (21)(31)(3)$.

The image of the residue map res is $\mathcal{H}^{m,\star}$, the set of 321-avoiding decreasing factorizations into m factors. In fact, res is a bijection from skew set-valued tableaux on the alphabet $[m]$ to $\mathcal{H}^{m,\star}$ up to shifts in the skew shape. For this purpose, let us describe the inverse of the residue map. Let $\mathbf{h} = h^m h^{m-1} \dots h^2 h^1 \in \mathcal{H}^{m,\star}$. Begin by filling m along the diagonals labeled by the letters that appear in h^m. As the resulting T is supposed to be of skew shape, the cells containing m along increasing diagonals need to go weakly down from left to right. If these diagonals are consecutive, then the cells have to be in the same row of T since T is semistandard. Continue the procedure above by filling i along the diagonals specified by h^i for all $i = m-1, m-2, \dots, 1$, applying the condition that the resulting filling should be semistandard.

Proposition 2.11. *If $\mathbf{h} \in \mathcal{H}^{m,\star}$, then the above algorithm is well-defined up to shifts along diagonals. It produces a skew semistandard set-valued tableau T such that $\mathsf{res}(T) = \mathbf{h}$.*

If the skew shape λ/μ of the tableau T is known, then one may simplify the procedure above noting that the filling of i specified by letters in h^i must occur along a horizontal strip for all $i = m, m-1, \dots, 1$. In this case, the recovered tableau T is unique.

Example 2.12. Let $\mathbf{h} = (61)(752)(75)(762)$. In the procedure to determine a suitable skew tableau whose residue map is $\mathbf{h}$, after filling 4's along the diagonals with labels 1 and 6 respectively, due to semistandardness, the 3 in diagonal 2 is below the 4 in diagonal 1, while the 3's in diagonals 5 and 7 are respectively to the left and below the 4 in diagonal 6. Continuing with the remaining fillings, we have two possibilities:

$$T_1 = \begin{array}{|c|c|c|c|} \hline 4_{1} & & & \\ \hline 13_{2} & & & \\ \hline & & 23_{5} & 4_{6} \\ \hline & & 1_{6} & 123_{7} \\ \hline \end{array} \in \mathsf{SVT}^4((4,4,1,1)/(2,2)) \text{ or } T_2 = \begin{array}{|c|c|c|} \hline 4_{1} & & \\ \hline 13_{2} & & \\ \hline & & \\ \hline & 23_{5} & 4_{6} \\ \hline & 1_{6} & 123_{7} \\ \hline \end{array} \in \mathsf{SVT}^4((3,3,1,1,1)/(1,1,1)).$$

Theorem 2.13. *The crystal on set-valued tableaux $\mathsf{SVT}^m(\lambda/\mu)$ and the crystal on decreasing factorizations $\mathcal{H}^{m,\star}$ intertwine under the residue map. That is, the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{SVT}^m(\lambda/\mu) & \xrightarrow{\mathsf{res}} & \mathcal{H}^{m,\star} \\ \downarrow f_k & & \downarrow f_k^\star \\ \mathsf{SVT}^m(\lambda/\mu) & \xrightarrow{\mathsf{res}} & \mathcal{H}^{m,\star}. \end{array}$$

Corollary 2.14. *The set $\mathcal{H}^{m,\star}$, together with crystal operators $e_i^\star$ and $f_i^\star$ for $1 \leqslant i < m$ and weight function* wt *defined in Definition 2.3, is a Stembridge crystal.*

3 Insertion algorithms

In this section, we discuss two insertion algorithms for decreasing factorizations in $\mathcal{H}^m$ (resp. $\mathcal{H}^{m,\star}$). The first is the Hecke insertion introduced by Buch et al. [3], which we review in Section 3.1. We prove a relationship between Hecke insertion and the residue map (see Theorem 3.5). In particular, this proves [13, Open Problem 5.8] for 321-avoiding permutations. The second insertion is a new insertion, which we call $\star$-insertion, introduced in Section 3.2. It goes from 321-avoiding decreasing factorizations in the 0-Hecke monoid to pairs of (transposes of) semistandard tableaux of the same shape and is well-behaved with respect to the crystal operators.

3.1 Hecke insertion

Hecke insertion was first introduced in [3] as column insertion. Here we state the row insertion version as in [16]. In this section, we represent a decreasing factorization $\mathbf{h} = h^m h^{m-1} \cdots h^1$, where $h^i = h^i_1 h^i_2 \dots h^i_{\ell_i}$, by a *decreasing Hecke biword*

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} m & \dots & m & \dots & 1 & \dots & 1 \\ h^m_1 & \dots & h^m_{\ell_m} & \dots & h^1_1 & \dots & h^1_{\ell_1} \end{bmatrix}.$$

We say that $[\mathbf{k}, \mathbf{h}]^t$ is 321-*avoiding* if $\mathbf{h}$ is 321-avoiding.

Example 3.1. Consider the decreasing Hecke factorization $\mathbf{h} = (2)(3)(31)(2)$. Then the corresponding biword $[\mathbf{k}, \mathbf{h}]^t$ is

$$\begin{bmatrix} \mathbf{k} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 & 2 & 1 \\ 2 & 3 & 3 & 1 & 2 \end{bmatrix}.$$

Definition 3.2. Starting with a decreasing Hecke biword $[\mathbf{k}, \mathbf{h}]^t$, we define *Hecke row insertion* from the right. The insertion sequence is read from right to left. Suppose there are n columns in $[\mathbf{k}, \mathbf{h}]^t$. Start the insertion with (P_0, Q_0) being both empty tableaux. We recursively construct (P_{i+1}, Q_{i+1}) from (P_i, Q_i). Suppose the $(n-i)$-th column in $[\mathbf{k}, \mathbf{h}]^t$ is $[y, x]^t$.

We describe how to insert x into P_i, denoted $P_i \leftarrow x$, by describing how to insert x into a row R. The insertion may modify the row and may produce an output integer, which will be inserted into the next row. First, we insert x into the first row R of P_i following the rules below:

1. If $x \geqslant z$ for all $z \in R$, then the insertion terminates in either of the following ways:

   (a) If we can append x to the right of R and obtain an increasing tableau, the result P_{i+1} is obtained by doing so; form Q_{i+1} by adding a box with y in the same position where x is added to P_i.

   (b) Otherwise $P_{i+1} = P_i$. Form Q_{i+1} by adding y to the existing corner of Q_i whose column contains the rightmost box of row R.

2. Otherwise, there exists a smallest z in R such that $z > x$.
   (a) If replacing z with x results in an increasing tableau, then do so. Let z be the output integer to be inserted into the next row.
   (b) Otherwise, row R remains unchanged. Let z be the output integer to be inserted into the next row.

The entire Hecke insertion terminates at (P_n, Q_n) after we have inserted every letter from the Hecke biword. The resulting insertion tableau P_n is an increasing tableau. If $\mathbf{k} = (n, n-1, \ldots, 1)$, the recording tableau Q_n is a standard set-valued tableau.

Example 3.3. Take $[\mathbf{k}, \mathbf{h}]^t$ from Example 3.1. Following the Hecke row insertion, we compute its insertion tableau and recording tableau:

$$\emptyset \to \begin{array}{|c|}\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&3\\\hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&3\\\hline\end{array} \to \begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array} = P,$$

$$\emptyset \to \begin{array}{|c|}\hline 1\\\hline\end{array} \to \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&2\\\hline\end{array} \to \begin{array}{|c|c|}\cline{1-1} 2\\\hline 1&23\\\hline\end{array} \to \begin{array}{|c|c|}\hline 2&4\\\hline 1&23\\\hline\end{array} = Q.$$

Example 3.4. Note that the recording tableau for the Hecke insertion of Definition 3.2 is not always a semistandard set-valued tableau. For example, for $\mathbf{h} = (21)(41)$ we have

$$P = \begin{array}{|c|c|}\cline{1-1} 4\\\hline 1&2\\\hline\end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|}\cline{1-1} 22\\\hline 1&1\\\hline\end{array}.$$

However, Theorem 3.5 below states that in certain cases it is.

Theorem 3.5. *Let $T \in \mathsf{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \mathsf{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q). Then $Q = T$.*

Remark 3.6. Combining Theorems 2.13 and 3.5 shows that Hecke insertion from right to left (as opposed to left to right as in [13]) intertwines the crystal on set-valued tableaux and the $\star$-crystal, even though in general it is not always well-defined (see Example 3.4). This resolves [13, Open Problem 5.8] when the decreasing factorizations are 321-avoiding. For more details see [14].

3.2 The $\star$-insertion

We define the new $\star$-insertion from 321-avoiding decreasing Hecke biwords $[\mathbf{k}, \mathbf{h}]^t$ to pairs of tableaux P and Q, denoted by $\star([\mathbf{k}, \mathbf{h}]^t) = (P, Q)$, as follows.

Definition 3.7. Fix a 321-avoiding decreasing Hecke biword $[\mathbf{k}, \mathbf{h}]^t$. The insertion is done by reading the columns of this biword from right to left. Begin with (P_0, Q_0) being a pair of empty tableaux. For every integer $i \geqslant 0$, we recursively construct (P_{i+1}, Q_{i+1}) from (P_i, Q_i) as follows. Let $[q, x]^t$ be the i-th column (from the right) of $[\mathbf{k}, \mathbf{h}]^t$. Suppose that we are inserting x into row R of P_i.

Case 1 If R is empty or $x > \max(R)$, form P_{i+1} by appending x to row R and form Q_{i+1} by adding q in the corresponding position to Q_i. Terminate and return (P_{i+1}, Q_{i+1}).

Case 2 Otherwise, if $x \notin R$, locate the smallest y in R with $y > x$. Bump y with x and insert y into the next row of P_i.

Case 3 Otherwise, $x \in R$, so locate the smallest y in R with $y \leqslant x$ and interval $[y, x]$ contained in R. Row R remains unchanged and y is to be inserted into the next row of P_i.

Set $(P, Q) = (P_\ell, Q_\ell)$ if $[\mathbf{k}, \mathbf{h}]^t$ has length ℓ. Define the $\star$-*insertion* by $\star([\mathbf{k}, \mathbf{h}]^t) = (P, Q)$.

Furthermore, denote by $P \leftarrow x$ the tableau obtained by inserting x into P. The collection of all cells in $P \leftarrow x$, where insertion or bumping has occurred is called the *insertion path* for $P \leftarrow x$. In particular, in Case 1 the newly added cell is in the insertion path, in Case 2 the cell containing the bumped letter y is in the insertion path, and in Case 3 the cell containing the same entry as the inserted letter is in the insertion path.

Example 3.8. Let

$$\begin{bmatrix}\mathbf{k}\\ \mathbf{h}\end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 & 6 & 5 & 5 & 4 & 2 & 2 & 1 \\ 4 & 2 & 1 & 4 & 3 & 2 & 2 & 5 & 4 & 4 \end{bmatrix}.$$

Then we have $\star([\mathbf{k}, \mathbf{h}]^t) = (P, Q)$, where

$$P = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 4 \\ \cline{1-2} 2 & 5 \\ \hline 2 & 3 & 4 \\ \hline 1 & 2 & 4 \\ \hline \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|} \hline 7 \\ \cline{1-1} 5 \\ \cline{1-2} 4 & 7 \\ \hline 2 & 5 & 7 \\ \hline 1 & 2 & 6 \\ \hline \end{array}. \quad \text{Furthermore} \quad P \leftarrow 4 = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 4 \\ \cline{1-1} 2 \\ \cline{1-2} 2 & 5 \\ \hline 2 & 3 & 4 \\ \hline 1 & 2 & 4 \\ \hline \end{array}.$$

The cells in the insertion path of $P \leftarrow 4$ are highlighted in yellow.

Lemma 3.9. *Let $[\mathbf{k}, \mathbf{h}]^t$ be a 321-avoiding decreasing Hecke biword. Suppose that $\star([\mathbf{k}, \mathbf{h}]^t) = (P, Q)$. Then: (1) P^t is semistandard and Q has the same shape as P. (2) Let x be an integer such that $x\mathbf{h}$ is 321-avoiding. Then the insertion path for $P \leftarrow x$ goes weakly to the left.*

Theorem 3.10. *The $\star$-insertion is a bijection from the set of all 321-avoiding decreasing Hecke biwords to the set of all pairs of tableaux (P, Q) of the same shape, where both P^t and Q are semistandard and* $\mathsf{row}(P)$ *is 321-avoiding.*

4 Properties of the $\star$-insertion

In this section, we show that the $\star$-insertion intertwines with the crystal operators. More precisely, the insertion tableau remains invariant on connected crystal components under the $\star$-insertion by employing certain micro-moves, see [14]. The $\star$-crystal on $\mathcal{H}^{m,\star}$

intertwines with the usual crystal operators on semistandard tableaux on the recording tableaux under the $\star$-insertion. We relate the $\star$-insertion to the uncrowding operation.

Proposition 4.1. *For $\mathbf{h} \in \mathcal{H}^{m,\star}$ such that $f_k^\star(\mathbf{h}) \neq 0$ for some $1 \leqslant k < m$, the $\star$-insertion tableau for $\mathbf{h}$ equals the $\star$-insertion tableau for $f_k^\star(\mathbf{h})$.*

Proposition 4.2. *Let $T \in \mathsf{SSYT}(\lambda)$ and $(P, Q) = \star \circ \mathsf{res}(T)$. Then $Q = T$.*

Theorem 4.3. *For $\mathbf{h} \in \mathcal{H}^{m,\star}$ let $(P^\star(\mathbf{h}), Q^\star(\mathbf{h})) = \star(\mathbf{h})$ be the insertion and recording tableaux under the $\star$-insertion of Definition 3.7. Then (1) $f_i^\star(\mathbf{h})$ is defined if and only if $f_i(Q^\star(\mathbf{h}))$ is defined and (2) if $f_i^\star(\mathbf{h})$ is defined, then $Q^\star(f_i^\star(\mathbf{h})) = f_i Q^\star(\mathbf{h})$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathsf{SSYT} \\ \big\downarrow f_i^\star & & \big\downarrow f_i \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \mathsf{SSYT}. \end{array}$$

Remark 4.4. Theorem 4.3 and an analysis of the lowest weight elements provide another proof via $\star$-insertion, in the case where w is 321-avoiding, of the Schur positivity of $\mathfrak{G}_w$ of [5] $\mathfrak{G}_w = \sum_\mu \beta^{|\mu|-\ell(w)} g_w^\mu s_\mu$, where $g_w^\mu = |\{T \in \mathsf{SSYT}^n(\mu') \mid w_C(T) \equiv w\}|$.

Let λ, μ be partitions such that $\lambda \subseteq \mu$ and $\lambda_1 = \mu_1$. A *flagged increasing tableau* of shape μ/λ is a tableau of shape μ with fillings by positive integers in the skew shape μ/λ such that for all $1 \leqslant i \leqslant \ell(\mu)$ all entries in the i-th row of the tableau are at most $i-1$ and such that this filling is both row strict and column strict. In particular, the bottom row is empty. Denote the set of all flagged increasing tableaux of shape μ/λ by $\mathcal{F}_{\mu/\lambda}$. Flagged increasing tableaux are also called *elegant fillings* by some authors [7, 1].

The following definition is based on the uncrowding operator introduced by Reiner, Tenner and Yong [17, Definition 3.8].

Definition 4.5. Let $T \in \mathsf{SVT}(\lambda)$. Define an *uncrowding operation* on T as follows. Identify the topmost row in T that contains cells with more than one letter and let x be the largest letter in this row in a cell containing more than one letter. Remove x from this cell and perform RSK row bumping with x into the rows above. The resulting tableau T' is the output of this operation. The *uncrowding map*, denoted $\mathsf{uncrowd}$, is defined as follows. Let $T \in \mathsf{SVT}(\lambda)$ with $\mathsf{ext}(T) = \ell$.

- Start with $\widetilde{P}_0 = T$ and $\widetilde{Q}_0 = F$, where F is the unique flagged increasing tableau of shape λ/λ.
- For each $1 \leqslant i \leqslant \ell$, $\widetilde{P}_i$ is obtained by performing the uncrowding operator on $\widetilde{P}_{i-1}$. Suppose that cell C was added to form $\widetilde{P}_i$ by removing the largest entry in cell B in $\widetilde{P}_{i-1}$. Add a cell with entry k to $\widetilde{Q}_{i-1}$ at the same position as C, where k is the difference in the row indices of cells B and C.

- Terminate and return $(\widetilde{P}, \widetilde{Q}) = (\widetilde{P}_\ell, \widetilde{Q}_\ell)$.

Example 4.6. Let T be the semistandard set-valued tableau

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 4 & 5 \\ \cline{1-3} 2 & 23 & 3 \\ \hline 1 & 1 & 1 & 12 & 234 & 5 \\ \hline \end{array}\,. \quad \text{Perform an uncrowding operation to obtain} \quad \begin{array}{|c|c|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 2 & 2 & 3 \\ \hline 1 & 1 & 1 & 12 & 234 & 5 \\ \hline \end{array}\,.$$

Proceeding with uncrowding the remainder of the excess entries and recording the changes, we have $\mathsf{uncrowd}(T) = (\widetilde{P}, \widetilde{Q})$, where

$$\widetilde{P} = \begin{array}{|c|c|c|c|c|c|} \hline 5 & 5 \\ \cline{1-2} 4 & 4 \\ \cline{1-3} 3 & 3 & 4 \\ \cline{1-4} 2 & 2 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 & 2 & 5 \\ \hline \end{array} \quad \text{and} \quad \widetilde{Q} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 4 \\ \cline{1-2} \blacksquare & 3 \\ \cline{1-3} \blacksquare & \blacksquare & \blacksquare \\ \cline{1-4} \blacksquare & \blacksquare & \blacksquare & 1 \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}\,.$$

We remark that $\mathsf{uncrowd}\colon \mathsf{SVT}^m(\lambda) \to \bigsqcup_\mu \mathsf{SSYT}^m(\mu) \times \mathcal{F}_{\mu/\lambda}$, where the disjoint union is taken over partitions μ with $\lambda \subseteq \mu$ and $\mu_1 = \lambda_1$, is a bijection with inverse given by the crowding map crowd. A proof of this fact can be found in [12] (see also [17]). In addition, Monical, Pechenik and Scrimshaw in [13, Theorem 3.12] proved that the crystal operators on $\mathsf{SVT}^m(\lambda)$ intertwine with those on $\mathsf{SSYT}^m(\mu)$ under $\mathsf{uncrowd}$. Here we relate uncrowding with the $\star$-insertion.

Theorem 4.7. *Let $T \in \mathsf{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \mathsf{uncrowd}(T)$, $(P, Q) = \star \circ \mathsf{res}(T)$. Then $Q = \tilde{P}$.*

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